Convolution, Separation and Concurrency

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Abstract

A notion of convolution is presented in the context of formal power series together with lifting constructions characterising algebras of such series, which usually are quantales. A number of examples underpin the universality of these constructions, the most prominent ones being separation logics, where convolution is separating conjunction in an assertion quantale; interval logics, where convolution is the chop operation; and stream interval functions, where convolution is used for analysing the trajectories of dynamical or real-time systems. A Hoare logic is constructed in a generic fashion on the power series quantale, which applies to each of these examples. In many cases, commutative notions of convolution have natural interpretations as concurrency operations.

Keywords: formal power series, convolution, semigroups, quantales, formal semantics, systems verification, concurrency, separation logics, interval logics, Hoare logics

1 Introduction

Algebraic approaches play a fundamental role in mathematics and computing. Algebraic axioms for groups, rings, modules or lattices, for instance, capture certain features of concrete models in an abstract uniform fashion. Fundamental constructions, such as products, quotients or adjunctions, can be presented and investigated in algebra in simple generic ways.

This article investigates the notion of *convolution* or *Cauchy product* from formal language theory [12, 5] as such a fundamental notion, supporting the generic construction of various models and calculi that are interesting to computing. This provides a unified structural view on various computational models known from the computer science literature.

Questions of summability and divergence aside, the operational content of convolution is simple: an entity is separated in all possible ways into two parts, two functions are simultaneously applied to these parts, their outputs are combined, and the sum over all possible combinations is taken. Suppose two functions f and g from an algebra S (with suitable multiplication $\circ$) into an algebra Q (with suitable multiplication $\odot$ and suitable summation Σ). Using the nomenclature of formal language theory, the convolution of f and g for an element $x \in S$ is defined as

$$(f \otimes g)\, x \;=\; \sum_{x = y \circ z} f\, y \odot g\, z.$$

Hence x is first separated in all possible ways into parts y and z. The function f is then applied to y and g to z. After that, the results of these applications are combined in Q. The convolution is indeed the sum of all possible splittings of x.

In formal language theory, functions $f : S \to Q$ are also known as power series—more precisely as formal or rational power series. This notion is slightly different from that commonly used in algebra, as are the notions of convolution or Cauchy product. In formal language theory, moreover, power series usually map elements of the free monoid $S = X^*$ over the finite alphabet X—the set of words or strings over X—into a semiring $(Q, +, \odot, 0)$. Since every word can only be split into finitely many prefix/suffix pairs, the summation occurring in convolution is finite and therefore well defined. A simple example of Q is the boolean semiring with $+$ as disjunction and $\odot$ as conjunction. Power series then become characteristic functions representing languages, telling us whether or not some word is in some language, and convolution becomes language product. In more general settings, Q can model probabilities or weights associated to words; a Handbook has been devoted to the subject [12]. This example alone underpins the power of power series and convolution.

Complementing this body of work, we generalise the typeof power series, rebalancing the assumptions on source algebras S and target algebras Q and thus shifting the focus to other applications. Among those, we show that, for suitable algebras S and Q, convolution becomes *separating conjunction* of separation logic (cf. [7]), or alternatively the *chop* operator of interval temporal logics [25]. Both can in fact be combined, for instance within interval logics, to provide new notions of concurrency for this setting. In addition, we use power series to capture, in a generic manner, the algebraic properties of convolution for wide classes of instances and show how Hoare-style compositional inference systems can be derived uniformly for all of them.

More concretely, the main contributions of this article are as follows.

- Considering power series that map arbitrary partial semigroups into quantales, we prove a generic lifting result showing that spaces of power series form quantales as well.

- This lifting result is generalised by making the target quantale partial, by considering bi-semigroups and bi-quantales with two multiplication operations, by mapping two separate semigroups into a bi-quantale, and by setting up source semigroups suitable for distinguishing between finite and infinite system behaviours.

- We show that algebras of state and predicate transformers arise as instances of the generic lifting theorem.
- Propositional Hoare calculi (without assignment axioms) are derived within the power series quantale in a generic fashion; and we discuss some ramifications of deriving concurrency rules in this setting.
- We provide a series of instances of the lifting result, showing how quantales of languages, binary relations, matrices and automata, sets of paths and traces as well as interval functions and predicates arise from a non-commutative notion of convolution.
- In the commutative case, we present the assertion quantales of separation logic with separation based on general resource monoids as well as multisets, sets with disjoint union and heaplets. We also present a separation operation on finite vectors, which leads to a notion of convolution-based parallelism for linear transformations.
- Both kinds of instances are combined into a new algebraic approach to stream interval functions and predicates, which allow the logical analysis of trajectories of dynamic and real time systems. This provides a convolution-based spatial concurrency operation in addition to the conventional temporal chop operator.
- We illustrate how convolution as separating conjunction allows us to derive the frame rule of separation logic by simple equational reasoning.

Our lifting results are generic in the following sense: after setting up a suitable partial semigroup—words under concatenation, closed intervals under chop, multisets under addition or resource monoids under resource aggregation—the space of all functions into a quantale automatically forms a quantale with convolution as multiplication. When the target quantale is formed by the booleans, power series can be identified with and predicates and characteristic functions for sets, as their extensions. Multiplication in the booleans becomes conjunction and convolution then reduces to

$$(f \otimes g)\, x = \sum_{x = y \circ z} f\, y \sqcap g\, z.$$

If S is a set of resources and $\circ$ a (commutative) notion of resource aggregation, then convolution is separating conjunction. If S is a set of closed intervals and $\circ$ splits an interval into two disjoint parts, then convolution is chop. In that sense, separating conjunction can be seen as a language product over resources and chop as a language product over intervals. Here and in all similar cases, our lifting result implies that the predicates of type $S \to \mathbb{B}$ form an assertion quantale; in the first case that of separation logic; in the second one that of interval logics. But our results cover models beyond the booleans, for instance probabilistic or weighted predicates or other kinds of functions. In general, the convolution has a strongly spatial and concurrent flavour whenever the operations $\circ$ and $\odot$ are commutative.

Similarly, for all instances of this lifting, the construction of Hoare logics is generic because it works for abitrary quantales [20]. Finally, due to the emphasis on functions instead of sets, the approach is constructive so long as the underlying source and target algebras are.

The remainder of this article is organised as follows. Section 2 recalls the basic algebraic structures needed. Section 3 introduces our approach to power series with partial semigroups as source algebras and quantales as target algebras; it also proves our basic lifting result. Section 4 discusses the case of power series into the boolean quantale, when convolution becomes a possibly non-commutative notion of separating conjunction. Section 5 and 6 present non-commutative and commutative instances of our lifting lemma; Section 5 discussing, among others, the chop operation over intervals and Section 6 focusing on variants of separating conjunction. Section 7 shows how state and predicate transformers arise in the power series setting. Section 8 presents a lifting result for power series into partial quantales with an example. Section 9 generalises the lifting result to bi-semigroups and bi-quantales and presents two examples. Section 10 generalises the result to power series from two semigroups into a bi-quantale; Section 11 presents in particular the quantale of stream interval functions, which is based on this generalisation. Section 12 further generalises the approach to applications with finite and infinite behaviours. Section 13 shows that the interchange laws of concurrent Kleene algebras fail in general power series quantales. Based on this, Section 14 discusses how generic Hoare logics can be developed over power series quantales. Section 15 shows how the approach can be used for deriving the frame rule of separation logic, using convolution as the algebraic notion of separating conjunction. Section 16 contains a conclusion.

2 Algebraic Preliminaries

In this section, we briefly recall the most important mathematical structures used in this article: partial semigroups and monoids, their commutative variants, semigroups and dioid as well as quantales. We also consider such structures with two operations of composition or multiplication, that is, bi-semigroups, bi-monoids, bi-semirings and bi-quantales.

Semigroups. A *partial semigroup* is a structure $(S, \cdot, \bot)$ such that $(S, \cdot)$ is a semigroup and $x \cdot \bot = \bot = \bot \cdot x$ holds for all $x \in S$. It follows that $\bot \notin S$, which is significant for various definitions in this article. A *partial monoid* is a partial semigroup with multiplicative unit 1. We often write $(S, \cdot)$ for partial semigroups and $(S, \cdot, 1)$ for partial monoids, leaving $\bot$ implicit. A (partial) semigroup S is *commutative* if $x \cdot y = y \cdot x$ for all $x, y \in S$. Henceforth, we use $\cdot$ for a general multiplication and $*$ for a commutative one.

An important property of semigroups is *opposition duality.* For every semigroup $(S, \cdot)$, the structure $(S, \odot)$ with $x \odot y = y \cdot x$ for all $x, y \in S$ forms a semigroup; the *opposite* of S. Similarly, the opposite of a monoid is a monoid.

The definitions of semigroups and monoids generalise to n operations, but we are mainly interested in the case $n = 2$. A *partial bi-semigroup* is a structure $(S, \circ, \bullet)$ such that $(S, \circ)$

and $(S, \bullet)$ are partial semigroups. *Partial bi-monoids* $(S, \circ, \bullet, 1, 1')$ can be obtained from them as standard.

Semirings. A *semiring* is a structure $(S, +, \cdot, 0)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot)$ a semigroup, and the distributivity laws $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ as well as the annihilation laws $0 \cdot x = 0$ and $x \cdot 0 = 0$ hold. A semiring is *unital* if the multiplicative reduct is a monoid (with unit 1). A *dioid* is an additively idempotent semiring S, that is, $x + x = x$ holds for all $x \in S$. The additive reduct of a dioid thus forms a semilattice with order defined by $x \leq y \Leftrightarrow x + y = y$. Obviously, the classes of semirings and dioids are closed under opposition duality.

A *bi-semiring* is a structure $(S, +, \circ, \bullet, 0)$ such that $(S, +, \circ, 0)$ and $(S, +, \bullet, 0)$ are semirings; a *trioid* is an additively idempotent bi-semiring. A bi-semiring or trioid is *unital* if the underlying bi-semigroup is a bi-monoid.

Quantales. A *quantale* is a structure $(Q, \leq, \cdot)$ such that $(Q, \leq)$ is a complete lattice, $(Q, \cdot)$ is a semigroup and the distributivity axioms

$$x \cdot (\sum_{i \in I} y_i) = \sum_{i \in I} (x \cdot y_i), \qquad (\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} (x_i \cdot y)$$

hold, where $\sum X$ denotes the supremum of a set $X \subseteq Q$. Similarly, we write $\prod X$ for the infimum of X. The distributivity laws imply, in particular, the isotonicity laws

$$x \leq y \Rightarrow z \cdot x \leq z \cdot y, \qquad x \leq y \Rightarrow x \cdot z \leq y \cdot z.$$

A quantale is *commutative* and *partial* if the underlying semigroup is as well; *unital* if the underlying semigroup is a monoid; and *distributive* if the infinite distributivity laws

$$x \sqcap (\sum_{i \in I} y_i) = \sum_{i \in I} (x \sqcap y_i), \qquad x + (\prod_{i \in I} y_i) = \prod_{i \in I} (x + y_i)$$

hold. A *boolean quantale* is a distributive quantale in which every element has a complement.

The boolean unital quantale $\mathbb{B}$, where multiplication $\cdot$ coincides with meet, plays an important role in this article.

A *bi-quantale* is a structure $(Q, \leq, \circ, \bullet)$ such that $(Q, \leq, \circ)$ and $(Q, \leq, \bullet)$ are quantales. It is unital if the two underlying semigroups are monoids.

It is easy to see that every (unital) quantale is a (unital) dioid and every (unital) bi-quantale a (unital) trioid. In particular, $0 = \sum \emptyset = \sum_{i \in \emptyset} x_i$ and annihilation laws as in dioids follow from this as special cases of distributivity.

3 Power Series Quantales

Formal (or rational) power series [5] have been studied in formal language theory for decades. For brevity, we call them *power series* in this article. In formal language theory, a power series is simply a function from the free monoid X^* over a finite alphabet X into a suitable algebra Q, usually a semiring or dioid $(Q, +, \cdot, 0, 1)$.

Operations on $f, g : X^* \to Q$ are defined as follows. Addition is lifted pointwise, that is, $(f + g)\,x = f\,x + g\,x$. Multiplication is given by the *convolution* or *Cauchy product*

$$(f \cdot g)\,x = \sum_{x=yz} f\,y \cdot g\,z,$$

where yz denotes word concatenation and the sum in the convolution is finite since finite words can only be split in finitely many ways into prefix/suffix pairs. Furthermore, the *empty power series* $\mathbb{0}$ maps every word to 0, whereas the *unit power series* $\mathbb{1}$ maps the empty word to 1 and all other words to 0.

We write Q^{X^*} for the set of power series from X^* to Q and, more generally, Q^S for the class of functions of type $S \to Q$. The following lifting result is well known.

Proposition 1. *If $(Q, +, \cdot, 0, 1)$ is a semiring (dioid), then so is $(Q^{X^*}, +, \cdot, \mathbb{0}, \mathbb{1})$.*

This construction generalises from free monoids over finite alphabets to arbitrary partial semigroups or monoids. The sums in convolutions then become infinite due to infinitely many possible decompositions of elements. Here, due to potential divergence, these sums may not exist. However, we usually consider target algebras in which addition is idempotent and sums corresponds to suprema. The existence of arbitrary suprema can then be covered by completeness assumptions.

We fix suitable algebraic structures S and Q. First, we merely assume that S is a set, but for more powerful lifting results it is required to be a partial semigroup or partial monoid.

For a family of functions $f_i : S \to Q$ and $i \in I$ we define

$$(\sum_{i \in I} f_i)\,x = \sum_{i \in I} f_i\,x,$$

whenever the supremum in Q at the right-hand side exists. This comprises

$$(f + g)\,x = f\,x + g\,x$$

as a special case. Since x ranges over S, the constant $\perp$ is excluded as a value. Another special case is

$$(\sum_{i \in \emptyset} f_i)\,x = (\sum \emptyset)\,x = \sum_{i \in \emptyset} f_i\,x = 0.$$

Hence, in particular, $\sum_{i \in \emptyset} f_i = \lambda x.\ 0$ and we write $\mathbb{0}$ for this function.

We define the convolution

$$(f \cdot g)\,x = \sum_{x = y \cdot z} f\,y \cdot g\,z,$$

where the multiplication symbol is overloaded to be used on S, Q and Q^S. Again, this requires that the supremum in the right-hand side exists in Q. In the expression $x = y \cdot z$, the constant $\perp$ is again excluded as a value. Undefined splittings of x are thus excluded from contributing to convolutions.

Finally, whenever S and Q are endowed with suitable units, we define $\mathbb{1} : S \to Q$ as

$$\mathbb{1}\, x = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise,} \end{cases}$$

as for formal languages.

Theorem 1, the main result in this section, shows that quantale laws lift from the algebra Q to the function space Q^S of power series under these definitions. On the way to this result we recall that semilattice and lattice structures lift to function spaces, a fundamental result of domain theory [1].

Lemma 1. *Let S be a set. If $(L, +, 0)$ is a semilattice with least element 0 then so is $(L^S, +, \mathbb{0})$. If L is a complete lattice, then so is L^S.*

Proof. The semilattice lifting is covered by Proposition 1. As usual, L^S is ordered by $f \leq g \Leftrightarrow f + g = g$, and $\mathbb{0} \leq f$ for all $f \in L^S$.

If arbitrary suprema exist in L, then completeness lifts to L^S by definition of $\sum_{i \in I} f_i$. Finally, every complete join-semilattice is a complete lattice. □

Infima, if they exist, are defined like suprema by pointwise lifting as

$$(\prod_{i \in I} f_i)\, x = \prod_{i \in I} f_i\, x,$$

thus $(f \sqcap g)\, x = flx \sqcap g\ x$. Lemma 1 can then be strengthened.

Lemma 2. *Let S be a set. If $(D, +, \sqcap, 0)$ is a (distributive) lattice with least element 0, then so is $(D^S, +, \sqcap, \mathbb{0})$. Completeness and infinite distributivity laws between infima and suprema lift from D to D^S.*

Proof. The join- and meet-semilattice laws for $+$ and $\sqcap$ follow from Lemma 1. We need to verify absorption and distributivity. Let $f, g, h : S \to D$ and $x \in S$.

- $(f \sqcap (f + g))\, x = f\, x \sqcap (f\, x + g\, x) = f\, x$ by absorption on D. The proof of $f + (f \sqcap g) = f$ is lattice dual.
- The finite distributivity laws are special cases of the infinite ones below.

Completeness is covered by Lemma 1. For infinite distributivity,

$$(f \sqcap \sum_{i \in I} g_i)\, x = f\, x \sqcap \sum_{i \in I} g_i\, x = \sum_{i \in I} f\, x \sqcap g_i\, x = \sum_{i \in I} (f \sqcap g_i)\, x = (\sum_{i \in I} f \sqcap g_i)\, x.$$

The other distributivity law then follows from lattice duality. □

The final lifting result in this section deals with multiplicative structure as well. This requires S to be a partial semigroup instead of a set.

Theorem 1. *Let $(S, \cdot)$ be a partial semigroup. If $(Q, \leq, \cdot)$ is a (distributive) quantale, then so is $(Q^S, \leq, \cdot)$. In addition, commutativity in Q lifts to Q^S if S is commutative; unitality in Q lifts to Q^S if S is a partial monoid.*

Proof. Since Q is a quantale, all infinite suprema and infima exist; in particular those needed for convolutions.

The lifting to complete (distributive) lattices is covered by Lemma 2. It therefore remains to check the multiplicative monoid laws, distributivity of multiplication and annihilation. For left distributivity, for instance,

$$(f \cdot \sum_{i \in I} g_i)\, x = \sum_{x = y \cdot z} f\, y \cdot \sum_{i \in I} g_i\, z = \sum_{\substack{x = y \cdot z, \\ i \in I}} f\, y \cdot g_i\, z = \sum_{i \in I} (f \cdot g_i)\, x.$$

The proof of right distributivity is opposition dual.

Left distributivity ensures associativity, the proof of which lifts as with rational power series (Proposition 1). The restriction to partial semigroups is insignificant as, in $x = y \cdot z$, the constraint $x \in S$ only rules out contributions of $y \cdot z = \perp$. The same holds for unitality proofs.

Commutativity lifts from S and Q as follows:

$$(f \cdot g)\, x = \sum_{x = y \cdot z} f\, y \cdot g\, z = \sum_{x = z \cdot y} g\, z \cdot f\, y = (g \cdot f)\, x.$$

□

Once more the distributivity laws on Q^S imply the annihilation laws $\mathbb{0} \cdot f = \mathbb{0}$ and $f \cdot \mathbb{0} = \mathbb{0}$ for all $f : S \to Q$. When only finite sums are needed, Q can be assumed to be a semiring or dioid instead of a quantale. The following corollary to Theorem 1 provides an example.

Corollary 1. *Let $(S, \cdot)$ be a finite partial semigroup. If $(Q, +, \cdot, 0)$ is a semiring, then so is $(Q^S, +, \cdot, \mathbb{0})$. In addition, idempotency in Q lifts to Q^S; commutativity in Q lifts to Q^S if S is commutative; unitality in Q lifts to Q^S if S is a partial monoid.*

As another specialisation, Proposition 1 is recovered easily when S is the free monoid over a given alphabet and Q a semiring or dioid.

4 Power Series into the Boolean Quantale

In many applications, the target quantale Q is formed by the booleans $\mathbb{B}$. Power series are then of type $S \to \mathbb{B}$ and can be interpreted as characteristic functions or predicates. In fact,

$\mathbb{B}^S$ is isomorphic to the power set of S, which, in turn is in one to one correspondence with the set of all predicates over S, identifying predicates with their extensions.

In this context, Theorem 1 specialises to the powerset lifting of a partial semigroup or monoid S. For each $x \in S$, the boolean value $f\,x$ expresses whether or not x is in the set corresponding to f. Powerset liftings have been studied widely in mathematics [15, 6]. They have various applications in program semantics, for instance as power domains (cf. [1]).

Corollary 2. *Let S be a partial (commutative) semigroup. Then $\mathbb{B}^S$ forms a (commutative) distributive quantale where $\mathbb{B}^S \cong 2^S$, $\leq$ corresponds to $\subseteq$ and convolution $\cdot$ to the complex product*

$$X \cdot Y = \{x \cdot y \mid x \in X \wedge y \in Y\}$$

for all $X, Y \subseteq S$. If S has unit 1, then $\mathbb{B}^S$ has unit $\{1\}$.

Various instances of Corollary 2 are discussed in Sections 5 and 6.

The quantale $\mathbb{B}^S$ carries a natural logical structure with elements of $\mathbb{B}^S$ corresponding to predicates, suprema to existential quantification, infima to universal quantification and the lattice order to implication. In particular, $+$ corresponds to disjunction and $\sqcap$ to conjunction.

More interesting is the logical interpretation of convolution

$$(f \cdot g)\,x = \sum_{x = y \cdot z} f\,y \cdot g\,z$$

in the boolean quantale $\mathbb{B}^S$. The expression $x = y \cdot z$ denotes the decomposition or separation of the semigroup element x into parts y and z. The composition $f\ y \cdot g\ z = f\ y \sqcap g\ z$ in $\mathbb{B}$ models the conjunction of predicate f applied to y with predicate g applied to z. Finally, the supremum $\sum$ models the existential quantification over these conjunctions with respect to all possible decompositions of x.

The commutative case of Corollary 2 is immediately relevant to separation logic. In this context, the partial commutative semigroup $(S, *)$ is know as the *resource semigroup* [7]; it provides an algebraic abstraction of the heap. Its powerset lifting $\mathbb{B}^S$ captures the algebra of resource predicates that form the assertions of an extended Hoare logic—the assertion quantale of separation logic. In this assertion quantale, separating conjunction is precisely convolution: the product $x = y * z$ on the resource semigroup S decomposes or separates the resource or heap x into parts of heaplets y and z and the product $f\ y * g\ z = f\ y \sqcap g\ z$ in $\mathbb{B}$ once more conjoins $f\ y$ and $g\ z$; hence $x = y * z$ separates whereas $f\ y * g\ z = f\ y \sqcap g\ z$ conjoins. The concrete case of the heap is considered in more detail in Example 12.

The power series approach thus yields a simple algebraic view on a lifting to function spaces in which the algebraic operation of convolution into the booleans allows various interpretations, including that of a complex product, that of separating conjunction—commutative or non-commutative—and that of separating conjunction as a complex product. In the commutative setting it gives a simple account of the category-theoretical approach to O'Hearn and Pym's logic of bunched implication [27] in which convolution corresponds to coends and the quantale lifting is embodied by Day's construction [9].

5 Non-Commutative Examples

After the conceptual development of the previous sections we now discuss a series of examples which underpin the universality and relevance of the notion of convolution in computing. All of them can be obtained as instances of Theorem 1 after setting up partial semigroups or monoids appropriately. For all these structures, the lifting to the function space is then generic and automatic. The booleans often form a particularly interesting target quantale.

This section considers only examples with a non-commutative notion of convolution; for commutative examples see Section 6.

Example 1 (Formal Languages)**.** Let $(X^*, \cdot, \varepsilon)$ be the free monoid generated by the finite alphabet X with ε denoting the empty word. Let Q form a distributive unital quantale. Then Q^{X^*} forms a distributive unital quantale as well by Theorem 1. More precisely, since suprema in convolutions are always finite, one obtains the unital dioid $(Q^{X^*}, +, \cdot, \mathbb{0}, \mathbb{1})$ by lifting from a dioid $(Q, +, \cdot, 0, 1)$. This is the well known rational power series dioid of formal language theory. For $Q = \mathbb{B}$ one obtains, by Corollary 2, the quantale $\mathbb{B}^{X^*}$ of formal languages over X. □

Example 2 (Binary Relations)**.** For a set A consider the partial semigroup $(A \times A, \cdot)$ with $\cdot$ defined, for all $a, b, c, d \in A$, by

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c, \\ \perp, & \text{otherwise.} \end{cases}$$

For $Q = \mathbb{B}$, Theorem 1 (or its Corollary 2) ensures that $(\mathbb{B}^{A\times A}, \leq, \cdot)$, which is isomorphic to $(2^{A\times A}, \subseteq, \cdot)$, is the quantale of binary relations under union, intersection, relational composition and the empty relation.

More specifically, with every power series f we associate a binary relation R_f defined by $(a, b) \in R_f \Leftrightarrow f\ (a, b) = 1$. The empty relation $\emptyset$ obviously corresponds to the power series defined by $\mathbb{0}\,(a, b) = 0$ for all $a, b \in A$. Relational composition is given by convolution

$$(f \cdot g)\,(a, b) = \sum_{c\in A} f\,(a, c) \cdot g\,(c, b).$$

It can then be checked that $R_{f\cdot g} = R_f \cdot R_g = \{(a, b) \mid \exists c.(a, c) \in R_f \wedge (c, b) \in R_g\}$.

The unit relation cannot be lifted from a unit in $A \times A$ because $A \times A$ has no unit. Instead it can be defined on $\mathbb{B}^{A\times A}$ directly as

$$\mathbb{1}\,(a, b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

□

The constructions for relations generalise, for instance, to probabilistic or fuzzy relations where $Q \neq \mathbb{B}$, but this is not explored any further. Instead we consider the case of matrices.

Example 3 (Matrices)**.** Matrices are functions $f : A_1 \times A_2 \to B$, where A_1 and A_2 are index sets and Q is a suitable coefficient algebra. For the sake of simplicity we restrict our attention to square matrices with $A_1 = A_2 = A$. General non-square matrices require more complex partiality conditions.

The development is similar to binary relations, but uses coefficient algebras beyond $\mathbb{B}$. It is easy to check that matrix addition is modelled by

$$(f + g)\,(i, j) = f\,(i, j) + g\,(i, j),$$

whereas matrix multiplication is given by convolution

$$(f \cdot g)\,(i, j) = \sum_{k \in A} f\,(i, k) \cdot g\,(k, j),$$

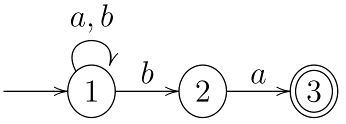
under suitable restrictions to guarantee the existence of sums, such as finiteness of A or idempotency of additionin Q. The zero and unit matrices are defined as in the relational case.

$$\mathbb{1}\,(i, j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbb{0}\,(i, j) = 0.$$

Theorem 1 then shows that quantales are closed under matrix formation. It can easily be adapted to showing that square matrices of finite dimension over a semiring form a semiring or that matrices over a dioid form a dioid. □

This example not only links matrices with power series, it also yields a simple explanation of the well known relationship between binary relations and boolean matrices. If a relation $R \subseteq A \times A$ is modelled as $f_R : A \times A \to \mathbb{B}$ defined by $f_R\,(a, b) = 1 \Leftrightarrow (a, b) \in R$ as indicated above, then it *is* a boolean matrix.

Example 4 (Finite Automata)**.** Suppose V is a set of state symbols, X an alphabet, $i \in V$ the initial state and $F \subseteq V$ a set of final states. Conway [8] has shown that transition relations δ of finite automata (V, X, δ, i, F) can be modelled in terms of finite matrices of type $V \times V \to \mathsf{Rex}(X)$ into the algebra of regular expressions $\mathsf{Rex}(X)$ over X, for instance a Kleene algebra with constants from X. Consider the following automaton and transition matrix as an example.



$$\begin{pmatrix} a+b & b & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

More generally, the full automaton, including its initial and final state information, is captured by the following triple.

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a+b & b & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

It is well known that the algebra of regular expressions forms a dioid, hence Theorem 1 applies, showing that transition matrices over the dioid of regular expressions form a dioid, as in Example 3. Other kinds of automata, such as probabilistic or weighted ones, can be modelled along this line. □

In fact, it has been shown that Kleene algebras are closed under matrix formation [24], but the neccessary treatment of the Kleene star is beyond the scope of this article. In addition, it is well known that regular languages need not be closed under general unions, hence do not form quantales.

Example 5 (Trace Functions)**.** Let V be a finite set of state symbols and X a finite set of transition symbols, as in a finite automaton. A *trace* [13] is a finite word over $(V \cup X)^*$ in which state and transition symbols alternate, starting and ending with state symbols. We write $T(V, X)$ for the set of traces over V and X. It is endowed with a partial monoid structure by defining, for $p_1\alpha_1q_1, p_2\alpha_2q_2 \in T(V, X)$, the *fusion product*

$$p_1\alpha_1q_1 \cdot p_2\alpha_2q_2 = \begin{cases} p_1\alpha_1q_1\alpha_2q_2, & \text{if } q_1 = p_2, \\ \perp, & \text{otherwise.} \end{cases}$$

Then convolution becomes

$$(f \cdot g)\,\tau = \sum_{\tau = p\alpha_1 r \cdot r\alpha_2 q} f\,p\alpha_1 r \cdot g\,r\alpha_2 q$$

and Theorem 1 implies that the set $Q^{T(V,X)}$ of trace functions into the distributive quantale Q forms a distributive quantale. If Q is unital, then $Q^{T(V,X)}$ becomes unital by defining

$$\mathbb{1}\,x = \begin{cases} 1, & \text{if } x \in V, \\ 0, & \text{otherwise.} \end{cases}$$

For $Q = \mathbb{B}$ we obtain the well known quantale of sets of traces.

Trace functions $\mathbb{B}^{T(X,V)}$ have a natural interpretation as trace predicates. Convolution $(f \cdot g)\,\tau$ indicates the various ways in which property f holds on a prefix of trace τ whereas property g holds conjunctively on the consecutive suffix, as for instance in temporal logics over computation traces or paths. □

Sets of traces generalise both languages and binary relations, which are obtained by forgetting structure in the underlying partial monoid. Another special case is given by sets of paths in a graph, which is obtained by forgetting state labels. The explicit construction of the corresponding paths quantale is straightforward and therefore not shown.

Example 6 (Interval Functions)**.** Let $(P, \leq)$ be a linear order and I_P the set of all closed intervals over P—the empty interval being open by definition. For an interval x, let x_{min} and x_{max} represent respectively the minimum and maximum value in x.

We impose a partial semigroup structure on I_P be defining the *fusion product* on I_P, similar to the case of binary relations, traces and matrices, as

$$x \cdot y = \begin{cases} x \cup y, & \text{if } x_{max} = y_{min}, \\ \bot, & \text{otherwise.} \end{cases}$$

An *interval function* is a function $f : I_P \to Q$ into a suitable algebra. Whenever Q is a (distributive) quantale, Theorem 1 applies and Q^{I_P} forms a (distributive) quantale, too. Convolution of interval functions is given by

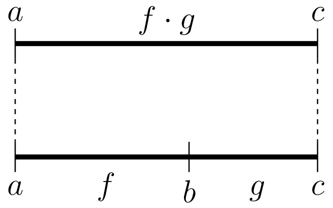
$$(f \cdot g)\, x = \sum_{x = y \cdot z} f\, y \cdot g\, z.$$

Like in the case of relations, the unit interval function is not lifted from I_P, but defined directly as

$$\mathbb{1}\,[a,b] = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

The quantale of interval functions then becomes unital.

Interval predicates are functions of type $I_P \to \mathbb{B}$. Convolution of interval predicates is known as the *chop* operation [25], where $(f \cdot g)\,[a,c]$ holds if it is possible to split interval $[a,c]$ into $[a,b]$ and $[b,c]$ such that $f\,[a,b]$ and $g\,[b,c]$ hold in conjunction.



The meaning of an interval predicate $f\,x$ can be defined in various ways. For instance f can hold somewhere (at some point) in x or (almost) everywhere (see [25, 30]), and it is even possible to define and use non-deterministic evaluators [18] that enable calculations of apparent states (see [11]). □

Naive use of interval predicates may have undesired effects: If $f\,x$ means that f holds at each point in interval x, then $(f \cdot \neg f)$ is always false, since both f and $\neg f$ would have to hold in at least one fusion point, which is impossible. An alternative definition of interval composition without fusion therefore seems desirable.

The duration calculus presents a solution in terms of an ‘almost everywhere’ operator, such that a property holds almost everywhere in an interval if it is false in the interval for a set of points of measure zero [30]. Others have defined ‘jump conditions’ leaving the possibility of both f and $\neg f$ holding at the fusion point open [21]. Here we model a third approach [11], with chop formalised over non-overlapping intervals, in the power series setting.

Example 7 (Intervals without Fusion)**.** We define a composition of contiguous intervals that avoids fusion. To this end we consider the set I_P of intervals of the form (a, b), $(a, b]$, $[a, b)$ and $[a, b]$, for $a, b \in P$. We include the empty interval $\emptyset$, which is by definition equal to (a, a), $(a, a]$ and $[a, a)$ for all $a \in P$. The interval x precedes the interval y, written $x \prec y$, if $\forall a \in x, b \in y.\ a < b$. The composition of intervals is defined as

$$x \cdot y = \begin{cases} x \cup y, & \text{if } x \cup y \in I_P \text{ and } x \prec y, \\ \bot, & \text{otherwise.} \end{cases}$$

Convolution $f \cdot g$ is then defined as usual. Theorem 1 ensures once more that Q^{I_P} forms a distributive quantale whenever Q does. The unit $\mathbb{1} : I_P \to Q$, however, requires modification. Defining

$$\mathbb{1}\, x = \begin{cases} 1, & \text{if } x = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

it is easy to check that $(\mathbb{1} \cdot f)\ x = f\, x = (f \cdot \mathbb{1})\, x$ for any interval x and the new definition of interval composition. This makes the quantale Q^{I_P} unital. □

The examples in this section show that the generic lifting construction in Theorem 1 allows a uniform treatment of a variety of mathematical objects, including relations, formal languages, matrices and sets of intervals. In each case, a (partial) composition on the underlying objects needs to be defined, e.g., on words, ordered pairs, index pairs of matrices, traces, paths or intervals. Lifting to the function space is then generic.

Such a generic lifting has been discussed previously for languages, relations, paths and traces in the context of an Isabelle/HOL library with models of Kleene algebras [3, 2]. Theorem 1 has, in fact, already been implemented in Isabelle. Based on this, the existing implementation of models of Kleene algebras can be unified and simplified considerably.

6 Commutative Examples

This section provides instances of Theorem 1 and Corollary 2 for the commutative case. As discussed in Section 4, this situation typically arises when the composition of the underlying semigroup $(S, *)$ is used to split resources, heaps, states, etc, in a spatial fashion, which is in contrast to the previous section where $f \cdot g$ meant that there was a dependency between f and g, which often carries a temporal meaning. One can often think of convolution instantiated to such a spatial separation in terms of parallelism or concurrency.

In particular we instantiate Theorem 1 to four kinds of resource monoids based on multisets under multiset union, sets under disjoint union, partial functions under union and vectors. Notions of separating conjunction as convolution arises in all these examples in a natural way. In the disjoint union and vector examples, the relationship between convolution, separation and concurrency becomes most apparent. Previously, this observation of separating conjunction as a notion of concurrency with a strongly spatial meaning has been one of the motivations for concurrent separation logic [7] and concurrent Kleene algebra [20].

As a preparation we show how multisets with multiset union and sets with disjoint union arise in the power series setting.

Example 8 (Multisets)**.** Let S be a set and let $f : S \to \mathbb{N}$ assign a multiplicity to elements of S. Consider the max/min-plus algebra over $\mathbb{N}$ [16], which forms a commutative distributive quantale. Define, rather artificially, a partial semigroup on S by stipulating

$$x * y = \begin{cases} x, & \text{if } x = y, \\ \bot, & \text{otherwise.} \end{cases}$$

Then $\mathbb{N}^S$ is the set of multisets over the set S which, by Theorem 1, forms a commutative distributive quantale under the operations

$$(f \uplus g)\, x = (f * g)\, x = \sum_{x = x * x} f\, x + g\, x = f\, x + g\, x,$$

$$(\sum_{i \in I} f_i)\, x = \max_{i \in I}(f_i\, x), \qquad (\prod_{i \in I} f_i)\, x = \min_{i \in I}(f_i\, x).$$

The "convolution" $\uplus$ is the usual multiset addition. For example,

$$a^2b^5c \uplus ab^3d^2 = a^3b^8cd^2,$$
$$a^2b^5c + ab^3d^2 = a^2b^5cd^2,$$
$$a^2b^5c \sqcap ab^3d^2 = ab^3.$$

Example 9 (Powersets)**.** Under the same conditions as in Example 8, suppose that $f : S \to \mathbb{B}$ is the characteristic function which determines the subsets of S. Then $\mathbb{B}^S \cong 2^S$ reduces to the complete distributive lattice of powersets of S; the ring of sets over S. In particular, $f \uplus g = \max(f, g)$. This lifting implements the powerset functor. □

Theorem 1 shows that the function space Q^S from a partial commutative semigroup S into a commutative quantale Q forms a commutative quantale. In addition, we have seen in Section 4, that, in that case, $\mathbb{B}^S$ may yield the quantale of resource predicates in which convolution is separating conjunction. We now discuss four special cases of separating conjunction.

Example 10 (Separating Conjunction on Multisets)**.** The free commutative monoid $(X^*, *, 0)$ generated by the alphabet X is isomorphic to the set of all multisets over X with $*$ being multiset addition $\uplus$. By Theorem 1, Q^{X^*} forms a commutative quantale if Q does; distributivity and unitality lift as usual.

Convolution $(f * g)\, x = \sum_{x = y * z} f\, y * g\, z$ separates the multiset or resource x in all possible ways and then applies the functions f and g to the result, depending on the interpretation of multiplication in Q. For $Q = \mathbb{B}$, $\mathbb{B}^{X^*}$ forms the resource predicate quantale over multisets. Convolution $f * g$ is separating conjunction as a complex product on sets of multisets based on multiset addition as a separator:

$$(f * g)\, x = \sum_{x = y \uplus z} f\, y \sqcap g\, z.$$

□

In many contexts, multisets form a paradigmatic data type for resources.

Example 11 (Separating Conjunction on Sets)**.** The free commutative idempotent monoid $(X^*, *, 0)$ generated by the alphabet X is isomorphic to 2^X with $*$ being union. More interesting in our context is the consideration of disjoint union, which is defined as

$$x \oplus y = \begin{cases} x \cup y, & \text{if } x \cap y = 0, \\ \bot, & \text{otherwise.} \end{cases}$$

Then $(X^*, \oplus, 0, \bot)$ forms a partial commutative monoid and, by Theorem 1, Q^{X^*} forms a commutative quantale. Convolution $(f * g)\, x$ now separates the set x into disjoint subsets and then applies the functions f and g to these subsets, depending on the interpretation of $*$ in the target quantale. For target quantale $\mathbb{B}$ we obtain the resource predicate quantale $\mathbb{B}^{X^*}$ on power sets based on disjoint union as a separator:

$$(f * g)\, x = \sum_{x = y \oplus z} f\, y \sqcap g\, z.$$

□

This kind of separating conjunction is particularly appropriate for (indexed) families.

Example 12 (Separating Conjunction on Heaplets)**.** Let $(S, *, 0)$ be the partial commutative monoid of partial functions $\eta : A \to B$ with empty function $0 : A \to B$ and composition defined by

$$\eta_1 * \eta_2 = \begin{cases} \eta_1 \cup \eta_2, & \text{if } dom(\eta_1) \cap dom(\eta_2) = \emptyset, \\ \bot, & \text{otherwise.} \end{cases}$$

The functions η are sometimes called *heaplets* and used to model a memory heap. As usual, by Theorem 1, Q^S forms a commutative distributive unital quantale whenever Q does. In particular, $\mathbb{B}^S$ forms an algebra of heap assertions with convolution as separating conjunction over the heap. □

Example 13 (Separating Conjunction on Vectors)**.** Consider a set S of vectors x of fixed dimension $|x| = n$. We turn this into a partial commutative semigroup by defining composition as

$$(x * y)_i = \begin{cases} x_i, & \text{if } y_i = 0, \\ y_i, & \text{if } x_i = 0, \\ \bot, & \text{otherwise.} \end{cases}$$

Also let $x = \bot$ if $x_i = \bot$ for some $1 \leq i \leq n$. It is obvious from this definition that the zero vector 0 is a unit with respect to $*$. For example,

$$\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} * \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} * \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \bot$$

Then Theorem 1 implies that Q^S forms a commutative distributive unital quantale whenever Q does, and $\mathbb{B}^S$ forms an assertion algebra with a vector-based notion of separating conjunction. □

The notion of separation on vectors, which splits vectors into disjoint blocks, lends itself to transforming such vectors in parallel fashion. This is further elaborated in Example 14.

In separation logic, a magic wand operation is often used. It is the upper adjoint of separating conjunction. In the quantale setting, this adjoint exists because separating conjunction distributes over arbitrary suprema by definition.

Additional notions of resource monoids and liftings to assertion algebras have been studied within the Views framework [10]. Whether their generic soundness results for Hoare logics can be reconstructed in the power series setting is left for future work.

7 Transformers and Bi-Quantales

The powerset lifting discussed in Section 3 suggests that state and predicate transformers could be modelled as power series as well. This section sketches how this can be achieved. A detailed analysis and the consideration of particular classes of predicate transformers is left for future work.

A *state transformer* $f_R : A \to 2^B$ is often associated with a relation $R \subseteq A \times B$ by defining

$$f_R\, a = \{b \mid (a, b) \in R\}.$$

State transformers are turned into *predicate transformers* $\hat{f}_R : 2^B \to 2^A$ by the Kleisli lifting

$$\hat{f}_R\, Y = \{x \mid f_R\, x \subseteq Y\}.$$

The following results are well known [4].

Proposition 2. *The state transformers in* $(2^B)^A$ *and the predicate transformers in* $(2^A)^{2^B}$ *form complete distributive lattices.*

Proof. $2^B \cong \mathbb{B}^B$ forms a complete distributive lattice by Lemma 2 because $\mathbb{B}$ forms a complete distributive lattice. The same argument applies to 2^A. It therefore follows that $(2^B)^A$ and $(2^A)^{2^B}$ are again complete distributive lattices by Lemma 2. □

Predicate transformers of type $2^A \to 2^A$ form a monoid with respect to function composition. It is also well known that the subalgebra of *completely additive* predicate transformers, which satisfy $f\,(\sum_{i\in I} X_i) = \sum_{i\in I}(f\, X_i)$, forms a distributive unital quantale in which the identity function is the multiplicative unit. However, the operation of infimum in this algebra is not the one that is lifted pointwise; instead it is induced by the operation of supremum [4]. A dual result holds for *completely multiplicative* predicate transformers, which satisfy $f\,(\prod_{i\in I} X_i) = \prod_{i\in I}(f\, X_i)$. In this case, the monoidal part of the quantale lifting is not obtained with the power series lifting technique either.

The cases of resource monoids, where assertion algebras contain a notion of separating conjunction, are more interesting.

Let S be a partial monoid. A *monoid transformer* is a function of type $S \to 2^S$. A *monoid predicate transformer* is a function of type $2^S \to 2^S$. Examples are *resource transformers* and *resource predicate transformers*, in which case S is a resource monoid. Such transformers have been studied in the context of abstract separation logic [7]. The following results follow immediately in our setting.

Proposition 3. *Let S be a partial monoid. Then the monoid transformers in $(2^S)^S$ and the monoid predicate transformers in $(2^S)^{2^S}$ form distributive unital quantales. In both cases, commutativity lifts from S.*

Proof. 2^S forms a distributive unital quantale according to Corollary 2. It is commutative whenever S is. Hence $(2^S)^S$ forms a distributive unital quantale by Theorem 1. Commutativity lifts again from S.

Similarly, $(2^S)^{2^S}$ is a distributive unital quantale by Theorem 1 because 2^S is and the multiplicative reduct of 2^S is a monoid. Commutativity lifts again from S. □

Proposition 3 can be combined with the previous observation about predicate transformer quantales.

Theorem 2. *Let S be a partial (commutative) monoid. Then $((2^S)^{2^S}, \subseteq, \cdot, \circ, id, \mathbb{1})$ forms weak a unital bi-quantale with (commutative) convolution $\cdot$ and function composition $\circ$ as well as the unit function id and unit power series $\mathbb{1}$.*

In this context, *weak* means that the left distributivity law $f \circ \sum_{i\in I} g_i = \sum_{i\in I} f \circ g_i$ need not hold in the space of predicate transformers. It holds, however, when predicate transformers are completely additive.

8 Partial Power Series Quantales

This section generalises Theorem 1 to situations in which the target algebras Q are assumed to be partial quantales in the sense that their semigroup retracts are partial. In this case, partiality of composition shows up not only in the splitting $x = y \cdot z$, but also in the product $f\,y \cdot g\,z$ in convolutions. It turns out that the quantale structure of the target algebra is preserved at the level of the function space, but the loss of totality in $f\,y \cdot g\,z$ causes the function space to be partial as well. Previous proofs must therefore be reconsidered.

As an example we consider linear transformations of vectors implemented by matrices, in which vectors that are separated as in Example 13 can be transformed in concurrent fashion by matrices which can be separated into non-zero blocks along the diagonal. This is a particular manifestation of the correspondence between separation and concurrency in the context of convolution.

Proposition 4. *Let $(S, \cdot)$ be a partial semigroup. If $(Q, \leq, \cdot)$ is a (distributive) partial quantale, then so is $(Q^S, \leq, \cdot)$. In addition, commutativity lifts from S and Q to Q^S and unitality lifts if S is a partial monoid.*

Proof. By Theorem 1, the (commutative) monoidal and distributivity laws need to be checked.

Suppose $(f \cdot (g \cdot h))\, x$ is defined. Then

$$(f \cdot (g \cdot h))\, x = \sum_{x = x_1 \cdot (x_2 \cdot x_3)} f\, x_1 \cdot (g\, x_2 \cdot h\, x_3).$$

Thus $x_1 \cdot (x_2 \cdot x_3)$ is defined and equal to $(x_1 \cdot x_2) \cdot x_3$ and $f\, x_1 \cdot (g\, x_2 \cdot h\, x_3)$ is defined and equal to $(f\, x_1 \cdot g\, x_2) \cdot h\, x_3$. Hence

$$\sum_{x = x_1 \cdot (x_2 \cdot x_3)} f\, x_1 \cdot (g\, x_2 \cdot h\, x_3) = \sum_{x = (x_1 \cdot x_2) \cdot x_3} (f\, x_1 \cdot g\, x_2) \cdot h\, x_3 = ((f \cdot g) \cdot h)\, x.$$

The situation where $((f \cdot g) \cdot h)\, x$ is defined is opposition dual. Hence Q^S forms a partial semigroup.

Suppose that $(f \cdot \sum_{i \in I} g_i)\, x$ is defined. Then

$$(f \cdot \sum_{i \in I} g_i)\, x = \sum_{x = y \cdot z} f\, y \cdot (\sum_{i \in I} g_i)\, z = \sum_{x = y \cdot z} \sum_{i \in I} (f\, y \cdot g_i\, z) = \sum_{i \in I} (f \cdot g_i)\, x.$$

The proof can be reversed if the $(f \cdot g_i)\ x$ are defined. The proof of right distributivity is opposition dual. This shows that Q^S forms a partial distributive quantale.

Suppose $(f \cdot g)\, x$ is defined and S and Q are both commutative. Then

$$(f \cdot g)\, x = \sum_{x = y \cdot z} f\, y \cdot g\, z = \sum_{x = z \cdot y} g\, z \cdot f\, y = (g \cdot f)\, x.$$

This lifts commutativity.

Finally, assume that S is a monoid and Q is unital and define the power series $\mathbb{1}$ as usual. Suppose that $(\mathbb{1} \cdot f)\, x$ is defined. Then

$$(\mathbb{1} \cdot f)\, x = \sum_{x = y \cdot z} \mathbb{1}\, y \cdot f\, z = 1 \cdot f\, x = f\, x.$$

Moreover, $f \cdot \mathbb{1} = f$ follows from opposition duality. This lifts unitality. □

Example 14 (Linear Transformations of Vectors)**.** Consider again the partial semigroup $(S, *)$ on n-dimensional vectors from Example 13. It is easy to check that S actually forms a partial commutative dioid with respect to $*$ as multiplication and standard vector addition. Distributivity $x * (y + z) = (x * y) + (x * z)$ follows immediately from the definition: the case of $x_i = 0$ holds trivially, the case of $(y + z)_i = 0$ requires that $y_i = z_i = 0$.

Proposition 4 then implies as a special case that the functions of type $S \to S$ form a commutative dioid; they form a trioid with the other multiplication being function composition. The sum in the convolution is obviously finite since there are only finitely many ways of splitting a vector of finite dimension. In addition, the functions f and g in a convolution are not only applied to separate parts y and z of vector x, but they must map to separate parts $f\,y$ and $g\,z$ of the resulting vector as well.

Unitality cannot be lifted as in Proposition 4 because the units of $+$ and $*$ coincide. It is easy to check that the unit with respect of $*$ on S^S is defined as

$$e\,x = \begin{cases} 0, & \text{if } x = 0, \\ \bot, & \text{otherwise.} \end{cases}$$

For further illustration consider the linear transformations on n-dimensional vectors given by multiplying n-dimensional vectors with an $n \times n$ matrix and adding an n-dimensional vector.

As a simple example of a term contributing to a convolution consider

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} * \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} a_1 x \\ c_1 y \end{pmatrix} * \begin{pmatrix} b_2 y \\ d_2 y \end{pmatrix} = \bot,$$

whereas

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} * \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} a_1 x \\ 0 \end{pmatrix} * \begin{pmatrix} 0 \\ d_2 y \end{pmatrix} = \begin{pmatrix} a_1 x \\ d_2 y \end{pmatrix}.$$

This shows that matrices contributing to convolutions must essentially consist of two non-trivial blocks along the diagonal modulo (synchronised) permutations of rows and columns. That is, they are of the form

$$\begin{pmatrix} M_1 & \mathbb{0} \\ \mathbb{0} & M_2, \end{pmatrix}$$

where $\mathbb{0}$ represents zero matrices of appropriate dimension. Each pair of vectors resulting from a decomposition can be rearranged such that the first vector consists of an upper block of non-zero coefficients and a lower block of zeros, whereas the second vector consists of an upper zero and a lower non-zero block, and such that the two non-zero blocks do not overlap. One must be able to decompose matrices and vectors of the linear transformation into the same blocks to make convolutions non-trivial.

The transformations implemented by the above block matrix on rearranged vectors, and more generally all linear transformations, can clearly be executed independently or in parallel by the matrices M_1 and M_2 parts of a vector if the convolution is non-trivial. In this sense the convolution $*$ on linear transformations is a notion of concurrent composition. □

9 Power Series over Bi-Semigroups

Our main lifting result (Theorem 1) shows that the quantale structure Q is preserved at the level of the function space Q^S provided that S is a partial semigroup. This can easily be adapted from partial semigroups S to partial n-semigroups and n-quantales with n operations of composition which may or may not be commutative. Here we restrict our attention to bi-semigroups and bi-quantales and we discuss several examples.

Proposition 5. *Let $(S, \circ, \bullet)$ be a partial bi-semigroup. If $(Q, \leq, \circ, \bullet)$ is a (distributive unital) bi-quantale, then so is $(Q^S, \leq, \circ, \bullet)$.*

It is obvious that properties such as commutativity and unitality lift as before.

Example 15 (Functions over Two-Dimensional Intervals)**.** Closed two-dimensional intervals over a linear order can be defined in a straightforward way. For intervals x and y, we write $x \times y$ for the box consisting of points with x-coordinates in x and y-coordinates in y.

$$\begin{aligned} x \times y &= \{(a, b) \mid a \in x \wedge b \in y\} \\ x \times \bot &= \bot \\ \bot \times y &= \bot \end{aligned}$$

We define the horizontal composition of two-dimensional intervals as

$$(x_1 \times y_1) \circ (x_2 \times y_2) = \begin{cases} (x_1 \cdot x_2) \times y_1, & \text{if } y_1 = y_2, \\ \bot, & \text{otherwise.} \end{cases}$$

and their vertical composition as

$$(x_1 \times y_1) \bullet (x_2 \times y_2) = \begin{cases} x_1 \times (y_1 \cdot y_2), & \text{if } x_1 = x_2, \\ \bot, & \text{otherwise.} \end{cases}$$

Whenever the target algebra forms a bi-quantale, Proposition 5 applies and the function space forms a bi-quantale as well. In particular, horizontal and vertical convolution are given by

$$\begin{aligned} (f \circ g)\,(x \times y) &= \sum_{x = x_1 \cdot x_2} f\,(x_1 \times y) \circ g\,(x_2 \times y), \\ (f \bullet g)\,(x \times y) &= \sum_{y = y_1 \cdot y_2} f\,(x \times y_1) \bullet g\,(x \times y_2). \end{aligned}$$

The situation easily generalises to n-dimensional intervals with n convolutions which may or may not be commutative. □

Example 16 (Series-Parallel Pomset Languages)**.** Let $(S, \cdot, *, 1)$ be a bi-monoid with non-commutative composition $\cdot$, commutative composition $*$ and shared unit 1. Furthermore, let $(Q, \leq, \cdot, *, 1)$ be a bi-quantale with non-commutative composition $\cdot$, commutative composition $*$ and shared unit 1. Then Q^S forms a bi-quantale according to Proposition 5 with a non-commutative convolution given by $\cdot$ and a commutative convolution given by $*$. For $\mathbb{B}^S$ and S being freely generated from a finite alphabet X, we obtain the *series-parallel pomset languages* or *partial word languages* over X, which have been studied by Grabowski, Gischer and others [17, 14]. They form a standard model of true concurrency. □

Example 17 (Square Matrices with Parallel Composition)**.** We define a partial commutative composition $*$ on square matrices as a generalisation of vector case, splitting matrices into blocks along the diagonal.

$$(f * g)\,(i,j) = \begin{cases} f\ (i,j), & \text{if } \forall k.\ g\ (i,k) = 0 \wedge g\ (k,j) = 0, \\ g\ (i,j), & \text{if } \forall k.\ f\ (i,k) = 0 \wedge g\ (k,j) = 0, \\ \bot, & \text{otherwise.} \end{cases}$$

Associativity and commutativity of this operation is easy to check; (infinite) distributivity holds as well. It follows that square matrices into suitable coefficient algebras form partial bi-quantales. □

Examples 16 and 17 thus show other situations where a commutative convolution gives rise to a notion of parallel or concurrent composition.

10 Two-Dimensional Power Series Bi-Quantales

We now extend the power series approach to two dimensions; an extension to n dimensions can be obtained along the same lines. We consider two separate partial semigroups or monoids $(S_1, \circ)$ and $(S_2, \bullet)$. In many cases, S_2 is assumed to be commutative. This differs from Section 9 in that two different semigroups algebras are lifted to a bi-quantale, whereas in Section 9 a bi-semigroup is lifted to a bi-quantale.

We consider functions $F : S_1 \to S_2 \to Q$ from the partial semigroups S_1 and S_2 into an algebra Q, usually a bi-quantale. Note that $A \to B \to C$ stands for $A \to (B \to C)$, and we write $(C^B)^A$ for the class of functions of that type.

The main construction is as follows. Theorem 1 can be applied to semigroup S_1 and target algebra Q^{S_2} to lift to $(Q^{S_2})^{S_1}$. Alternatively, S_2 and Q^{S_1} can be lifted to $(Q^{S_1})^{S_2}$. The algebras Q^{S_1} and Q^{S_2} can be obtained by lifting as well; they can be considered as partial evaluations of a power series $F : S_1 \to S_2 \to Q$ to power series $F^y : S_1 \to Q$ and $F^x : S_2 \to Q$ where

$$F^y = \lambda x.\ F\,x\,y, \qquad\qquad F^x = \lambda y.\ F\,x\,y$$

This construction can be iterated n times for power series $F : S_1 \to \cdots \to S_n \to Q$.

It is well known that the function spaces obtained are isomorphic: in general $(C^A)^B \cong (C^B)^A \cong C^{A\times B} \cong C^{B\times A}$ under the Curry-Howard isomorphism. A categorical framework is provided by the setting of symmetric monoidal closed categories [23], which we do not explore further in this article. Instead we move freely between isomorphic function spaces.

By analogy to the one-dimensional case of power series we define operations on the function space $Q^{S_1\times S_2}$ which lift the corresponding operations on Q. Ultimately our aim is to show that bi-quantale axioms lift from Q to $Q^{S_1 \times S_2}$. We define

$$\begin{aligned}
(\sum_{i\in I} F_i)\,x\,y &= \sum_{i\in I}(F_i\,x\,y),\\
(\prod_{i\in I} F_i)\,x\,y &= \prod_{i\in I}(F_i\,x\,y),\\
(F \circ G)\,x\,y &= \sum_{x=x_1\circ x_2} F\,x_1\,y \circ G\,x_2\,y,\\
(F \bullet G)\,x\,y &= \sum_{y=y_1\bullet y_2} F\,x\,y_1 \bullet G\,x\,y_2.
\end{aligned}$$

As in the one dimensional case, $\mathbb{0} = \sum_{i\in\emptyset} F_i$. The convolution $F \circ G$ acts on the first parameter whereas $F \bullet G$ acts on the second one; $\sum_{i\in I} F_i$ and $\prod_{i\in I} F_i$ are defined by pointwise lifting on both arguments.

We now show how two-dimensional lifting results can be obtained in a modular fashion from one-dimensional ones with Theorem 1. By currying consider the functions $F^y : S_1 \to Q$ and $F^x : S_2 \to Q$. For these we can reuse the definitions of suprema, infima and convolution from the one dimensional case in Section 3. Suprema, for instance, are given by

$$(\sum_{i\in I} F_i^y)\,x = \sum_{i\in I}(F_i^y\,x), \qquad (\sum_{i\in I} F_i^x)\,y = \sum_{i\in I}(F_i^x\,y).$$

The equations for infima are lattice dual. Convolutions are given by

$$(F^y \circ G^y)\,x = \sum_{x=x_1\circ x_2} F^Y\,x_1 \circ G^y\,x_2, \qquad (F^x \bullet G^x)\,y = \sum_{y=y_1\bullet y_2} F^x\,y_1 \bullet G^x\,y_2.$$

The relationship between operations of different dimensions is captured by the following lemma.

Lemma 3. *The maps $\varphi_1 : Q^{S_1\times S_2} \to Q^{S_2}$ and $\varphi_2 : Q^{S_1\times S_2} \to Q^{S_1}$ defined by*

$$\varphi_1 = \lambda X.\ (X)^y, \qquad \varphi_2 = \lambda X.\ (X)^x$$

are homomorphisms.

(a) $(\sum_{i\in I} F_i)^y = (\sum_{i\in I} F_i^y)$ *and* $(\sum_{i\in I} F_i)^x = (\sum_{i\in I} F_i^x)$,

(b) $(\prod_{i\in I} F_i)^y = (\prod_{i\in I} F_i^y)$ *and* $(\prod_{i\in I} F_i)^x = (\prod_{i\in I} F_i^x)$,

(c) $(F \circ G)^y = (F^y \circ G^y)$ *and* $(F \bullet G)^x = (F^x \bullet G^x)$.

Proof. We only provide proofs for the first conjunct of (a) and for (c). The remaining proofs are similar. For addition we calculate

$$(\sum_{i\in I} F_i)^y\,x = (\sum_{i\in I} F_i)\,x\,y = \sum_{i\in I}(F_i\,x\,y) = \sum_{i\in I}(F_i^y\,x) = (\sum_{i\in I} F_i^y)\,x.$$

For composition $\circ$,

$$\begin{aligned}(F\circ G)^y\,x &= (F\circ G)\,x\,y\\ &= \sum_{x=x_1\circ x_2}(F\,x_1\,y)\circ(G\,x_2\,y)\\ &= \sum_{x=x_1\circ x_2}(F^y\,x_1)\circ(G^y\,x_2)\\ &= (F^y\circ G^y)\,x.\end{aligned}$$

□

If $(S_1,\circ,1_\circ)$ and $(S_2,\bullet,1_\bullet)$ are partial monoids and the bi-quantale Q has units 1^y and 1^x with respect to $\circ$ and $\bullet$ (overloading notation), we define units on $Q^{S_1\times S_2}$ as

$$\mathbb{1}_\circ = \lambda x,y.\begin{cases}1_\circ, & \text{if } x=1_\circ,\\ 0, & \text{otherwise,}\end{cases}\qquad \mathbb{1}_\bullet = \lambda x,y.\begin{cases}1_\bullet, & \text{if } y=1_\bullet,\\ 0, & \text{otherwise.}\end{cases}$$

The following result links these binary units with the unary units $(\mathbb{1}^y)_\circ : S_1\to Q$ and $(\mathbb{1}^x)_\bullet : S_2\to Q$, as defined in Section 3.

Lemma 4.

(a) $(\mathbb{1}_\circ)^y = (\mathbb{1}^y)_\circ$,

(b) $(\mathbb{1}_\bullet)^x = (\mathbb{1}^x)_\bullet$.

Proof. For (a),

$$(\mathbb{1}_\circ)^y\,x = \mathbb{1}_\circ\,x\,y = \begin{cases}1_\circ, & \text{if } x=1_\circ,\\ 0, & \text{otherwise.}\end{cases} = (\mathbb{1}^y)_\circ\,x.$$

The proof of (b) is similar. □

By Lemmas 3 and 4, a lifting from Q can be decomposed into a lifting to Q^{S_2} and, if the lifted property is preserved, a function application in $(Q^{S_2})^{S_1}$. Alternatively one can lift to Q^{S_1} and then use function application in $(Q^{S_1})^{S_2}$. In the above constructions, there are two kinds of liftings: pointwise liftings from Q to Q^{S_1} or Q^{S_2} and lifting by convolution for Q^{S_1} and Q^{S_2}.

Proposition 6. *Let $(S_1,\circ)$ be a partial semigroup and S_2 a set. If $(Q,\leq,\circ)$ is a (distributive) quantale, then so is $(Q^{S_1\times S_2},\leq,\circ)$. Unitality and commutativity lift from S_1 and Q to $Q^{S_1\times S_2}$.*

Proof. If S_1 is a partial semigroup and Q a (distributive) quantale, then Q^{S_1} is a (distributive) quantale by Theorem 1, and by λ-abstraction for $F=\lambda y.\ F^y\ x$ and the homomorphic properties of $(.)^y$ in Lemma 3. For example,

$$\begin{aligned}((F \circ G) \circ H)\, x\, y &= ((F \circ G) \circ H)^y\, x \\ &= ((F^y \circ G^y) \circ H^y)\, x \\ &= (F^y \circ (G^y \circ H^y))\, x \\ &= (F \circ (G \circ H))^y\, x \\ &= (F \circ (G \circ H))\ x\ y.\end{aligned}$$

If the quantale Q is unital, then so is Q^{S_1}, again by Theorem 1. As previously, this follows by λ-abstraction and the homomorphic properties of $(.)^y$ by Lemmas 3 and 4. For instance,

$$(\mathbb{1}_\circ \circ F)\, x\, y = (1_\circ \circ F)^y\ x = (1_\circ^y \circ F^y)\, x = F^y\, x = F\, x\, y.$$

If S_1 and Q are both commutative, then

$$(F \circ G)\, x\, y = (F \circ G)^y\, x = (F^y \circ G^y)\, x = (G^y \circ F^y)\, x = (G \circ F)^y\ x = (G \circ F)\, x\, y$$

with the homomorphism properties of $(.)^y$ and commutativity on Q^{S_1} due to Theorem 1. □

The next statement is immediate since $Q^{S_1 \times S_2}$ and $Q^{S_2 \times S_1}$ are isomorphic.

Corollary 3. *Let S_1 be a set and $(S_2, \bullet)$ a partial semigroup. If $(Q, \leq, \bullet)$ is a (distributive) quantale, then so is $(Q^{S_1 \times S_2}, \leq, \bullet)$. Unitality and commutativity lift from S_2 and Q to $Q^{S_1 \times S_2}$.*

Proposition 6 and Corollary 3 can therefore be combined into the following lifting theorem for two-dimensional power series.

Theorem 3. *Let $(S_1, \circ)$ and $(S_2, \bullet)$ be partial semigroups. If $(Q, \leq, \circ, \bullet)$ is a (distributive) bi-quantale, then so is $(Q^{S_1 \times S_2}, \leq, \circ, \bullet)$. It is unital whenever Q is unital and S_1 and S_2 are partial monoids. A convolution on $Q^{S_1 \times S_2}$ is commutative if the corresponding composition on S_i and Q are commutative.*

Remember that a unital bi-quantale may have different units for its two compositions.

As already mentioned, the construction of the bi-quantale of two-dimensional power series generalises immediately to n underlying partial semigroups $(S_i, \circ_i)$, n-dimensional power series $F : S_1 \to \cdots \to S_n \to Q$ and convolutions

$$(F \circ_i G)\ \ldots\ x_i\ \ldots = \sum_{x_i = y \circ_i z} (F\ \ldots\ y\ \ldots) \circ_i (G\ \ldots\ z\ \ldots).$$

We do not pursue this generalisation in this article; the lifting arguments apply without modification.

11 Examples

As examples of two-dimensional bi-quantales we present two interval based models that distinguish between time and space dimensions. The monoidal operators may be used to separate these two dimensions independently; time is separated using chop, space using separating conjunction as a notion of concurrent composition. The consideration of such algebras with both kinds of separation was the starting point of this article. In the second example of vector stream interval functions, spatial or concurrent splitting is of course commutative, whereas temporal splitting is not.

Example 18 (Stream Interval Functions)**.** Let $(S_1, \cdot)$ be the partial semigroup $(I_P, \cdot)$ of closed intervals I_P under interval function as in Example 6 and let S_2 be the set of all functions of type $P \to A$ for an arbitrary set A. It follows from Proposition 6 that $Q^{I_P \times A^P}$ forms a distributive quantale, whenever Q is a distributive quantale. A unit can be adjoined to $Q^{I_P \times A^P}$ along the lines of Example 6, but with a second parameter.

As a typical interpretation, consider $P = \mathbb{R}$ with the standard order on reals as a model of time and let functions $f : \mathbb{R} \to A$ model the temporal behaviour or trajectories of some system. For instance, f could be the solution of a differential equation. In that case, $F\ x\ f$ evaluates the behaviour of system f in the interval x. Such kinds of functions have been called *stream interval functions* [11]. The convolution

$$(F \cdot G)\, x\, f = \sum_{x = y \cdot z} (F\, y\, f) \cdot (G\, z\, f)$$

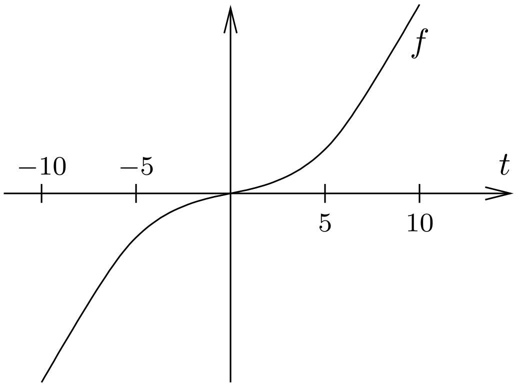
splits the interval x into all possible prefix/suffix pairs y and z, applies F to the behaviour of f on interval y and G to the behaviour of f on interval z and then combines these results. There are different ways in which the application of stream interval functions can be realised. Moreover, the situation generalised to arbitrary finitely bounded intervals without fusion.

As in the case of interval functions, our prime example of stream interval functions are *stream interval predicates*, where $Q = \mathbb{B}$. Then convolution becomes a generalised version of chop or non-commutative separating conjunction:

$$(F \cdot G)\, x\, f = \sum_{x = y \cdot z} (F\, y\, f) \sqcap (G\, z\, f).$$

A predicate F could, for instance, test the values of a function f over an interval x—at all points of x, at some points of x, at almost all points of x, at no points of x and so on. It could, for instance, test, whether the trajectory of system f evolves within given boundaries, that is a flight path is within a given corridor or that a train moves according to a given time schedule.

More concretely, let $P = A = \mathbb{R}$ and that $f\ t = t^3$ as shown below. Note that the diagram is not drawn to scale.



Let

$$F\ x\ f = \forall t \in x.\ f\ t \geq 0, \qquad\qquad G\ x\ f = \forall t \in x.\ f\ t < 0.$$

Then $F\ [0,10]\ f = 1$ and $G\ [-7,-1]\ f = 1$, but $F\ [-2,-1]\ f = 0$ and $G\ [-7,0]\ f = 0$. □

Stream interval predicates have been used to reason about real-time systems [11], but their interpretation in terms of power series is new. It is worth noting that P may be instantiated to other partial orders (e.g., $\mathbb{Z}$), allowing one to model both discrete and continuous systems.

Using Theorem 3, one may further develop this approach with rules for system-level reasoning by decomposing systems along a time and space dimension. To the best of our knowledge, our treatment is the first to offer both decompositions and to add a natural notion of concurrency to interval logics. Exploration of these rules in concrete models as well as their application towards verification of example systems is left as future work. Here we present one single example which is based on vectors of functions.

Example 19 (Vector Stream Interval Functions)**.** Let f from the previous example now be a vector or product of functions f_i such that $f : P \to A^n$, or more concretely $f : \mathbb{R} \to A^n$. One can then split $f(t)$ as in Example 13 with respect to the commutative operation $*$ on A^n. For functions $f, g : P \to A^n$ we define

$$(f * g)\, p = f\, p * g\, p$$

by pointwise lifting. This turns $(S_2, *) = ((A^n)^P, *)$ into a partial commutative semigroup, whereas $(S_1, \cdot)$ is again the partial semigroup $(I_P, \cdot)$. According to Theorem 3, $Q^{S_1 \times S_2}$ forms a distributive bi-quantale with commutative convolution $*$ whenever Q does.

The stream interval predicates in the case of $Q = \mathbb{B}$ yield once more an interesting special case. Now a vector of functions, for instance the solution to a system of differential equations, is applied to arguments ranging over an interval and the stream interval predicates evaluate the behaviour modelled by this vector of functions on the interval.

The convolution

$$(F \cdot G)\, x\, f = \sum_{x = y \cdot z} (F\, y\, f) \sqcap (G\, z\, f)$$

can be seen as a horizontal composition. It evaluates the full vector of functions to splittings of the interval x, using F for the prefix part of the splitting and G for its suffix part. In the context of interval logics this corresponds to a chop operation, which has a temporal flavour.

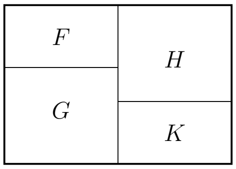
The convolution or separating conjunction

$$(F * G)\,x\,f = \sum_{f=g*h} (F\,x\,g) \sqcap (G\,x\,h)$$

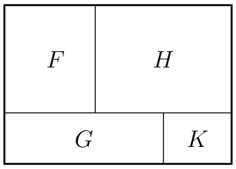
can be seen as a vertical composition. It evaluates the conjunction of F and G, which is obtained by separating the vector f into all possible parts g and h, over the full interval x. Applied to vectors this adds an algebraic notion of concurrent composition to interval calculi; it clearly has a spatial flavour.

The two types of convolution may be distinguished using diagrams such as the ones below, where time occupies the x-axis and space the y-axis.

$(F * G) \cdot (H * K)$



$(F \cdot H) * (G \cdot K)$



The left diagram depicts $(F * G) \cdot (H * K)$, where the convolution first splits the stream interval function along the x-axis (time dimension) to give us formulae $F * G$ and $H * K$. Each of these is then split along the y-axis (space dimension). On the other hand the right diagram depicts $(F \cdot H) * (G \cdot K)$, where the space dimension is split first to give $F \cdot H$ and $G \cdot K$, followed by a split along the time dimension. □

The examples in Section 6 suggest that other notions of spatial separation, for instance those based on disjoint unions for families of functions, or more specific notions such as separating conjunction on heaps, can be used instead of vector separation. Theorem 3 is modular in this regard. We therefore do not present these examples in detail.

12 Power Series over Futuristic Monoids

This section adapts the power series approach to a case which is appropriate, for instance, for languages with finite and infinite words and for intervals which may be semi-infinite in the sense that they have no upper bounds. Such approaches are, for instance, appropriate for total correctness reasoning, where termination cannot be assumed or for reactive (concurrent) systems. We model these cases abstractly with monoids which, due to lack of better nomenclature, we call futuristic.

Formally, a partial semigroup $(S, \cdot)$ is *futuristic* if $S = S^u \cup S^b$, $S^u \cap S^b = \emptyset$ and $x \cdot y$ is undefined whenever $x \in S^u$. Thus, S^u and S^b correspond to the unbounded and bounded elements of S, respectively. For S^b, we require that if $x \cdot y \in S^b$, then $x \in S^b$.

In that case, for $f, g : X \to Y$, we define

$$(f \cdot g)\, x = \sum_{x=y\cdot z} (f\, y) \cdot (g\, z) + \begin{cases} f\, x, & \text{if } x \in S^u, \\ 0, & \text{if } x \in S^b. \end{cases}$$

Lemma 5. *Let $(S, \cdot)$ be a futuristic partial semigroup. If Q is a (distributive) quantale, then Q^S is a (distributive) quantale with $\mathbb{0} : S \to Q$ not necessarily a right annihilator and left distributivity holding only for non-empty suprema.*

Proof. We need to verify the laws involving '$\cdot$' with our new multiplication. It suffices to consider the cases where $x \in S^u$; the others are covered by Theorem 1. For left distributivity we calculate, for $I \neq \emptyset$,

$$\begin{aligned}(f \cdot \sum_{i\in I} g_i)\, x &= f\, x + \sum_{x=y\cdot z} f\, y \cdot \sum_{i\in I} (g_i\, z) \\ &= (\sum_{i\in I} f\, x) + \sum_{i\in I} \sum_{x=y\cdot z} (f\, y \cdot g_i\, z) \\ &= \sum_{i\in I} (f\, x + \sum_{x=y\cdot z} (f\, y \cdot g_i\, z)) \\ &= (\sum_{i\in I} (f \cdot g_i))\, x.\end{aligned}$$

For $I = \emptyset$, however $(f \cdot \mathbb{0})\, x = f\, x$ if $x \in S^u$, hence in this case left distributivity fails.

For right distributivity, which is no longer opposition dual, we calculate

$$\begin{aligned}((\sum_{i\in I} f_i) \cdot g)\, x &= (\sum_{i\in I} f_i\, x) + \sum_{x=y\cdot z} (\sum_{i\in I} f_i\, y) \cdot g\, z \\ &= (\sum_{i\in I} f_i\, x) + \sum_{i\in I} \sum_{x=y\cdot z} (f_i\, y \cdot g\, z) \\ &= \sum_{i\in I} (f_i\, x + \sum_{x=y\cdot z} (f_i\, y \cdot g\, z)) \\ &= (\sum_{i\in I} (f_i \cdot g))\, x.\end{aligned}$$

Left annihilation is as usual a special case of right distributivity. We calculate explicitly

$$(\mathbb{0} \cdot f)\, x = \mathbb{0}\, x + \sum_{x=y\cdot z} \mathbb{0}\, y \cdot f\, z = 0 + 0 = 0$$

Finally, for associativity, we calculate

$$\begin{aligned}(f \cdot (g \cdot h))\, x &= f\, x + \sum_{x = y \cdot z} f\, y \cdot (g\, z + \sum_{z = u \cdot v} g\, u \cdot h\, v)\\ &= f\, x + (\sum_{x = y \cdot z} f\, y \cdot g\, z) + \sum_{x = y \cdot z} f\, y \cdot (\sum_{z = u \cdot v} g\, u \cdot h\, v)\\ &= (f \cdot g)\, x + \sum_{x = y \cdot u \cdot v} f\, y \cdot g\, u \cdot h\, v\\ &= (f \cdot g)\, x + \sum_{x = w \cdot v} (\sum_{w = y \cdot u} f\, y \cdot g\, u) \cdot h\, v\\ &= (f \cdot g)\, x + \sum_{x = w \cdot v} (f \cdot g)\, w \cdot h\, v\\ &= ((f \cdot g) \cdot h)\, x.\end{aligned}$$

The last but first step uses the fact that $w \in S^b$ □

Proposition 7. *Let $(S_1, \circ)$ be a futuristic partial semigroup and S_2 a set. If $(Q, \leq, \circ)$ is a (distributive) quantale, then $(Q^{S_1 \times S_2}, \leq, \circ)$ is a (distributive) quantale with $\mathbb{0}$ not necessarily a right annihilator and left distributivity holding only for non-empty suprema. Unitality lifts from Q to $Q^{S_1 \times S_2}$ with unit $\mathbb{1}_\circ$ if S_1 is a partial monoid.*

The proof adapts that of Proposition 6 to Lemma 5. A treatment of historistic intervals is dual, that is, left annihilation fails. Proposition 7 can be extended further into an analogue of Theorem 3. We do not explicitly display this statement.

Example 20 (Formal Languages with Infinite Words)**.** Let X be a finite alphabet. Let X^*, as previously, denote the set of finite words over X and X^ω the set of all infinite words, which are sequences of type $\mathbb{N} \to X$. Let $X^\infty = X^* \cup X^\omega$. Then $X^* \cap X^\omega = \emptyset$ by definition. Every language $L \subseteq X^\infty$ may contain finite as well as infinite words and we write $\mathsf{fin}(L)$ and $\mathsf{inf}(L)$ for the sets of all finite and infinite words in L.

In this context it is natural to disallow the concatenation of an infinite word with another word, hence X^∞ is endowed with a futuristic partial monoid structure. In addition, the product of $L_1, L_2 \subseteq X^\infty$ is commonly defined as

$$L_1 \cdot L_2 = \mathsf{inf}(L_1) \cup \{vw \mid v \in \mathsf{fin}(L_2) \wedge w \in L_2\}.$$

This is captured by the futuristic product with $Y = \mathbb{B}$. It then follows from Lemma 5 that X^∞ forms a distributive quantale in which $L \cdot \emptyset = \emptyset$ need not hold and left distributivity holds only for non-empty suprema. In fact, the absence of right annihilation can be verified with the singleton stream $L = \{aaa \dots\}$. □

Models with finite/infinite paths and traces can be built in a similar fashion.

Example 21 (Functions and Predicates over Futuristic Intervals)**.** Let $(P, \leq)$ be a linear order without right endpoint. Let I_P^f stand for the set of all non-empty closed intervals over X and let I_X^i denote the set of all *futuristic intervals* $[a, \infty] = \{b \mid b \geq a\}$. This does not mean that we add an explicit element ∞ to X; ∞ is merely part of our naming conventions. Then $I_X = I_X^f \cup I_X^i$ and $I_X^f \cap I_X^i = \emptyset$. The fusion product of intervals can now be redefined as

$$x \cdot y = \begin{cases} x, & \text{if } x \in I_X^i, \\ [x_{\min}, y_{\max}], & \text{if } x \in I_X^f \text{ and } x_{\max} = y_{\min}, \\ \bot, & \text{otherwise,} \end{cases}$$

where $y_{\max} = \infty$ is included as an option. It then follows from Lemma 5 that Q^{I_P} forms a distributive quantale in which $\mathbb{0}$ is not necessarily a right annihilator. In fact, $f \circ \mathbb{0} = \mathbb{0}$ can be falsified with any interval $x = [a, \infty]$ and interval predicate $f = \lambda x.\ a \in x$. □

An example of closed and open intervals without fusion can be obtained along the same lines. Examples of bi-quantales based on stream functions over futuristic intervals with a notion of separating conjunction can be obtained in a straightforward way.

13 Interchange Laws

Algebras in which a spatial or concurrent separation operation interact with a temporal or sequential one have already been studied, for instance, in the context of concurrent Kleene algebra [20]. In addition to the trioid or bi-quantale laws, these algebras provide interesting interaction laws between the two compositions, which in this context are interpreted as concurrent and sequential composition. Such laws are, obviously, of general interest.

More concretely, the following *interchange laws* hold in concurrent Kleene algebras:

$$(x * y) \cdot z \leq x * (y \cdot z),$$
$$x \cdot (y * z) \leq (x \cdot y) * z,$$
$$(w \cdot x) * (y \cdot z) \leq (w \cdot y) * (x \cdot z).$$

We call the first two laws *small interchange laws* and the last one *weak interchange law*. These laws hold in models of concurrency including shuffle languages and certain classes of partially ordered multisets [14]. It has been shown that one of the small interchange laws is equivalent to a separation logic style frame rule in a certain encoding of Hoare logics [19]. The weak interchange law, in turn, is equivalent to one of the standard concurrency rules for Hoare logic, which is similar to those considered in Owicki and Gries' logic [28] or in concurrent separation logic [7]. This relationship is considered further in Section 14.

The close relationship between power series and separation logic and the similarity between two-dimensional power series and concurrent Kleene algebras make it worth considering the interchange laws in this setting. However we obtain mainly negative results.

To start with a positive result, we establish interchange laws between other kinds of operations.

Lemma 6. *In every quantale, the following interchange laws hold:*

$$(w \sqcap x) \cdot (y \sqcap z) \leq (w \cdot y) \sqcap (x \cdot z), \qquad (w \sqcap x) * (y \sqcap z) = (w * y) \sqcap (x * z).$$

It turns out, however, that the small and weak interchange laws between sequential and concurrent composition do not hold in general. This is established by the counterexamples which support the following lemma.

Proposition 8. *There are $F, G, H, K : S_1 \to S_2 \to \mathbb{B}$ such that the following holds.*

(a) $F \cdot G \nleq F * G$,

(b) $(F * G) \cdot H \nleq F * (G \cdot H)$,

(c) $F \cdot (G * H) \nleq (F \cdot G) * H$,

(d) $(F * G) \cdot (H * K) \nleq (F \cdot H) * (G \cdot K)$.

Proof. First, note that $\leq$ can be interpreted as $\Rightarrow$ for stream interval predicates, and recall that parallel composition of predicates is separating conjunction when f is a vector of functions.

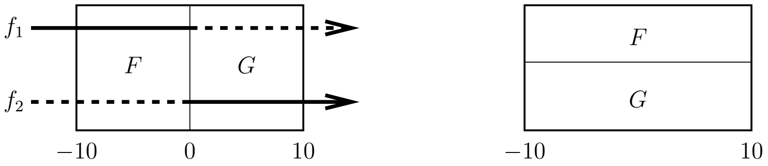
(a) To refute $F \cdot G \leq F * G$, let $x = [-10, 10]$, $f = (f_1, f_2)$ with

$$f_1\, t = \begin{cases} 1, & t \leq 0, \\ 0, & t > 0, \end{cases} \qquad f_2\, t = \begin{cases} 0, & t \geq 0, \\ 1, & t < 0, \end{cases}$$

and

$$F\, x\, f = \forall t \in x.\ f_1\, t = 1, \qquad G\, x\, f = \forall t \in x.\ f_2\, t = 1.$$

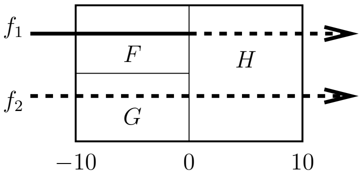
Then $(F \cdot G)\, x\, f = 1$, splitting interval x at $t = 0$, whereas $(F * G)\, x\, f = 0$ since neither F nor G holds on the entire interval x. This may be visualised using the diagrams below, where dashed lines represent that the corresponding function has value 0, and solid lines represent a value 1. For the right diagram, there is not possible way for the vectors f_1 and f_2 to go through F and G.

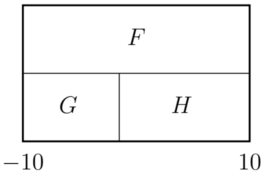


(b) To refute $(F * G) \cdot H \leq F * (G \cdot H)$, let $x = [-10, 10]$, f_1 as in (a) and $f_2 = \lambda t.\ 0$, where

$$\begin{aligned} F\, f\, x &= \forall t \in x.\ f_1\, t = 1, \\ G\, f\, x &= \forall t \in x.\ f_2\, t = 0, \\ H\, f\, x &= \forall t \in x.\ f_1\, t = 0 \vee f_2\, t = 0. \end{aligned}$$

This makes the left hand side 1 and the right hand side 0. This is visualised by the diagram below—neither f_1 nor f_2 may go through F.





(c) $H \cdot (G * F) \leq (H \cdot G) * F$ can be refuted by function

$$f_1'\, t = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0, \end{cases}$$

and f_2 as in (b), exploiting opposition duality between the two interchange laws and realising that f_1' is the "time reverse" of f_1.

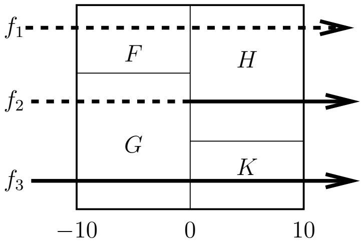
(d) To refute $(F * G) \cdot (H * K) \leq (F \cdot H) * (G \cdot K)$, consider $f = (f_1, f_2, f_3)$ where

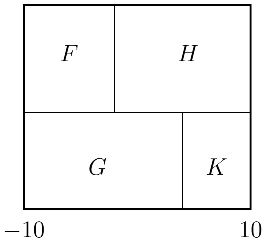
$$f_1\, t = 0, \qquad f_2\, t = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0, \end{cases} \qquad f_3\, t = 1$$

and

$$\begin{aligned} F\, f\, x &= \forall t \in x.\ f_1\, t = 0, \\ G\, f\, x &= \forall t \in x.\ f_2\, t < f_3\, t, \\ H\, f\, x &= \forall t \in x.\ f_1\, t < f_2\, t, \\ K\, f\, x &= \forall t \in x.\ f_3\, t = 1. \end{aligned}$$

For $x = [-10, 10]$, the diagram on the left below shows that the left hand side $(F * G) \cdot (H * K)$ holds. However, in the diagram on the right, which represents $(F \cdot H) * (G \cdot K)$, there is no possible combination of horizontal and vertical splits that satisfy f. In particular, f_1 must go through F, and similarly f_3 must go through K. We have a choice of placing f_2 above the horizontal line (through F and H), or below (through G and K), however, neither choice is appropriate.





□

Imposing addition algebraic restrictions, which would allow the derivation of interchange laws, is left for future work. A promising candidate is the consideration of locality assumptions, as in separation logic [7], which are briefly explained in the following section, or the inclusion of dependency relations [20] in the definition of the semigroup operations.

14 Hoare Logics from Power Series Quantales

One benefit of algebras is that they support the development of verification systems. It is well known, for instance, that quantales can be endowed with Hoare logics [20], more precisely *propositional* Hoare logics, in which data flow rules such as assignment rules are missing. This section illustrates how this leads to propositional Hoare logics over power series.

But before that we briefly recall how notions of iteration arise in the quantale setting, since these are needed for while rules in Hoare logic.

Since quantales are complete lattices, least and greatest fixpoints of isotone functions exist. Moreover, due to their infinite distributivity laws, functions such as $\lambda\alpha.\ x+\alpha$, $\lambda\alpha.\ x\cdot\alpha$ and $\lambda\alpha.\ \alpha\cdot x$ are continuous and the first one is even co-continuous in distributive quantales. This means that in particular the least fixpoints built by using combinations of these functions can be obtained by iteration from 0 to the first limit ordinal.

More specifically, the function $\varphi = \lambda\alpha.\ 1+x\cdot\alpha$ is continuous, hence has the least fixpoint $\mu\varphi = x^* = \sum_{i\in\mathbb{N}}\varphi^i(0) = \sum_{i\in\mathbb{N}} x^i$. This notion of finite iteration is needed for deriving a while-rule for a finite loop in a partial correctness setting.

More generally, the unfold and induction rules

$$\begin{aligned} 1+x\cdot x^* &= x^*, \qquad & z+x\cdot y \le y &\Rightarrow x^*\cdot z \le y,\\ 1+x^*\cdot x &= x^*, \qquad & z+y\cdot x \le y &\Rightarrow z\cdot x^* \le y \end{aligned}$$

can be used for reasoning about the star. In a total correctness setting, a notion of possibly infinite iteration is preferable, which corresponds to the greatest fixpoint of φ. Infinite iteration is also useful for futuristic monoids Section 12, for example, when reasoning about reactive systems, and Hoare rules for these can be developed. However, because these follow a similar pattern to finite iteration, we leave their full treatment as future work.

Equipped with the star in the power series quantale we can now follow [20] in setting up a propositional Hoare logic. The development is slightly non-standard, in that there is no distinction between assertions and programs at the level of algebra. It follows the lines of a previous approach by Tarlecki [29].

For a quantale Q and elements $x, y, z \in Q$, we define validity of a Hoare triple Tarlecki-style as

$$\vdash \{x\}y\{z\} \Leftrightarrow x\cdot y \le z.$$

In Tarlecki's original article, this encoding has been used for a relational semantics where not only the program, but also its pre- and postconditions are modelled as relations. It is equally suitable for trace or language based extensions of Hoare logic to concurrency, such as the rely-guarantee method [22].

The proof of the following proposition is then straightforward and generic for quantales.

Proposition 9 ([20])**.** *Let Q be a unital quantale with unit* 1*. The following rules of propo-*

sitional Hoare logic are derivable, for all $w, w_1, w_2, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2 \in Q$.

$$\vdash \{x\}1\{x\} \qquad \frac{x_1 \leq x_2 \quad \vdash \{x_2\}y\{z_2\} \quad z_2 \leq z_1}{\vdash \{x_1\}y\{z_1\}}$$

$$\frac{\vdash \{x\}y_1\{z\} \quad \vdash \{x\}y_2\{z\}}{\vdash \{x\}y_1 + y_2\{z\}} \qquad \frac{\vdash \{w\}x_1\{z\} \quad \vdash \{z\}x_2\{y\}}{\vdash \{w\}x_1 \cdot x_2\{y\}}$$

$$\frac{\vdash \{x\}y\{x\}}{\vdash \{x\}y^*\{x\}}$$

We can strengthen the choice and star rule as follows.

$$\frac{\vdash \{x \cdot w_1\}y_1\{z\} \quad \vdash \{x \cdot w_2\}y_2\{z\}}{\vdash \{x\}w_1 \cdot y_1 + w_2 \cdot y_2\{z\}} \qquad \frac{\vdash \{x \cdot w_1\}y\{x\}}{\vdash \{x\}(w_1 \cdot y)^* \cdot w_2\{x \cdot w_2\}}$$

The proof of the first one is essentially that of the choice rule. For the second one suppose $x \cdot w_1 \cdot y \leq x$. Then $x \cdot (w_1 \cdot y)^* \leq x$ by star induction and $x \cdot (w_1 \cdot y)^* \cdot w_2 \leq x \cdot w_2$ by isotonicity. If w_1 and w_2 are, in some sense, complemented, then this yields the standard conditional rule and while rule of Hoare logic.

Instantiating Proposition 9 to power series quantales automatically yields Hoare calculi for virtually all the examples discussed in this article. The instantiation to the binary relations quantale reproduces Tarlecki's original soundness result. Other instances yield, in a generic way, Hoare logics over computationally meaningful semantics based on finite words (traces in the sense of concurrency theory), paths in graphs (sequences of events in concurrency theory), paths in the sense of automata theory, or pomsets. We also obtain generic propositional Hoare logics for reasoning about interval and stream interval predicates in algebraic variants of interval logics.

In addition, Proposition 9 covers commutative quantales, where the Tarlecki-style encoding of the validity of Hoare triples might make less sense.

The rules covered by Proposition 9, however, are entirely sequential. For applications involving concurrency, such as the vector stream interval functions in Example 19, additional rules are desirable. In concurrent Kleene algebra, Owicki-Gries-style concurrency rules and frame rules in the style of separation logic can be derived. The same derivation, however, is ruled out in the quantale context, because the concurrency rule obtained is equivalent to the weak interchange law and the frame rule to one of the small interchange laws, both of which have been refuted in Proposition 8.

Instead we can use the interchange laws provided by Lemma 6.

Lemma 7. *In quantale Q the following concurrency rule is derivable, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in Q$.*

$$\frac{\vdash \{x_1\}y_1\{z_1\} \quad \vdash \{x_2\}y_2\{z_2\}}{\vdash \{x_1 \sqcap x_2\}y_1 \sqcap y_2\{z_1 \sqcap z_2\}}$$

Proof. Suppose $x_1 \cdot y_1 \leq z_1$ and $x_2 \cdot y_2 \leq z_2$. Then

$$(x_1 \sqcap x_2) \cdot (y_1 \sqcap y_2) \leq (x_1 \cdot y_1) \sqcap (x_2 \cdot y_2) \leq z_1 \sqcap z_2$$

by weak interchange (Lemma 6) and the assumptions. □

Once more this rule is available automatically in all examples discussed in this article.

As an alternative to conjunction-based notions of concurrency, it might still be possible to derive concurrency and frame rules under additional syntactic restrictions, for instance, those capturing the synchronisation between sequential and concurrent compositions, or in particular models. An investigation is left for future work.

15 The Frame Rule in a Power Series Context

Section 6 shows that the assertion quantales which underlie separation logic—implementing the boolean operations together with a notion of separation logic on predicates over a resource monoid—can be modelled in the power series setting. Predicate transformers, which yield another way of deriving Hoare logics over assertion algebras, can be modelled in that setting as well (Section 7).

In this section we sketch how a combination of these results allows us to derive the frame rule of separation logic by equational reasoning. Convolution plays a central part in the proof. Previously, algebraic proofs of the frame rule have been given in a state transformer context [7] as well as in the context of concurrent Kleene algebra [20].

It is well known that in the predicate transformer setting, validity of Hoare triples can be encoded as

$$\vdash \{p\}R\{q\} \Leftrightarrow p \leq \hat{f}_R\, q,$$

which is essentially an adjunction, using the notation of Section 7, but writing $p, q, \dots$ for predicates, which are elements of the assertion quantale of separation logic. It is also well known that the rules of Hoare logic can be derived in this setting, assuming that predicate transformers are isotone. A result of separation logic states that the frame rule can be derived whenever the predicate transformer f under consideration is *local*, that is, it satisfies

$$f * id \leq f.$$

Intuitively, locality means that the effect of a transformer can always be localised on part of the state. For a detailed discussion see [7].

Before deriving the frame rule we use properties of power series and convolution to prove a point-wise analogue of locality which simplfies the proof.

Lemma 8. *f is local if and only if $(f\,p) * q \leq f\,(p * q)$.*

Proof. Let $(f\,p) * q \leq f\,(p * q)$. Then $(f\,p) * (id\,q) = (f\,p) * q \leq f\,(p * q)$ and therefore

$$(f * id)\, r = \sum_{r = p * q} (f\,p) * (id\,q) \leq \sum_{r = p*q} f\,(p * q) = f\,r.$$

Let f be local. Then

$$(f * id)\, r = \sum_{r = p*q} (f\,p) * q \leq f\,r = f\,(p * q),$$

whence $(f\,p) * q \leq f\,(p * q)$. □

Lemma 9. *Let $\hat{f}_R$ be a local predicate transformer associated to program R. Then the following frame rule holds.*

$$\frac{\vdash \{p\}R\{q\}}{\vdash \{p * r\}R\{q * r\}}$$

Proof. Let $\vdash \{p\}R\{q\}$, that is, $p \le \hat{f}_R\, q$. Then $p * r \le (\hat{f}_R\, q) * r \le \hat{f}_R\,(q * r)$ by Lemma 8 and therefore $\vdash \{p * r\}R\{q * r\}$. □

A deeper investigation of Hoare logics, inference rules for separation logic, and extensions to concurrency in this setting is left for future work.

16 Conclusion

The aim of this article is to demonstrate that convolution is a versatile and interesting construction in mathematics and computer science. Used in the context of power series and integrated into lifting results, it yields a powerful tool for setting up various mathematical structures and computational models and calculi endowed with generic algebraic properties.

Beyond the language models known from formal language theory, these include assertion quantales of separation logic (which can be lifted from an underlying resource monoid), assertion quantales of interval logics (which can be lifted from an underlying semigroup of intervals) and stream interval functions (which have applications in the analysis of dynamic and real-time systems). For all these examples, the power series approach provides a simple new approach. For the latter two, new kinds of concurrency operations are provided.

In addition, the modelling framework based on power series has been combined with a verification approach by deriving, in generic fashion, propositional Hoare logics for virtually all the examples considered. In particular, state, predicate or resource transformers, which can be used for constructing these logics, arise as instances of power series.

This article focused mainly on the proof of concept of the relevance of convolution. Many of the modelling examples and verification approaches featured require further investigation. This includes in particular the derivation of more comprehensive sets of Hoare-style inference rules for concurrency verification, separation logic and interval temporal logics, and more detailed case studies with separation, inverval and stream interval algebras, and with concurrent systems with infinite behaviours.

For all these case studies, the formalisation of the power series approach and the implementation of modelling tools plays and important role. In fact, the basic lifting lemma and a detailed predicate transformer approach based on power series have already been formalised within the Isabelle/HOL proof assistant [26]. The development of a power series based verification tool for separation logic, and even concurrent separation logic, will be the next step in the tool chain.

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