Stability analysis of T-S fuzzy control systems by using set theory

Jiuxiang Dong Member, IEEE, Guang-Hong Yang, Senior Member, IEEE and Huaguang Zhang, Senior Member, IEEE

Abstract—This paper is concerned with the stability analysis for T-S fuzzy control systems. By exploiting the property of the structure of fuzzy inference engine, an equivalence relation on index set of the product of fuzzy rule weights is defined. Further, a new stability criterion is proposed by using the equivalence relation, and formulated into progressively less conservative sets of linear matrix inequalities. By using an extension of Pólya’s Theorem, the new criterion is proved to be with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. A numerical example is given to illustrate the effectiveness of the proposed method.

Index Terms—T-S fuzzy control systems, stability analysis, equivalence class, set theory, linear matrix inequalities (LMIs).

I. INTRODUCTION

SINCE the terminology of the fuzzy set was proposed by Zadeh in 1965 [35], it has been found extensive applications in the areas of industrial and economical systems and so on. In particular, by constructing Takagi-Sugeno (T-S) fuzzy models of nonlinear control systems, various systematic mathematical techniques are successfully developed for guaranteeing the stability and performance of nonlinear systems. T-S fuzzy systems can be viewed as some locally linear time-invariant systems connected by IF-THEN rules. As a result, the conventional linear system theory can be applied for nonlinear control systems.

In recent years, stability analysis and synthesis of T-S fuzzy systems have been well studied [31], [6], [5], [32], [4], [34], [30], [37], [21], where quadratic Lyapunov function approaches [20], [9], [28], [18] are widely employed. Since a common Lyapunov matrix is used for all local models of fuzzy systems, the quadratic Lyapunov function approach often leads to conservative results. Then parameter dependent Lyapunov functions (or called fuzzy Lyapunov functions) [24], [11], [19], [17], [36], piecewise Lyapunov functions [13], [23] and k-sample variation Lyapunov functions [16] are respectively proposed for reducing the conservatism introduced by using quadratic Lyapunov functions. On the other hand, by sharing the same fuzzy rules with the fuzzy models, parallel distributed compensation (PDC) control schemes [29] are often used for designing fuzzy controllers in the existing literature. In addition, a number of alternative control schemes are also proposed for less conservative design, such as non-PDC control schemes [11], [33], switching constant controller gain schemes [10], local nonlinear feedback control schemes [7] and so on.

The above-mentioned results have made significant progress in stability analysis and synthesis of T-S fuzzy control systems, and they are applicable for the T-S fuzzy systems with any membership function and any fuzzy inference engine, which implies that they are independent of the actual membership shape and the choice of fuzzy inference engines. Hence, they might be conservative if specific knowledge of the fuzzy membership or fuzzy inference is available, then the properties of fuzzy membership shapes or fuzzy inference engines are exploited by many researchers, and some less conservative conditions for the stability analysis and synthesis of T-S fuzzy control systems are presented. For example, by incorporating shape information in the form of polynomial constraints, a stability and performance condition for polynomial-in-membership Takagi-Sugeno fuzzy systems is proposed in [27]. A stability analysis condition based on some inequalities in the form of a p-dimensional fuzzy summation is given in [25]. By using the property of pseudotrapezoid membership functions, a class of Lyapunov functions and fuzzy control schemes depending on dominant fuzzy membership functions are presented in [8]. By constructing tensor product T-S fuzzy models and using the property of the tensor product of membership functions, modelling and control based on a recursive algorithm are given in [2] and [1], respectively. By utilizing the extreme points in each partition to address the constraints of the fuzzy weights and their derivatives, a switching control law based on the partition is achieved in [15].

Motivated by the above works, where the properties about the shape of membership functions or the structure of fuzzy rule weights are exploited for less conservative conditions, we will further study the stability analysis problem for T-S fuzzy control systems by using some new properties of rule weights with a fuzzy product inference engine. By partitioning index set of the product of rule weights with the aid of an
equivalence relation on the index set, a new stability analysis criterion is acquired and the new criterion is composed of a family of linear matrix inequalities with progressively less conservatism. In particular, by using an extension of the Pólya’s Theorem, it is shown that the criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. Moreover, it is proved that the class of new approaches are not only with less conservatism but also with a lighter computational burden than the existing approaches in [20]. The comparisons with the existing approaches in [29], [20], [9], [24], [28] by a numerical example further illustrate that the new conditions have the potential to give less conservative results.

The rest of this paper is organized as follows. Section II gives some necessary preliminaries on set theory. T-S fuzzy models are given in Section III. By defining an equivalence relation on index set of the product of rule weights and using the equivalence relation, a new stability analysis condition is proposed in Section IV. In Section V, a numerical example is given to illustrate the effectiveness of the proposed methods. Section VI concludes the paper.

II. PRELIMINARIES AND TECHNICAL LEMMAS

Set theory is one of the most fundamental branches of mathematics. In this section, some related notations and terminologies of elementary set theory are recalled. Further, some new technical lemmas are proposed, which are useful for obtaining a stability analysis criterion of T-S fuzzy control systems.

A. Notation, conception and some existing lemmas

- $\mathbb{Z}_+$ denotes the positive integer set.
- $\emptyset$ denotes empty set.
- $|X|$ denotes the number of elements (cardinality) of a set $X$.
- $X_1, X_2, \ldots, X_n$ are sets,

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$$

$$\left\{ \left( x_1, \ldots, x_n \right) : x_1 \in X_1 \land \cdots \land x_n \in X_n \right\} \quad (1)$$

where $(x_1, x_2, \ldots, x_n)$ is an ordered $n$-tuple, and represents a classic logical operator “conjunction”.

We also use the permutation $x = x_1 x_2 \cdots x_n$ to denote the ordered $n$-tuple $(x_1, x_2, \ldots, x_n)$. Use $x_{[i]}$ to represent the $i$-th element of $x$, i.e., an element $\tau$ belongs to $\prod_{j=1}^{p} X_j$, which means that $\tau = \tau_1 \tau_2 \cdots \tau_{p}$ and $\tau_j \in X_j, j = 1, \ldots, p$.

For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{h_1+\cdots+h_p} \in \prod_{j=1}^{p} S_{h_j}$, where $h_i, i = 1, \ldots, p$ are positive integers, we define two maps as follows:

$$\chi_j : \prod_{i=1}^{p} S_{h_i} \rightarrow S_{h_j}, \quad \text{for } j = 1, \ldots, p$$

$$\varrho_j : \prod_{i=1}^{p} S_{h_i} \rightarrow \prod_{i=1}^{p} S_{l_i}, \quad \text{for } j = 1, \ldots, g,$$

$$g = \min\{ h_i : 1 \leq i \leq p \}$$

with

$$\chi_1(\sigma) = \sigma_1 [\sigma_2 [ \cdots [ \sigma_{h_1} \cdots ] \cdots ]$$

$$\chi_2(\sigma) = \sigma_{h_1+1} [\sigma_{h_1+2} [ \cdots [ \sigma_{h_1+h_2} \cdots ] \cdots ]$$

$$\vdots$$

$$\chi_p(\sigma) =$$

$$\sigma_{h_1+\cdots+h_{p-1}+1} [\sigma_{h_1+\cdots+h_{p-1}+2} [ \cdots [\sigma_{h_1+\cdots+h_{p-1}+h_p} ] \cdots ]$$

$$\varrho_1(\sigma) = \sigma_1 [\sigma_{h_1+2} [ \cdots [ \sigma_{h_1+h_2+1} \cdots ] \cdots ]$$

$$\varrho_2(\sigma) = \sigma_{h_1+3} [\sigma_{h_1+h_2+2} [ \cdots [ \sigma_{h_1+\cdots+h_{p-1}+1} ] \cdots ]$$

$$\vdots$$

$$\varrho_g(\sigma) = \sigma_{h_1+h_g} [\sigma_{h_1+h_{g+1}} [ \cdots [ \sigma_{h_1+\cdots+h_{p-1}+g} ] \cdots ]$$

and denote $\chi_i(\sigma)$ by $\varrho^i_1(\sigma)$, $\varrho^i_j(\sigma)$ by $\varrho^i_j(\sigma)$.

For function $\mu_j(v(t)), 1 \leq i \leq p, j = 1, \ldots, g$, we define

$$\mu_\tau = \prod_{j=1}^{p} \prod_{i=1}^{l} \mu_j(v\tau_{i}(t))$$

$$= \prod_{j=1}^{p} \prod_{i=1}^{l} \mu_j(v_{\tau_{i}}(t)) \quad (3)$$

where $\tau \in \prod_{i=1}^{p} S_{h_i}$.

B. Equivalence class and inequality

In this subsection, a relation on index set is defined, and it is proved to be an equivalence relation. By using the equivalence relation, a new condition is proposed for converting a parameter dependent inequality into parameter independent inequalities.

Let a set $S_0 \subset \mathbb{Z}_+$ with $|S_0| < \infty$ ($|S_0|$ denotes the cardinality of the set $S_0$). If $(i_1, i_2, \ldots, i_{h_0}) \in S_0^{h_0}$, we can view the element $(i_1, i_2, \ldots, i_{h_0})$ of $S_0^{h_0}$ as an $h_0$-ary permutation $i_1 i_2 \cdots i_{h_0}$. We define a map $st(\bullet)$ from $S_0^{h_0}$ to $S_0^{h_0}$

$$st(i_1 i_2 \cdots i_{h_0}) = l_1 l_2 \cdots l_{h_0} \quad (4)$$

as an arrangement of the permutation $i_1 i_2 \cdots i_{h_0}$ with $l_1 \leq l_2 \leq \cdots \leq l_{h_0}$.

Based on the mapping $st(\bullet)$, we define a binary relation on $S_0^{h_0}$ as follows:

$$\mathbb{R}_{h_0} = \left\{ (i_1 i_2 \cdots i_{h_0}, j_1 j_2 \cdots j_{h_0}) : st(i_1 i_2 \cdots i_{h_0}) = st(j_1 j_2 \cdots j_{h_0}) \right\}$$

(5)

From the definition of the relation $\mathbb{R}_{h_0}$, we can easily verify that $\mathbb{R}_{h_0}$ is reflexive, symmetric, and transitive, i.e., $\mathbb{R}_{h_0}$ is an equivalent relation over the set $S_0^{h_0}$.

Denote $S_0^{h_0} / \mathbb{R}_{h_0}$ as the quotient of the equivalent relation $\mathbb{R}_{h_0}$, i.e., $S_0^{h_0} / \mathbb{R}_{h_0}$ is formed of all equivalence classes of $\mathbb{R}_{h_0}$. By Lemma 7 (see Appendix), we have the quotient set $S_0^{h_0} / \mathbb{R}_{h_0}$ is a partition of the set $S_0^{h_0}$, i.e.,

$$S_0^{h_0} = \bigcup_{y \in S_0^{h_0} / \mathbb{R}_{h_0}} S_0$$

with for all $x \neq y \in S_0^{h_0} / \mathbb{R}_{h_0}, x \cap y = \emptyset$.
For example, $X = \{11, 12, 21, 22\}$, then $\sum_{\tau \in R} M_\tau = M_{11} + M_{12} + M_{21} + M_{22}$. Define a binary relation on $X$ as

$$R = \{(i_1i_2, j_1j_2) : st(j_1j_2) = st(i_1i_2)\}$$

where $st(\cdot)$ is the same as in (4).

Then the quotient set

$$X/R = \{[11]_R, [12]_R, [21]_R, [22]_R\}$$

with all the equivalence classes of $R_{oh_0}$ as follows:

- $[11]_R = \{11\}$
- $[12]_R = \{12, 21\} = [21]_R$
- $[22]_R = \{22\}$

The following fact can easily be obtained

$$\bigcup_{s_0 \in X/R} s_0 = [11]_R \bigcup [12]_R \bigcup [22]_R = X$$

which further validates Lemma 7, i.e., $X/R$ is a partition of $X$.

Then

$$\sum_{s \in X/R} \sum_{\tau \in s} M_\tau = \sum_{\tau \in [11]_R} M_\tau + \sum_{\tau \in [12]_R} M_\tau + \sum_{\tau \in [22]_R} M_\tau$$

$$= M_{11} + M_{12} + M_{21} + M_{22} = M = \sum_{\tau \in \mathbb{R}} M_\tau$$

**Lemma 1:** Let $S_i \subset \mathbb{Z}_+^p$, with $|S_i| < \infty$, $1 \leq l \leq p$, then $S_{i_1}^1 \times S_{i_2}^2 \times \cdots \times S_{i_p}^p / R_{ih_i}$ is a partition of $S_{i_1}^1 \times S_{i_2}^2 \times \cdots \times S_{i_p}^p$, where

$$R_{ih_i} = \{(i_1i_2 \cdots i_l, j_1j_2 \cdots j_{lh_i}) : st(j_1j_2 \cdots j_{lh_i}) = st(i_1i_2 \cdots i_l)\}, \quad 1 \leq l \leq p$$

(6)

and $st(\cdot)$ is the same as in (4).

**Proof:** See Appendix.

Based on Lemma 1, the following useful lemma can be obtained

**Lemma 2:** Let $S_i \subset \mathbb{Z}_+^p$, with $|S_i| < \infty$, $1 \leq l \leq p$, and

$$\mu_{j_1j_l}(v_j(t)) \geq 0, \quad \text{and} \quad \sum_{i_j \in S_j} \mu_{j_1j_l}(v_j(t)) = 1, \quad \text{for } i_j \in S_j,$$

$$j = 1, \ldots, p$$

if

$$\sum_{\sigma \in \mathbb{S}_i} M_\sigma < 0, \quad \text{for } \mathbb{S} \in \prod_{i=1}^p \{S_i^h / R_{ih_i}\}$$

(8)

then

$$\sum_{\sigma \in \mathbb{P}_{i=1}^p S_i^h} \mu_\sigma M_\sigma < 0$$

(9)

where $\mu_\sigma$ and $R_{ih_i}$ are the same as in (3) and (6), respectively.

**Proof:** See Appendix.

### III. System Description

**T-S fuzzy system**

The nonlinear system under consideration is described by the following fuzzy system model:

**Plant Rule** $(i_1i_2 \cdots i_p)$:

- **IF** $v_1(t)$ is $M_{i_1}$, and $v_2(t)$ is $M_{i_2}$, \ldots, $v_p(t)$ is $M_{i_p}$

- **THEN** $x(t) = A_{i_1i_2 \cdots i_p} x(t) + B_{i_1i_2 \cdots i_p} u(t)$

(10)

$x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^n$ is the control input vector, $v(t) = [v_1(t) \ v_2(t) \ \cdots \ v_p(t)]^T \in \mathbb{R}^p$, $v_i(t)$, $i = 1, \ldots, p$, are the premise variables and assumed to be measurable, $M_{i_1j_1}, j = 1, \ldots, p, i_j = 1, \ldots, r_j$ denotes an $v_j(t)$-based fuzzy set and they are linguistic terms characterized by fuzzy membership functions $M_{i_1j_1}(v_j(t))$, where $r_j$ is the number of $v_j(t)$-based fuzzy sets. Then, the fuzzy rule base consists of $r = \prod_{i=1}^p r_i$ IF-THEN rules.

By using a singleton fuzzifier, a product inference engine and a center average defuzzifier, the T-S fuzzy model is obtained as: Let

$$\mu_{j_1j_k}(v_j(t)) = \frac{r_j}{\sum_{i_j=1}^{r_j} M_{j_1j_k}(v_j(t))}, \quad \text{for } 1 \leq j \leq p, 1 \leq i_j \leq r_j$$

(12)

Combining it and (11), the fuzzy system can be written as follows:

$$\dot{x}(t) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \prod_{j=1}^p \mu_{j_1j_k}(v_j(t)) \times$$

$$A_{i_1i_2 \cdots i_p}x(t) + B_{i_1i_2 \cdots i_p}u(t)$$

(13)

From (12), it is resulted that

$$\sum_{i_j=1}^{r_j} \mu_{j_1j_k}(v_j(t)) = 1, \quad \text{for } 1 \leq j \leq p$$

(14)

By using set theory, (13) can be rewritten as follows:

$$\dot{x}(t) = \sum_{\tau \in \mathbb{P}_{i=1}^p S_i} \mu_\tau (A_\tau x(t) + B_\tau u(t))$$

(15)

where $\mu_\tau$ is the same as in (3) and

$$S_i = \{1, 2, \cdots, r_i\}, \quad i = 1, 2, \cdots, p$$

(16)

**Fuzzy controller**

In the existing literature, there are many fuzzy control schemes for T-S fuzzy systems, for example, parallel distributed compensation (PDC) control schemes [29], non-PDC control schemes [11], switching constant gain control schemes [10], dominant dependent fuzzy control schemes [8] and so on. This paper focuses on how to use the property of the product of rule weights based on the equivalence class in set theory for obtaining a better stability analysis condition, and any control scheme is applicable in this paper. In particular, the PDC controller is adopted in this paper as follows:

**Control Rule** $(i_1i_2 \cdots i_p)$:

- **IF** $v_1(t)$ is $M_{i_1}$, and $v_2(t)$ is $M_{i_2}$, \ldots, $v_p(t)$ is $M_{i_p}$

- **THEN** $x(t) = A_{i_1i_2 \cdots i_p} x(t) + B_{i_1i_2 \cdots i_p} u(t)$

(10)
\[ \dot{x}(t) = \frac{\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \left( \prod_{j=1}^{p} M_{ji}(v_j(t)) \right) \left( A_{i_1i_2\cdots i_p}x(t) + B_{i_1i_2\cdots i_p}u(t) \right)}{\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \prod_{j=1}^{p} M_{ji}(v_j(t))} \]  

**IF** \( v_1(t) \) is \( M_{1i_1} \) and \( v_2(t) \) is \( M_{2i_2}, \ldots, v_p(t) \) is \( M_{pi_p} \)

**THEN** \( u(t) = K_{i_1i_2\cdots i_p}x(t) \)

By using a singleton fuzzifier, a product inference engine and a center average defuzzifier, the final output of the fuzzy controller is inferred as follows:

\[ u(t) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \prod_{j=1}^{p} \mu_{i_1i_2\cdots i_p}(v_j(t)) K_{i_1i_2\cdots i_p}x(t) \]  

(17)

Its substitutional description based on set theory is

\[ u(t) = \sum_{\tau \in \prod_{i=1}^{p} S_i} \mu_{\tau} \sum_{x(t) \in S_i} x(t) \]  

(18)

where \( \mu_{\tau} \) and \( S_i \) are the same as in (3) and (16), respectively.

**Closed-loop fuzzy system**

Now we substitute (18) into (15), then we have

\[ \dot{x}(t) = \sum_{\sigma \in \prod_{i=1}^{p} S_i} \mu_{\sigma} A_{\sigma} x(t) \]

\[ + \sum_{\chi \in \prod_{i=1}^{p} S_i} \sum_{\eta \in \prod_{i=1}^{p} S_i} \mu_{\chi} B_{\eta} \left( \sum_{\xi \in \prod_{i=1}^{p} S_i} \mu_{\xi} K_{\xi} x(t) \right) \]  

(19)

where the definitions of \( \mu_{\sigma}, \mu_{\eta}, \mu_{\chi}, \mu_{\xi} \) refer to (3) and \( S_i = \{1, 2, \ldots, r_i\}, i = 1, 2, \ldots, p \).

Combining (14) and (19), it follows that

\[ \dot{x}(t) = \sum_{\xi \in \prod_{i=1}^{p} S_i} \mu_{\xi} A_{\xi} x(t) \]

\[ + \sum_{\xi \in \prod_{i=1}^{p} S_i} \mu_{\xi} B_{\xi} \left( \sum_{\tau \in \prod_{i=1}^{p} S_i} \mu_{\tau} K_{\tau} x(t) \right) \]  

(20)

where the relation of \( \xi \) and \( \xi^{\epsilon_1} \) (or \( \xi^{\epsilon_2} \)) is given in (2).

Let

\[ \Lambda_{\xi} = A_{\xi} + B_{\xi} K_{\xi} \]  

(21)

then the closed-loop system (20) can be rewritten as:

\[ \dot{x}(t) = \sum_{\xi \in \prod_{i=1}^{p} S_i^2} \mu_{\xi} \mu_{\xi^{\epsilon_1}} \Lambda_{\xi} x(t) \]  

(22)

**Description of fuzzy system by using fuzzy basis functions**

(13) can be further re-described by fuzzy basis functions

\[ \mu_{i_1i_2\cdots i_p}(v(t)) = \prod_{j=1}^{p} M_{ji}(v_j(t)) \]

(23)

as follows:

\[ \dot{x}(t) = \sum_{\xi \in \prod_{i=1}^{p} S_i} \mu_{\xi} A_{\xi} x(t) \]

\[ + \sum_{\xi \in \prod_{i=1}^{p} S_i} \mu_{\xi} \left( \sum_{\tau \in \prod_{i=1}^{p} S_i} \mu_{\tau} K_{\tau} x(t) \right) \]  

(24)
\[
\dot{B}_q(t) = B_r, \quad \tilde{K}_q(t) = K_r
\]

then the closed-loop system (23) can be rewritten as follows:

\[
\dot{x}(t) = \sum_{r \in \Pi_{i=1}^p S_i} \alpha_i(v(t)) \left( \dot{A}_r x(t) + \tilde{B}_r u(t) \right)
\]

which is equivalent to

\[
\dot{x}(t) = \sum_{i=1}^r \alpha_i(v(t)) \left( \dot{A}_i x(t) + \tilde{B}_i u(t) \right)
\]

Along the lines of the above technique, the fuzzy controller (17) can also be rewritten as follows:

\[
u(t) = \sum_{i=1}^r \alpha_i(v(t)) \tilde{K}_i x(t)
\]

Moreover, we can easily obtain \(0 \leq \alpha_i(v(t)) \leq 1, \ i = 1, \ldots, r, \ \sum_{i=1}^r \alpha_i(v(t)) = 1\).

The fuzzy system description (27) with (28) is widely used in the existing literature, and there are various stability analysis conditions based on the description, see [29], [20], [28], and the reference therein, where the condition in [29] is with the least computational complexity based on LMIs, and the condition in [28] is asymptotically necessary and sufficient for quadratic stability analysis of T-S fuzzy control systems with any possible membership function and inference engine. In order to give the comparisons with the existing methods by theoretical proof, some existing conditions are recalled as follows:

**Lemma 3:** [29] If there exists a matrix \(\bar{P} = \bar{P}^T > 0\) satisfying

\[
\text{He}(\bar{P}G_{ij} + \bar{P}G_{ji}) < 0, \quad \text{for } 1 \leq i \leq j \leq r
\]

where

\[
G_{ij} = \bar{A}_i + \bar{B}_i \bar{K}_j
\]

then the fuzzy system (27) with (28) is asymptotically stable.

**Lemma 4:** [20] If there exist matrices \(\bar{P} = \bar{P}^T > 0, \ \bar{Y}_{ij}\), \(1 \leq i \leq j \leq r\) satisfying

\[
\text{He}(\bar{P}G_{ij} + \bar{P}G_{ji}) \leq \bar{Y}_{ij} + (\bar{Y}_{ij})^T, \quad \text{for } 1 \leq i \leq j \leq r
\]

\[
[\bar{Y}_{ij}] < 0
\]

then the fuzzy system (27) with (28) is asymptotically stable.

**Lemma 5:** [24] Assume that \(\bar{\alpha}_i(v(t)) \leq \bar{\phi}_i, \ 1 \leq i \leq r, \) if there exist matrices \(X = X^T, \ P_i = P_i^T, \ 1 \leq i \leq r, \) satisfying the following LMIs

\[
P_i > 0, \quad 1 \leq i \leq r
\]

\[
P_i + X > 0, \quad 1 \leq i \leq r
\]

\[
\bar{P} + \frac{1}{2}\text{He}(P_i G_{ij} + P_i G_{ji}) < 0, \quad 1 \leq i \leq l \leq r, \ 1 \leq j \leq r
\]

where \(\tilde{P} = \sum_{l=1}^r \phi_i (P_l + X), \ \phi_i \) are scalars, then the fuzzy system (27) with (28) is asymptotically stable.

**IV. STABILITY CRITERION**

In this section, a new stability analysis criterion for T-S fuzzy systems is proposed with progressively less conservatism. It is proved that the new criterion is with less conservatism and complexity than Lemma 4. Moreover, by using an extension of Pólya’s Theorem, it is shown that the criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. Before main results are presented, some provades are following:

Since \(S_i, \ i = 1, \ldots, p, \) are with finite elements, and \( |S_i| = r_i, \) then \( |\prod_{i=1}^p S_i| = |\prod_{i=1}^p r_i| . \) Further, we can define a 1-1 mapping from the set \(\prod_{i=1}^p S_i\) to the set \(\{1, 2, \ldots, \tilde{r}\}, \) where \(\tilde{r} = \prod_{i=1}^p r_i .\)

A particular \(q\) can be chosen as

\[
q(\tau) = 1 + \sum_{i=1}^{h_i} (\tau_{(i-1)} - 1)r_i^{i-1} + \sum_{i=2}^{h_i} (\tau_{(i-1)} - 1)^2 r_i^{i-1} + \ldots + \sum_{i=p}^{h_i} (\tau_{(i-1)} - 1) p\prod_{j=1}^{p-1} r_j^{-1}
\]

Let

\[
\bar{\alpha}_q(v(t)) = \mu(\tau) = \prod_{i=1}^p \prod_{j=1}^{h_i} \mu_{\tau_{(i-1)}}(v_j(t)), \quad \text{for } \tau \in \prod_{i=1}^p S_i
\]

Denote \(\bar{\alpha}_q(v(t))\) as \(\bar{\alpha}_q(\tau),\) then

\[
\sum_{\tau \in \prod_{i=1}^p S_i} \bar{\alpha}_q(\tau) = \sum_{\tau \in \prod_{i=1}^p S_i} \mu(\tau)
\]

From (14), we have

\[
\mu(\tau) = \prod_{\tau \in \prod_{i=1}^p S_i} \left( \sum_{\tau \in S_j} \mu_{j_{ij}}(v_j(t)) \right)
\]

Combining it and (34), then we have

\[
\sum_{i=1}^r \bar{\alpha}_q(\tau) = 1, \quad 0 \leq \bar{\alpha}_q(\tau) \leq 1
\]
For $\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}$, define

$$\sigma^{\beta_{1}} = \sigma(1)^{2} \sigma(2) \cdot \cdot \cdot \sigma(h_{1} + 1)^{\sigma(2h_{1} + 2)} \cdot \cdot \cdot \sigma(h_{2} + 2)$$

$$\cdot \cdot \cdot \sigma(2 \sum_{i=1}^{p} h_{i} + 1)^{\sigma(2 \sum_{i=1}^{p} h_{i} + 2)} \cdot \cdot \cdot \sigma(2 \sum_{i=1}^{p} h_{i} + p)^{\sigma(2 \sum_{i=1}^{p} h_{i} + 2)}$$

$$\sigma^{\beta_{2}} = \sigma(h_{1} + 1)^{\sigma(h_{1} + 2)} \cdot \cdot \cdot \sigma(h_{1} + p)^{\sigma(h_{1} + 2)} \cdot \cdot \cdot \sigma(h_{2} + 2) \cdot \cdot \cdot \sigma(2 \sum_{i=1}^{p} h_{i} + 1)^{\sigma(2 \sum_{i=1}^{p} h_{i} + 2)}$$

$$\cdot \cdot \cdot \sigma(2 \sum_{i=1}^{p} h_{i} + p)^{\sigma(2 \sum_{i=1}^{p} h_{i} + 2)}$$

$$\sigma^{(2 \sum_{i=1}^{p} h_{i})}$$

(36)

then $\sigma^{\beta_{1}}$ and $\sigma^{\beta_{2}}$ belong to $\prod_{i=1}^{p} S_{h_{i}}^{b_{i}}$.

**Theorem 1**: Given $h_{j} \in 2Z_{+}$ ($2Z_{+}$ denotes even set) with $h_{j} \geq 2$, $j = 1, \cdots, p$, binary relations $R(\cdot, h_{i})$, $i = 1, \cdots, p$, which are the same as in Lemma 2. If there exist matrices $P = P^{T} > 0$, $Y_{\sigma}, \sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}$, with $Y_{\sigma} = (Y_{\sigma})^{T}$ for $\sigma^{\beta_{1}} = \sigma^{\beta_{2}}$, $\sigma^{\beta_{2}} = \sigma^{\beta_{1}}$, satisfying the following LMIs

$$\sum_{\sigma \in S_{\sigma}^{h_{j}}} M_{\sigma} \leq \sum_{\sigma \in S_{\sigma}^{h_{j}}} Y_{\sigma}, \text{ for } \bar{s} \in \prod_{i=1}^{p} (S_{h_{i}}^{b_{i}} / R_{i})$$

(37)

where

$$M_{\sigma} = PA_{\sigma} + A_{\sigma}^{T} P, \text{ for } \sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}$$

(39)

and $A_{\sigma}$ is the same as in (21), $H_{\sigma}(\sigma^{\beta_{1}}, \sigma^{\beta_{2}}) = Y_{\sigma}, q(\cdot)$ is the same as in (32), then the continuous time fuzzy system (13) is asymptotically stable.

**Proof**: Applying Lemma 2 to (37), then we have

$$\sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma} M_{\sigma} \leq \sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma} Y_{\sigma}$$

(40)

Let $h_{j} = 2 h_{i}$, and define $q(\cdot)$ and $\alpha_{i}, i = 1, \cdots, \bar{r}$ by (32) and (33). It can be obtained from (38) that

$$\begin{bmatrix} \alpha_{1} \cr \alpha_{2} \cr \vdots \cr \alpha_{\bar{r}} \end{bmatrix}^{T} \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1\bar{r}} \\ H_{21} & H_{22} & \cdots & H_{2\bar{r}} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\bar{r}1} & H_{\bar{r}2} & \cdots & H_{\bar{r}\bar{r}} \end{bmatrix} \begin{bmatrix} \alpha_{1} \cr \alpha_{2} \cr \vdots \cr \alpha_{\bar{r}} \end{bmatrix} = 0$$

i.e.,

$$\sum_{i=1}^{\bar{r}} \sum_{j=1}^{\bar{r}} \bar{\alpha}_{i} \bar{\alpha}_{j} H_{ij} < 0$$

Combining it and the definition of $q(\cdot)$, it yields that

$$\sum_{i=1}^{\bar{r}} \sum_{j=1}^{\bar{r}} \bar{\alpha}_{i} \bar{\alpha}_{j} H_{ij} = \sum_{q(\sigma^{\beta_{1}}) = 1}^{h_{j} - 2} \bar{\alpha}_{q(\sigma^{\beta_{1}})} \bar{\alpha}_{q(\sigma^{\beta_{2}})} H_{q(\sigma^{\beta_{1}})q(\sigma^{\beta_{2}})}$$

$$= \sum_{q(\sigma^{\beta_{1}}) = 1}^{h_{j} - 2} \bar{\alpha}_{q(\sigma^{\beta_{1}})} \bar{\alpha}_{q(\sigma^{\beta_{2}})} Y_{\sigma}$$

$$= \sum_{\sigma^{\beta_{1}} \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma^{\beta_{1}}} \mu_{\sigma^{\beta_{2}}} Y_{\sigma}$$

Combining it and (40), we can obtain

$$\sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma} M_{\sigma} < 0$$

(41)

which is equivalent to

$$\sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) < 0$$

(42)

Choose a quadratic Lyapunov function

$$V(t) = x^{T}(t) P x(t)$$

then it follows from (20) that

$$V(t) = 2 x^{T}(t) P \dot{x}(t)$$

$$= 2 x^{T}(t) P \sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}}} \mu_{\sigma^{\beta_{1}}} \mu_{\sigma^{\beta_{2}}} (A_{\sigma^{\beta_{1}}} + B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) x(t)$$

$$= x^{T}(t) \sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \mu_{\sigma^{\beta_{1}}} \mu_{\sigma^{\beta_{2}}} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}})$$

(43)

Consider

$$\sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \mu_{\sigma} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) x(t)$$

for $1 \leq j \leq p$.

Combining it and (14), we have

$$\sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \mu_{\sigma} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) x(t) = 1, \text{ for } 1 \leq j \leq p$$

From it and (43), we can obtain

$$\ddot{V}(t) = x^{T}(t) \left( \prod_{j=1}^{p} \left( \sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \mu_{\sigma} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) \right) \right) x(t)$$

$$= x^{T}(t) \left\{ \sum_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \left( \prod_{j=1}^{h_{j} - 2} \mu_{\sigma^{\beta_{1}}} \mu_{\sigma^{\beta_{2}}} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) \right) \right\} x(t)$$

$$\times \left( \prod_{\sigma \in \prod_{i=1}^{p} S_{h_{i}}^{b_{i}} \times} \mu_{\sigma} \mathbf{H}(P A_{\sigma^{\beta_{1}}} + P B_{\sigma^{\beta_{1}}} K_{\sigma^{\beta_{2}}}) \right) x(t)$$
Lemma 10 in Appendix. Moreover, the value of \( \sigma \) by the following mapping,
\[
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) = \begin{cases} 
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) - \epsilon I, & \tau(1) = \tau(2), \sigma^{\beta_1} = \sigma^{\beta_2} \\
\hat{H}_{q(\hat{\sigma})}(q(\sigma)), & \text{others}
\end{cases}
\]
(49)

Choose \( \check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma)) \), then
\[
\check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma))
\]

For arbitrary \( r^2 \prod_{i=1}^{p} \bar{V}_{i}^{h_i} = \bar{r} \)-dimension vector \( z = [z_1 \ z_2 \ \cdots \ z_p]^T \neq 0 \), pre- and post-multiplying \([H_{ij}] \) by \( z^T \) and \( z \), then it follows that
\[
z^T[H_{ij}]z
\]

Let \( \check{\sigma} = \Psi(\tau, \sigma) \), and
\[
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) = \begin{cases} 
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) - \epsilon I, & \tau(1) = \tau(2), \sigma^{\beta_1} = \sigma^{\beta_2} \\
\hat{H}_{q(\hat{\sigma})}(q(\sigma)), & \text{others}
\end{cases}
\]
(49)

Choose \( \check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma)) \), then
\[
\check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma))
\]

For arbitrary \( r^2 \prod_{i=1}^{p} \bar{V}_{i}^{h_i} = \bar{r} \)-dimension vector \( z = [z_1 \ z_2 \ \cdots \ z_p]^T \neq 0 \), pre- and post-multiplying \([H_{ij}] \) by \( z^T \) and \( z \), then it follows that
\[
z^T[H_{ij}]z
\]

Let \( \check{\sigma} = \Psi(\tau, \sigma) \), and
\[
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) = \begin{cases} 
\hat{H}_{q(\hat{\sigma})}(q(\sigma)) - \epsilon I, & \tau(1) = \tau(2), \sigma^{\beta_1} = \sigma^{\beta_2} \\
\hat{H}_{q(\hat{\sigma})}(q(\sigma)), & \text{others}
\end{cases}
\]
(49)

Choose \( \check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma)) \), then
\[
\check{\mathbf{Y}}_{\sigma} = \hat{H}_{q(\hat{\sigma})}(q(\sigma))
\]

For arbitrary \( r^2 \prod_{i=1}^{p} \bar{V}_{i}^{h_i} = \bar{r} \)-dimension vector \( z = [z_1 \ z_2 \ \cdots \ z_p]^T \neq 0 \), pre- and post-multiplying \([H_{ij}] \) by \( z^T \) and \( z \), then it follows that
\[
z^T[H_{ij}]z
\]
Let \( Z_q(\sigma) = \sum_{\tau \in S_2} z_{q(\sigma \tau)} \) and \( \prod_{i=1}^{p} P_i = \bar{r} \), then from (53), we have that

\[
z^T[H_{ij}]z < \sum_{\sigma_1 \in \prod_{i=1}^{p} S_i} \sum_{\sigma_2 \in \prod_{i=1}^{p} S_i} Z_{q(\sigma_1)} Z_{q(\sigma_2)} \tilde{H}_{q(\sigma_1)q(\sigma_2)} \times 
\]

where

\[
\tilde{H}_{q(\sigma_1)q(\sigma_2)} = \sum_{\tau \in S_2} z_{q(\sigma_1 \tau)} \times \sum_{\tau \in S_2} z_{q(\sigma_2 \tau)}
\]

It follows from (37), (50) and (54) that

\[
\sum_{\sigma \in S_2^{d+2}} M_{\sigma} \leq \sum_{\sigma \in S_2^{d+2}} Y_{\sigma}, \quad \text{for } \bar{s} \in (S_1^{2d+2}/R_1(2d+2)) \times 
\]

\[
\prod_{i=2}^{p} (S_i^{2d_i}/R_i(2d_i)) 
\]

i.e., (37) holds for \( h_1 = 2d_1 + 2 \), \( h_i = 2d_i, \) \( i = 2, 3, \ldots, p \).

We have proved that if the condition of Theorem 1 holds for \( h_1 = 2d_1, \) \( i = 1, 2, \ldots, p \), then the condition of Theorem 1 also holds for \( h_1 = 2d_1 + 2, \) \( h_i = 2d_i, \) \( i = 2, \ldots, p \). Further, it is easily obtained that the condition of Theorem 1 also holds for \( h_1 = 2d_1 \geq 2d_1, \) \( h_i = 2d_i, \) \( i = 2, \ldots, p \).

Adopt the same technique for only \( h_i \) increasing for \( i = 2, \ldots, p \). Finally, we can obtain that the condition of Theorem 1 holds for \( h_1 = 2d \geq 2d_1, \) \( i = 1, \ldots, p \). Thus the proof is complete.

**Remark 1:** Theorem 1 collects the interactions of the product of membership functions in a single matrix. The similar technique for dealing with the interactions of the fuzzy rule weights has been proposed in [20]. What it follows, it is proved that the condition of Theorem 1 is more relaxed than Lemma 4 and with a lighter computational burden, see the following theorem and Remark 2.

**Theorem 3:** If the condition of Lemma 4 holds, then the condition of Theorem 1 holds.

**Proof:** If there exists a matrix \( \bar{P} = PT > 0 \), satisfying (30) and (31), then we have that

\[
\text{He}(PA_{\sigma_1} + \bar{P}B_{\sigma_1} K_{\sigma_2} + \bar{P}A_{\sigma_2} + \bar{P}B_{\sigma_2} \bar{K}_{\sigma_1}) \times 
\]

\[
< Y_{\sigma} \times (Y_{\sigma})^T, \quad \text{for } \sigma \in \prod_{i=1}^{p} S_i^2 
\]

\[
[H_{ij}] < 0 
\]

where \( \star_{e_1} \) and \( \star_{e_2} \) are the same as in (2), \( H_{q(\sigma_1)q(\sigma_2)} = Y_{\sigma} = Y_{q(\sigma_1)q(\sigma_2)}, q(\cdot) \) is defined in (25).

Define a binary relation \( \bar{R} \) over the set \( \prod_{i=1}^{p} s_i \in \prod_{i=1}^{p} (S_i^2/R_i), \) where \( R \) is given as follows:

\[
\bar{R} = \left\{ (\pi, \theta) : (\pi e_1) = \theta e_2 \text{ and } \pi e_2 = \theta e_1 \right\} 
\]

It is easily obtained that the relation \( \bar{R} \) is reflexive, symmetric, and transitive, i.e., it is an equivalence relation. Further, we have that the set \( \prod_{i=1}^{p} s_i/\bar{R} = \{[\theta]_{\bar{R}} : \theta \in \prod_{i=1}^{p} s_i \} \) is a partition of the set \( \prod_{i=1}^{p} s_i \).

Therefore,

\[
\text{He}(PA_{\sigma_1} + \bar{P}B_{\sigma_1} \bar{K}_{\sigma_2} + \bar{P}A_{\sigma_2} + \bar{P}B_{\sigma_2} \bar{K}_{\sigma_1} - Y_{\sigma}) \times 
\]

\[
= \sum_{\sigma \in \prod_{i=1}^{p} s_i} \text{He}(PA_{\sigma_1} + \bar{P}B_{\sigma_1} \bar{K}_{\sigma_2} + \bar{P}A_{\sigma_2} + \bar{P}B_{\sigma_2} \bar{K}_{\sigma_1} - Y_{\sigma}) 
\]

---

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On the other hand, if $S \in (\prod_{i=1}^{p} s_i) / \mathbb{R}$, for any $\vartheta, \pi \in S$, we have that $\vartheta e_1 = \pi e_2 = \pi e_1$ or $\vartheta = \pi$, which implies that $|S| = 1$ or 2.

For all $S$, assume some $\vartheta \in S$, from $S \subseteq \prod_{i=1}^{p} s_i \subseteq \prod_{i=1}^{p} \mathbb{S}^2$, we have that $\vartheta \in \prod_{i=1}^{p} \mathbb{S}^2$. For the $\vartheta$, by virtue of (55), we can obtain

$$\text{He}(PA_{\vartheta e_1} + PB_{\vartheta e_1} K_{\vartheta e_2} + PA_{\vartheta e_2} + PB_{\vartheta e_2} K_{\vartheta e_1} - Y_\vartheta) < 0$$

which implies that

$$\sum_{\vartheta \in S} \text{He}(PA_{\vartheta e_1} + PB_{\vartheta e_1} K_{\vartheta e_2} + PA_{\vartheta e_2} + PB_{\vartheta e_2} K_{\vartheta e_1} - Y_\vartheta) < 0$$

for $S \in (\prod_{i=1}^{p} s_i) / \mathbb{R}$, then

$$\sum_{\sigma \in \prod_{i=1}^{p} s_i} M_\sigma = \sum_{\vartheta \in (\prod_{i=1}^{p} s_i) / \mathbb{R}} \sum_{\sigma \in S} M_\sigma$$

$$< \sum_{\vartheta \in (\prod_{i=1}^{p} s_i) / \mathbb{R}} Y_\vartheta = \sum_{\sigma \in \prod_{i=1}^{p} s_i} Y_\vartheta.$$

Combining it and (56), we have that (37) and (38) hold for $h_1 = h_2 = \cdots = h_p = 2$. Further, by virtue of Theorem 2, we have that the condition of Theorem 1 with $h_i \geq 2$, $i = 1, 2, \cdots, p$ holds. Thus, the proof is complete.

**Remark 2:** Note that Theorem 3 shows that the condition of Theorem 1 is more relaxed than one of Lemma 4. In particular, the number of LMIs in Theorem 1 is $\prod_{i=1}^{p} \left( \frac{h_i + r_i - 1}{h_i} \right) + 2$, and the number of LMIs in Lemma 4 is $\left( \frac{1 + \prod_{i=1}^{p} r_i}{2} \right) + 2$.

For the case of $h_i = 2$, the number of LMIs in Theorem 1 is $\prod_{i=1}^{p} \left( \frac{1 + r_i}{2} \right) + 2$ and we can prove that $\prod_{i=1}^{p} \left( \frac{1 + r_i}{2} \right) \leq \left( \frac{1 + \prod_{i=1}^{p} r_i}{2} \right)$ (see Lemma 8 (ii)), which implies that the number of LMIs of Theorem 1 is smaller than Lemma 4. On the other hand, the number and size of variables in Theorem 1 with $h_i = 2$ are the same in Lemma 4, therefore, Theorem 1 with $h_i = 2$ is with a lighter computational burden than Lemma 4.

Note that we have shown that the conservativeness of Theorem 1 becomes less along with increasing $h_i, i = 1, \cdots, p$. In fact, if the $h_i$ is sufficiently large, the conditions of Theorem 1 is with no conservatism for any possible membership. The fact will be illustrated in Theorem 4. In order to obtain the proof of Theorem 4, the useful knowledge about standard $r_q$-simplex is necessary.

We write $\Delta_q$ for the standard $r_q$-simplex

$$\Delta_q = \left\{ [\mu_{q1}, \mu_{q2}, \cdots, \mu_{qr_q}] \in R^{rq} : \sum_{i=1}^{rq} \mu_{qi} = 1, 0 \leq \mu_{qi} \leq 1 \right\},$$

for $q = 1, \cdots, p$.

The following Lemma is an extension as the Pólya’s Theorem.

**Lemma 6:** Let $M(\mu) = M(\mu_1, \mu_2, \cdots, \mu_{1+1}, \mu_{2+1}, \mu_{2+2}, \cdots, \mu_{p+1}, \mu_{p+2}, \cdots, \mu_{p+r}, \mu_{p+r})$ is a homogeneous matrix-valued polynomial on $\Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_p}$, then $M(\mu) > 0$ for $\mu \in \Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_p}$ if and only if there exists a sufficiently large positive integer $d$, such that

$$\prod_{i=1}^{p} \left( \sum_{j=1}^{r_j} \mu_{ij} \right)^d M(\mu)$$

has all its coefficients positive.

Based on Lemma 6, we can obtain the following theorem.

**Theorem 4:** For arbitrary possible membership function $\mu_{ij}(v_j(t)), j = 1, \cdots, p, i = 1, \cdots, r_j$, $M(\mu) = \sum_{\sigma \in \prod_{i=1}^{p} s_i^2} \mu_\sigma M_\sigma < 0$ if and only if there exists a sufficiently large positive integer $d$, such that

$$\sum_{\sigma \in S} M_\sigma < 0, \text{ for } \sigma \in \prod_{i=1}^{p} (S_i^{d+2} / R_i(d+2))$$

**Proof:** If we consider the membership functions $\mu_{ij}, j = 1, \cdots, p, i = 1, \cdots, r_j$, as the variables of the matrix-value polynomial

$$M(\mu) = \sum_{\sigma \in \prod_{i=1}^{p} s_i^2} \mu_\sigma M_\sigma$$

where $M_\sigma$ are matrices, and $\mu_\sigma$ is the same as in (3) and from the property of membership function, we have that $\mu_\sigma \in \Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_p}$ is a monomial with variables $\mu_{j(2j-1)}, j = 1, 2, \cdots, p$.

From (14), it follows that

$$M(\mu) = \prod_{j=1}^{p} \left( \sum_{i=1}^{r_j} \mu_{ij} \right)^d M(\mu) = \sum_{\sigma \in \prod_{i=1}^{p} s_i^{d+2}} \mu_\sigma M_\sigma$$

(60)

Note that the like terms in (60) are not collected, in fact, if the term $\mu_\sigma M_\sigma$ and the term $\mu_\sigma M_\eta$ are like terms, which implies that $\mu_\sigma = \mu_\eta$. Because $\prod_{i=1}^{p} (S_i^{d+2} / R_i(d+2))$ is a partition of $\prod_{i=1}^{p} (S_i^{d+2} / R_i(d+2))$ by Lemma 1, there exists $\sigma \in \prod_{i=1}^{p} (S_i^{d+2} / R_i(d+2))$ such that $\sigma \in \tilde{S}$. From the definition of the equivalence relations $R_i h_i$, we have that $\eta \in \tilde{S}$. On the other hand, if some element $\tilde{\sigma} \in \tilde{S}$, it follows that $\mu_\sigma = \mu_\eta$, therefore, the coefficients of like terms of $\mu_\sigma$ is $\sum_{\sigma \in S} M_\sigma$. By virtue of Lemma 6, $-M(\mu) = - \sum_{\sigma \in \prod_{i=1}^{p} s_i^2} \mu_\sigma M_\sigma > 0$ if and only if there exists a sufficiently large positive integer $d$, such that $- \sum_{\sigma \in \prod_{i=1}^{p} s_i^{d+2}} M_\sigma > 0, \text{ for } \sigma \in \prod_{i=1}^{p} (S_i^{d+2} / R_i(d+2))$. Thus, the proof is complete.

**Remark 3:** From Theorem 4, it follows that if $h_i, i = 1, 2, \cdots, p$ are sufficiently large, the condition of Theorem 1 is sufficient and necessary for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. We should point out that if the properties of the shape of membership function or the firing probability of fuzzy rules are considered, then
less conservative results can be obtained, however this paper focuses on how to use the property of fuzzy product inference engine for less conservative and lighter computational burden conditions, then these properties about the shape and the firing probability are not used in this paper.

V. Example

In this section, a numerical example is given, the conditions of Theorem 1, Corollary 1 and the ones in [29], [20], [9], [28] are applied for illustrating the effectiveness of the new methods. All experiments are implemented in MATLAB, version 7.0.0 (R14) using the packages Yalmip [22] and SeDuMi 1.1R3. The computer used is an Intel (R) Core (TM)2 Quad CPU Q9400 (2.66 GHz), 3.5GB RAM, Windows XP Professional 2002 SP3.

Consider a continuous-time T-S fuzzy system (10) with \( p = 2, r_1 = r_2 = 2 \), where

\[
A_{11} = \begin{bmatrix} a & -10 \\ 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & -10 \\ 1 & 2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 2 & -10 \\ 1 & 1 \end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix},
\]

\[
B_{21} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}.
\]

The local feedback gains \( K, \tau \in \{(11), (12), (21), (22)\} \) are determined by selecting \([-2, -2]\) as the eigenvalues of the subsystems in the PDC controller (17). Figs. 1-10 show the feasible areas of \( a \) and \( b \) satisfying the conditions of Lemmas 3 and 4 in this paper, Theorem 5 in [9], Theorem 5 in [28] and Lemma 5 with \( A_1 = A_{11}, A_2 = A_{12}, A_3 = A_{21}, A_4 = A_{22}, B_1 = B_{11}, B_2 = B_{12}, B_3 = B_{21}, B_4 = B_{22} \), Theorem 1 with \( h_1 = h_2 = 2,4 \), Corollary 1 with \( h_1 = h_2 = 2,3,4 \), respectively.

It can be seen from Figs. 8 and 9 that the condition of Theorem 1 becomes more relaxed along with increasing \( h_1, h_2 \), which verifies Theorem 2. Note that Lemma 5 is based on fuzzy Lyapunov functions, and Fig. 10 shows the stability area obtained by Lemma 5 with the assumption of \( \dot{\alpha}_i(v(t)) \leq 0.85, 1 \leq i \leq 4 \). Comparing Figs. 2-4, 8, 9 with Fig. 10, it can be seen that the stability areas obtained by Theorem 1 and Corollary 1 are larger than the one by Lemma 5, though Theorem 1 and Corollary 1 are based on a single Lyapunov function. The numerical complexity of LMI conditions is closely related to the number of lines \( \mathcal{L} \) and decision variables \( \mathcal{D} \) in the LMIs to be solved, and LMI conditions can be solved in polynomial time with complexity proportional \( \mathcal{C} = \mathcal{D}^3 \mathcal{L} \) [7]. The numerical values of \( \mathcal{L}, \mathcal{D}, \mathcal{C} \) and the CPU time of the different methods are collected in Table I for illustrating the numerical complexity of different LMI conditions.

From Table I, it can be seen that the condition in Corollary 1 with \( h_1 = h_2 = 2 \) is of the least numerical complexity among these methods and has larger feasible area than Lemma 3. For \( 7 \leq a \leq 10, b = 3.4 \), the conditions of Lemmas 3, 4, 5, Theorem 5 in [9], Theorem 5 in [28] are unfeasible, however, the condition of Corollary 1 is feasible. It implies that the condition of Corollary 1 may give less conservative results than the existing conditions and with less numerical complexity.

Moreover, it can also be seen that the condition of Theorem 1 are with larger feasible area than the existing conditions and Corollary 1, which implies that the condition of Theorem 1 is more relaxed than the existing ones.

Compare Fig. 2 with Fig. 8, Fig. 4 with Fig. 9, it can be found that the feasible area of Corollary 1 is smaller than one of Theorem 1 for the same \( h_i \), which implies that Theorem 1 can effectively reduce conservatism than Corollary 1.

VI. Conclusion

In this paper, we have addressed the problem of the stability analysis for T-S fuzzy control systems. By constructing an equivalence relation on the index set of the product of fuzzy rule weights, a new stability analysis criterion of T-S fuzzy systems is proposed based on equivalence classes in set theory and the new criterion is stated as progressively less conservative sets of linear matrix inequalities. Further, it is proved that the new criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. A numerical example has been given to illustrate the effectiveness of the proposed method. Dynamic output feedback control problem of T-S fuzzy control systems will
TABLE I: \( \mathcal{L}, \mathcal{D} \) and \( \mathcal{C} = \mathcal{D}^3 \mathcal{L} \)

<table>
<thead>
<tr>
<th>Methods</th>
<th>( \mathcal{L} )</th>
<th>( \mathcal{D} )</th>
<th>( \mathcal{C} )</th>
<th>CPU time</th>
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<td>3</td>
<td>504</td>
<td>0.0409</td>
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<tr>
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<td>3</td>
<td>540</td>
<td>0.0313</td>
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<tr>
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<td>3</td>
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<td>0.0025</td>
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<td>CPU time</td>
<td>0.0409</td>
<td>0.0313</td>
<td>0.0025</td>
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</tr>
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</table>

\[
\mathcal{L} = \mathcal{D}^3 \mathcal{L}
\]

**Fig. 3**: Stability area by Corollary 1 with \( h_1 = h_2 = 3 \)

**Fig. 4**: Stability area by Corollary 1 with \( h_1 = h_2 = 4 \)

**Fig. 5**: Stability area by Lemma 4

**Fig. 6**: Stability area by Theorem 5 in [9]

be exploited by using set theory in the future. We also plan to apply set theory to fuzzy fault tolerant control problems.

**APPENDIX**

**Definition 1**: [12], [26],

- A \( n \)-ary relation \( \mathcal{R} \) is a set of ordered \( n \)-tuples, denoted by \( (x_1, \ldots, x_n) \) is the ordered collection of elements that has \( x_1 \) as its first element, \( x_2 \) as its second element, \ldots, and \( x_n \) as its \( n \)th element. Two \( n \)-tuples are equal, if each corresponding pair of their elements is equal. \( \mathcal{R} \) is a \( n \)-ary relation on \( \mathcal{X} \) if \( \mathcal{R} \subseteq \mathcal{X}^n \). It is customary to write \( \mathcal{R}(x_1, \ldots, x_n) \) instead of \( (x_1, \ldots, x_n) \in \mathcal{R} \) and in case that \( \mathcal{R} \) is binary, then we also use \( x\mathcal{R}y \) instead of \( (x, y) \in \mathcal{R} \).

- A binary relation \( \mathcal{R} \) on \( \mathcal{X} \) is **reflexive** if \( x\mathcal{R}x \) for every element \( x \) of \( \mathcal{X} \), i.e.,

\[
\mathcal{R} \text{ is reflexive } \iff \forall x(x \in \mathcal{X} \rightarrow x\mathcal{R}x)
\]

- A binary relation on \( \mathcal{X} \) is **symmetric**, if \( x\mathcal{R}y \), then \( y\mathcal{R}x \), i.e.,

\[
\mathcal{R} \text{ is symmetric } \iff \forall x\forall y(x \in \mathcal{X} \land y \in \mathcal{X} \land x\mathcal{R}y \rightarrow y\mathcal{R}x)
\]
Let $R$ be an equivalence relation on $X$, then the set $X/R = \{[x]_R : x \in X\}$ is a partition of $X$. Conversely, for each partition of $X$, there exists an equivalence relation $R_o$ on $X$, such that $X/R_o = \{[x]_{R_o} : x \in X\}$ is the partition.

**Lemma 7:** [12] (pp. 12) If $R$ is an equivalence relation on $X$, then the set $X/R = \{[x]_R : x \in X\}$ is a partition of $X.

**Lemma 8:** (i): Let $a, b \in \mathbb{Z}_+$, then
\[
\frac{(a+1)(b+1)}{2} \leq ab + 1 \tag{61}
\]

(ii) Let $r_i \in \mathbb{Z}_+$, $i = 1, \cdots, p$, then
\[
\prod_{i=1}^{p} \left( \frac{1 + r_i}{2} \right) \leq \left( \frac{1 + \prod_{i=1}^{p} r_i}{2} \right) \tag{62}
\]

**Proof:** (i): Consider two cases: (1) one of $a,b$ is 1, (2) $a \geq 2$, $b \geq 2$.

For the case one of $a,b$ is 1, then it is easily obtained that (61) holds. For the case $a \geq 2$, $b \geq 2$, we have that $ab \geq \max\{2a, 2b\} \geq a + b$, which implies that
\[
ab + a + b + 1 \leq 2ab + 2
\]

i.e.,
\[
\frac{(a+1)(b+1)}{2} \leq ab + 1
\]

Thus, the proof is complete.

(ii): We use mathematical induction, it is easily obtained that (62) holds for $p = 2$ from (i). Assume (62) holds for $p = k$, then we have
\[
\prod_{i=1}^{k} \left( \frac{1 + r_i}{2} \right) \leq \left( \frac{1 + \prod_{i=1}^{k} r_i}{2} \right)
\]

which implies that
\[
\prod_{i=1}^{k} \frac{1 + r_i}{2} \leq \frac{1 + \prod_{i=1}^{k} r_i}{2}
\]
Combining it and (63), then (62) holds for $p = k + 1$. Thus, by virtue of mathematical induction, the proof is complete.

**Lemma 9:** Let $S \subseteq \mathbb{Z}_+$ with $|S| < \infty$, $\xi R$ is an equivalence class of $S^{h+1}$ with \[ R = \{ (i_1 i_2 \cdots i_{k+1}) | s.t. (j_1 j_2 \cdots j_{h+1}) = st(i_1 i_2 \cdots i_{h+1}) \} , \] where $st(\cdot)$ is the same as in (4). For the set $\xi R$, we define a binary relation as \[ \tilde{R} = \{ (\eta_1 \eta_2 \cdots \eta_{h+1}, \gamma_1 \gamma_2 \cdots \gamma_{h+1}) \in (S^{h+1})^2 | st(\eta_1 \eta_2 \cdots \eta_{h+1}, \eta_{h+1} = \gamma_{h+1} \} \]

Then the relation $\tilde{R}$ is an equivalence relation and $\xi R$ is a partition of the set $\xi R$.

**Proof:** The proof is easily obtained and omitted.

**The proof of Lemma 1**

Proof: For any element $(i_1, i_2, \cdots, i_p) \in S_1^{h_1} \times S_2^{h_2} \times \cdots \times S_p^{h_p}$, we have $i_l \in S_l^{h_l}$, $1 \leq l \leq p$. Because $S_l^{h_l}$ is a partition of set $\xi R_h$, then there exists an equivalence class $[i_l]_{R_h}$, such that $i_l \in [i_l]_{R_h}$. Therefore, $(i_1, i_2, \cdots, i_p) \in [i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}}$. So we have

\[
S_1^{h_1} \times S_2^{h_2} \times \cdots \times S_p^{h_p} \subseteq \bigcup_{[i_1]_{R_{h_1}} \subseteq S_1^{h_1}} [i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}}
\]

Since $[i_l]_{R_{h_l}} \subseteq S_l^{h_l}$, $1 \leq l \leq p$ ,

\[
S_1^{h_1} \times S_2^{h_2} \times \cdots \times S_p^{h_p} \subseteq \bigcup_{[i_1]_{R_{h_1}} \subseteq S_1^{h_1}} [i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}}
\]

Combining it and (64), it follows that

\[
S_1^{h_1} \times S_2^{h_2} \times \cdots \times S_p^{h_p} = \bigcup_{[i_1]_{R_{h_1}} \subseteq S_1^{h_1}} [i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}}
\]


On the other hand, note that $[i_l]_{R_{h_l}}$ and $[j_l]_{R_{h_l}}$ are both equivalence classes on $S_l^{h_l}$, then $[i_l]_{R_{h_l}} = [j_l]_{R_{h_l}}$ or $[i_l]_{R_{h_l}} \cap [j_l]_{R_{h_l}} = \emptyset$.

There are the following two possible cases for sets $[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}}$ and $[j_1]_{R_{h_1}} \times [j_2]_{R_{h_2}} \times \cdots \times [j_p]_{R_{h_p}}$.

- Case 1: If there exits some $l$ satisfying $[i_l]_{R_{h_l}} \cap [j_l]_{R_{h_l}} = \emptyset$, then

\[
[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}} \cap [j_1]_{R_{h_1}} \times [j_2]_{R_{h_2}} \times \cdots \times [j_p]_{R_{h_p}} = \emptyset
\]

- Case 2: If there doesn’t exit $l$ satisfying $[i_l]_{R_{h_l}} \cap [j_l]_{R_{h_l}} = \emptyset$, which implies that $[i_l]_{R_{h_l}} = [j_l]_{R_{h_l}}$ for all $l, 1 \leq l \leq p$. It means that

\[
[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}} = [j_1]_{R_{h_1}} \times [j_2]_{R_{h_2}} \times \cdots \times [j_p]_{R_{h_p}}
\]

Therefore, it follows from the Cases 1 and 2 that $[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}} \cap [j_1]_{R_{h_1}} \times [j_2]_{R_{h_2}} \times \cdots \times [j_p]_{R_{h_p}} = \emptyset$ or $[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}} = [j_1]_{R_{h_1}} \times [j_2]_{R_{h_2}} \times \cdots \times [j_p]_{R_{h_p}}$. From the fact and (65), we can obtain that $[i_1]_{R_{h_1}} \times [i_2]_{R_{h_2}} \times \cdots \times [i_p]_{R_{h_p}} : [i_l]_{R_{h_l}} \subseteq S_l^{h_l}, 1 \leq l \leq p$ is a partition of the set $S_1^{h_1} \times S_2^{h_2} \times \cdots \times S_p^{h_p}$. Thus, the proof is complete.

**The proof of Lemma 2**

Proof: From Lemma 1, it follows that

\[
\sum_{\sigma \in \tilde{\bar{S}}} \mu_{\sigma} M_{\sigma} = \sum_{\bar{s} \in \tilde{\bar{S}}} \sum_{\sigma \in S} \mu_{\sigma} M_{\sigma} \quad (66)
\]

where $\bar{s} = \bigcap_{i=1}^P s_i$, with $s_i \in S_i^{h_i}/R_{h_i}$.

From the property of equivalence class in set theory, we can choose an arbitrary element in the equivalence class as its representative element. Let $s_j \in s_j$, then we choose $s_j$ as the representative element of the equivalence class $s_j$, and denote $s_j$ as $[s_j]_{R_{h}}$. Further, it follows from the definition of the equivalence relation $R_{h}$ that

\[
\prod_{i=1}^p \mu_{\sigma_{s_j}}(s_j) = \prod_{i=1}^p \mu_{\sigma_{s_j}}(s_j) \quad \text{for all } \tau \in \bar{s} = [s_j]_{R_{h}}
\]

Then for $\sigma \in \bar{s} = \bigcap_{i=1}^P s_i$

\[
\sum_{\sigma \in \bar{s}} \mu_{\sigma} M_{\sigma} = \sum_{\sigma \in \bar{s}} \mu_{\sigma} M_{\sigma} = \sum_{\sigma \in \bar{s}} \prod_{i=1}^p \mu_{\sigma_{s_j}}(s_j) M_{\sigma} = \prod_{j=1}^P \prod_{i=1}^p \mu_{\sigma_{s_j}}(s_j) M_{\sigma}
\]

\[
= \prod_{j=1}^P \prod_{i=1}^p \mu_{\sigma_{s_j}}(s_j) M_{\sigma} = \mu_{\sigma} \sum_{\sigma \in \bar{s}} M_{\sigma} \quad (67)
\]
where
\[
\mu_s = \prod_{j=1}^{p} \prod_{i_j=1}^{h_i} \mu_{s_{ij}}, \quad \text{with} \quad \bar{s} = \prod_{i=1}^{p} s_i = \prod_{i=1}^{p} \|s_i\|_{\mathbb{R}^{1 \times 1}},
\]
(68)
From (66) and (67), yields that
\[
\sum_{\sigma \in \prod_{i=1}^{p} \mathbb{Z}^{h_i}} \mu_{\bar{s}} M_\sigma = \sum_{\bar{s} \in \prod_{i=1}^{p} \mathbb{Z}^{h_i} / \{s_i \} \sigma \in \Sigma} \mu_{\bar{s}} M_\sigma = \sum_{\bar{s} \in \prod_{i=1}^{p} \mathbb{Z}^{h_i} / \{s_i \}} \mu_{\bar{s}} \sum_{\sigma \in \Sigma} M_\sigma
\]
Combining it and (7), (8), it follows that (9) holds. Thus, the proof is complete.

**Lemma 10:** If the 1-1 mapping \( g(\cdot) \) in (32) is respectively chosen as \( q_a(\cdot) \) and \( q_b(\cdot) \), then (38) in Theorem 1 respectively becomes
\[
[H_{ij}^a] < 0, \quad \text{with} \quad H_{q_a(\sigma_{i1})q_a(\sigma_{i2})} = Y_\sigma
\]
(69) and
\[
[H_{ij}^b] < 0, \quad \text{with} \quad H_{q_b(\sigma_{i1})q_b(\sigma_{i2})} = Y_\sigma
\]
(70) then (69) is equivalent to (70).

**Proof:** Define a mapping \( \varpi \) from the set \( \{1, 2, \cdots, r\} \) to itself with \( \varpi(\cdot) = q_b(q_a^{-1}(\cdot)) \). Since \( q_a(\cdot) \) and \( q_b(\cdot) \) are both 1-1 mappings, the inverse mapping of \( q_a(\cdot) \) exists and \( \varpi \) is also a 1-1 mapping. From (69) and (70), we have that
\[
H_{ij}^a = H_{(\varpi^{(i)})^{(j)}}
\]
Then (69) can be rewritten as
\[
\begin{bmatrix}
H_{\varpi(1)^{(1)}}^{b} & H_{\varpi(1)^{(2)}}^{b} & \cdots & H_{\varpi(1)^{(r)}}^{b} \\
H_{\varpi(2)^{(1)}}^{b} & H_{\varpi(2)^{(2)}}^{b} & \cdots & H_{\varpi(2)^{(r)}}^{b} \\
\vdots & \vdots & \ddots & \vdots \\
H_{\varpi(r)^{(1)}}^{b} & H_{\varpi(r)^{(2)}}^{b} & \cdots & H_{\varpi(r)^{(r)}}^{b}
\end{bmatrix}
< 0
\]
(71)
Since \( \varpi \) is also a 1-1 mapping, there exists a permutation matrix \( T \), such that
\[
[\varpi(1) \varpi(2) \cdots \varpi(r)] T = \begin{bmatrix} 1 & 2 & \cdots & r \end{bmatrix}
\]
Let \( T = T \otimes I_{n_x \times n_x} \), then
\[
\begin{bmatrix}
H_{11}^{b} & H_{12}^{b} & \cdots & H_{1r}^{b} \\
H_{21}^{b} & H_{22}^{b} & \cdots & H_{2r}^{b} \\
\vdots & \vdots & \ddots & \vdots \\
H_{r1}^{b} & H_{r2}^{b} & \cdots & H_{rr}^{b}
\end{bmatrix}
T^T
\]
\[
\begin{bmatrix}
H_{\varpi(1)^{(1)}}^{b} & H_{\varpi(1)^{(2)}}^{b} & \cdots & H_{\varpi(1)^{(r)}}^{b} \\
H_{\varpi(2)^{(1)}}^{b} & H_{\varpi(2)^{(2)}}^{b} & \cdots & H_{\varpi(2)^{(r)}}^{b} \\
\vdots & \vdots & \ddots & \vdots \\
H_{\varpi(r)^{(1)}}^{b} & H_{\varpi(r)^{(2)}}^{b} & \cdots & H_{\varpi(r)^{(r)}}^{b}
\end{bmatrix}
< 0
\]
which implies that
\[
\begin{bmatrix}
H_{11}^{b} & H_{12}^{b} & \cdots & H_{1r}^{b} \\
H_{21}^{b} & H_{22}^{b} & \cdots & H_{2r}^{b} \\
\vdots & \vdots & \ddots & \vdots \\
H_{r1}^{b} & H_{r2}^{b} & \cdots & H_{rr}^{b}
\end{bmatrix}
= [H_{ij}^b] < 0
\]
Then we have that (69) is equivalent to (70). ■

**REFERENCES**


Jiuxiang Dong received the B.S. degree in mathematics and applied mathematics, the M.S. degree in applied mathematics from Liaoning Normal University, China, in 2001 and 2004, respectively. He received the Ph.D. degree in navigation guidance and control from Northeastern University, China, in 2009. He is currently a professor at the College of Information Science and Engineering, Northeastern University. His research interests include fuzzy control, robust control and reliable control. Dr. Dong is an Associate Editor for the International Journal of Control, Automation, and Systems (IJCAS).

Guang-Hong Yang (SM’04) received the B.S. and M.S. degrees from Northeast University of Technology, Liaoning, China, in 1983 and 1986, respectively, and the Ph.D. degree in Control Engineering from Northeastern University, China (formerly, Northeast University of Technology), in 1994. He was a Lecturer/Associate Professor with Northeastern University from 1986 to 1995. He joined the Nanyang Technological University in 1996 as a Postdoctoral Fellow. From 2001 to 2005, he was a Research Scientist/Senior Research Scientist with the National University of Singapore. He is currently a professor at the College of Information Science and Engineering, Northeastern University. His current research interests include fault-tolerant control, fault detection and isolation, non-fragile control systems design, and robust control. Dr. Yang is an Associate Editor for the International Journal of Control, Automation, and Systems (IJCAS), the International Journal of Systems Science (IJSS), the IET Control Theory & Applications, and the IEEE Transactions on Fuzzy Systems.

Huaguang Zhang (SM’04) received the B.S. degree and the M.S. degree in control engineering from Northeast Dianli University of China, Jilin City, China, in 1982 and 1985, respectively. He received the Ph.D. degree in thermal power engineering and automation from Southeast University, Nanjing, China, in 1991.

He joined the Department of Automatic Control, Northeastern University, Shenyang, China, in 1992, as a Postdoctoral Fellow for two years. Since 1994, he has been a Professor and Head of the Institute of Electric Automation, School of Information Science and Engineering, Northeastern University, Shenyang, China. His main research interests are fuzzy control, stochastic system control, neural networks based control, nonlinear control, and their applications. He has authored and coauthored over 280 journal and conference papers, six monographs and co-invented 90 patents.

Dr. Zhang is Chair of the Adaptive Dynamic Programming & Reinforcement Learning Technical Committee on IEEE Computational Intelligence Society. He is an Associate Editor of Automatica, IEEE Transactions on Neural Networks, IEEE Transactions on Cybernetics, and Neurocomputing, respectively. He was an Associate Editor of IEEE Transactions on Fuzzy Systems (2008-2013). He was awarded the Outstanding Youth Science Foundation Award from the National Natural Science Foundation Committee of China in 2003. He was named the Cheung Kong Scholar by the Education Ministry of China in 2005. He is a recipient of the IEEE Transactions on Neural Networks 2012 Outstanding Paper Award.