Compressed Spectrum Estimation for Cognitive Radios

Jens P. Elsner, Martin Braun, Holger Jäkel and Friedrich K. Jondral
Universität Karlsruhe (TH), Germany, {elsner, braun, jaekel, fj}@int.uni-karlsruhe.de

***Abstract*— Cognitive Radios observe the spectral environment through spectrum estimation. In particular, reconfigurable Software Defined Radio architectures generate large amounts of sample data, which can be computationally expensive to analyse. Compression by time-shifted random preintegration prior to spectrum estimation reduces the amount of data to be processed. It is simple to implement and scalable. This paper shows that linear compression with time-shifted random preintegration is equivalent to compressed sensing with Toeplitz-structured random matrices and preserves autocorrelation properties, which allows for efficient joint compressed spectrum estimation and compressed signal detection in Cognitive Radio terminals.**

I. Introduction

Digital spectrum estimation is traditionally based on equidistant sampling of a time-continuous signal at Nyquist frequency. In certain applications sampling frequency or computational resources are limited, but one would still like to detect signals or to estimate the frequency content of a signal in very wide frequency band. Such an application is real-time spectrum estimation in Cognitive Radios.

Compressed Sensing (CS) is a recently popularized methodology to aquire signals at sub-Nyquist rate [1]. For Cognitive Radio applications, CS holds the promise to allow new tradeoffs between sampling rate and signal to noise ratio for estimation and detection [2]. Compression is implemented by a linear projection. If this projection approximately preserves the euclidean distances between any two vectors of the original signal space, the projection matrix is said to have the restricted isometry property (RIP). Under certain preconditions, the original signal can then be reconstructed with high probability through linear programming [1]. For practical applications, random Toeplitz-structured matrices, which can be implemented efficiently and have the RIP for time sparse signals and other bases of interest, have been proposed by Haupt et al. [3], [4]. In this paper the use of Toeplitz-structured CS matrices or, equivalently, time-shifted (random) preintegration is proposed for spectrum estimation as the projection preserves autocorrelation properties. This allows to apply fast linear spectrum estimation techniques to the compressed vector.

The paper is structured as follows. After showing the connection between compression matrices and random time-shifted preintegration in Section II, deterministic linear compression with matrices constructed from sequences with perfect periodic autocorrelation is introduced in Section III. Simulation results are presented in Section IV. Section V concludes.

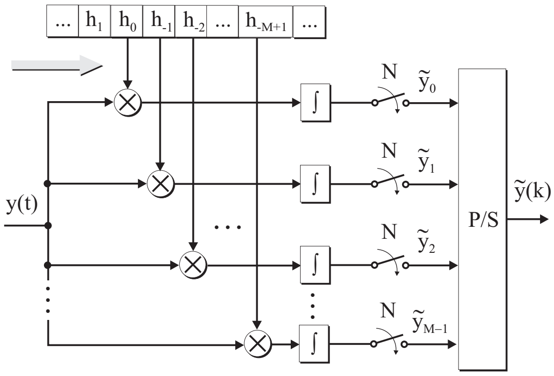


Fig. 1. Proposed time-continuous implementation of time-shifted random preintegration. Integration is reset after N samples.

II. Toeplitz-structured Compressed Sensing Matrices and Time-Shifted Random Preintegration

A. Preservation of Autocorrelation

A Toeplitz-structured compression matrix $A \in \mathbb{C}^{M \times N}$ with entries h_i can be written as

$$\mathbf{A} = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{N-1} \\ h_{-1} & h_0 & h_1 & \cdots & h_{N-2} \\ h_{-2} & h_{-1} & h_0 & \cdots & h_{N-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h_{-M+1} & \cdots & \cdots & h_{N-M-1} & h_{N-M} \end{pmatrix}, \quad (1)$$

such that $A_{i,j} = h_{j-i}$. The linear compression of a signal vector $\mathbf{y} \in \mathbb{R}^N$ into the vector $\tilde{\mathbf{y}} \in \mathbb{R}^M$ can then be written as

$$\tilde{\mathbf{y}} = \mathbf{A}\mathbf{y}\,. \quad (2)$$

This matrix multiplication can be implemented in a time-continuous block-processing fashion as a series of M time-shifted preintegrators shown in Figure 1, yielding the following system model:

$$\tilde{y}(lM + k) = \sum_{i=0}^{N-1} h_{i-(k \bmod M)}\, y(lN + i)\,. \quad (3)$$

where $l = \lfloor \frac{k}{M} \rfloor$ denotes the current block.

Assume zero mean and stationarity for $\mathbf{y}$ within a block length (short time stationarity assumption). Linear projections,

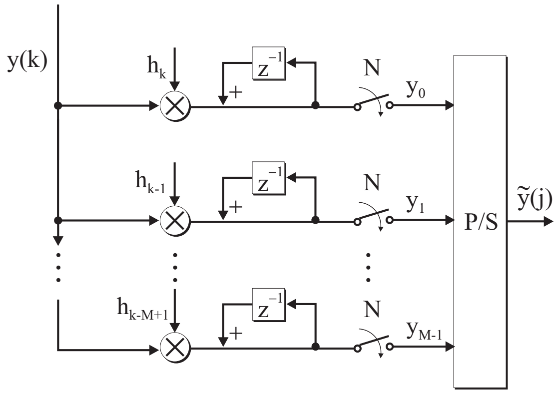


Fig. 2. Proposed time-discrete implementation of time-shifted random preintegration for data reduction. Integration is reset after N samples.

such as the mapping by $\mathbf{A}$, preserve stationarity. The autocorrelation function $\phi_{\tilde{y}\tilde{y}}(n)$ of $\tilde{y}$ is hence defined as

$$\phi_{\tilde{y}\tilde{y}}(n) = E\left[\tilde{y}(k)\tilde{y}(k+n)\right] . \tag{4}$$

Inserting (3) with $l = 0$, this yields, assuming zero mean and uncorrelated h_i,

$$\begin{aligned}\phi_{\tilde{y}\tilde{y}}(n) &= \sum_i \sum_j E\left[h_{i-k}y(i)h_{j-k-n}y(j)\right] \\ &= \sum_i \sum_j E\left[h_{i-k}h_{j-k-n}\right] E\left[y(i)y(j)\right] \\ &= \sum_i E\left[h_{i-k}h_{i-k}\right] E\left[y(i)y(i+n)\right] \\ &= \sum_i \alpha_{h_i}\phi_{yy}(n) .\end{aligned} \tag{5}$$

Hence, with, e.g., $h_i \in \{-1/\sqrt{N}, 1/\sqrt{N}\}$ autocorrelation is preserved.

B. Efficient Hardware Implementations

Precompression can be implemented efficiently in hardware as time shifted preintegration.

Two implementations are proposed. Figure 1 shows a proposed implementation for a time-continuous input with compressed sensing. Since the signal processing algorithms operate on the compressed signal $\tilde{y}$, a new data acquisition sub-system must be introduced. In order not to unnecessarily increase the complexity, the structure of the compression matrix can be used to optimize the acquisition.

The input signal is passed onto M multipliers. These multiply the input signal each with one of the elements $h_k, \ldots, h_{k+M}$ of the compression matrix, which come from a shift register. The limited alphabet from which the elements of the compression sequence are chosen makes an analog multiplication feasible.

The multiplier outputs are passed on to integrators. Once the shift register has been advanced M times, the integrator output is digitized and the integrator is reset. This results in M digital values, which form one compressed vector $\tilde{\mathbf{y}}$. The shift register clock is determined by the Nyquist frequency of the input signal, i.e., the shift frequency must be chosen such that it could be used a valid sampling frequency for regular A/D conversion of $y(t)$. Such a time-continuous implementation has the advantage that instead of using one fast A/D converter, M converters with a rate of $1/N$th of the original rate can be used.

In some cases, the original signal might also be of interest, and $y(t)$ is digitized before the compression. Here, the implementation shown in Figure 2 has a structure which can be implemented onto a field-programmable gate array (FPGA). The algorithmic advantage lies in precompression without multiplications prior to more sophisticated signal analysis.

C. Random vs. Deterministic Compression

If the power spectrum density (PSD) is estimated based on the compressed vector the spectral estimate is biased for small N and random Toeplitz $\mathbf{A}$ due to residual autocorrelation terms of h. To achieve betters results with regard to spectrum estimation, deterministic compression matrices with perfect periodic autocorrelation properties can be constructed. In compressed sensing terms, a circulant compression matrix $\mathbf{A}$ formed from sequences with perfect periodic autocorrelation, such as Zadoff-Chu or Ipatov sequences [5], is maximally incoherent with the Fourier basis and preserves the PSD. In contrast to random Toeplitz matrices, which have been shown to be a good choice for CS applications (cf. [3], [4] and references therein), the isometry properties of these matrices are subject to research. They offer, however, best performance for compressed spectrum estimation.

III. Deterministic Compression Matrices from Sequences with Perfect Periodic Autocorrelation

A. Sequences with perfect periodic autocorrelation

A sequence $\{a_n\}$ with perfect periodic autocorrelation and period N, normalized to unit energy, satisfies [5]

$$R_a(\tau) = \sum_{n=0}^{N-1} a_n a_{n+\tau}^* = \begin{cases} 1 & : \tau = kN \\ 0 & : \tau \neq kN \end{cases} \tag{6}$$

where the index is to be interpreted modulo N. Such a perfect sequence has a constant discrete periodic spectrum, which follows directly from the Wiener-Khinchin relationship between the periodic autocorrelation and its Fourier transform [5] and (6):

$$\mathrm{DFT}(R_a(\tau)) = |\mathrm{DFT}(a_n)|^2 = 1 . \tag{7}$$

The construction of perfect sequences is non-trivial. Two constructions of sequences suited for implementation of precompression, ternary and polyphase sequences[1], are cited in the following. For further constructions, refer to, e.g., surveys by Fan and Darnell [5] or Lüke et al. [6] and references therein.

[1] Unfortunately perfect sequences with small phase alphabet are rare. The longest known perfect binary sequence is $\{a_n\} = \{1, 1, 1, -1\}$, which means $N = 4$ [6]. The longest known quaternary perfect sequence yields $N = 16$ [6].

Perfect Ternary Sequences: Perfect ternary sequences are perfect sequences with a ternary alphabet. Of special interest are sequences with $a_n \in \{-1, 0, 1\}$. Ipatov constructs such perfect ternary sequences from a maximum length sequence (m-sequence) $\{b_n\}$ with $b_n \in \mathbb{F}_q$ [5]. The period of such an m-sequence is $q^m - 1$. Any nonzero element of $\mathbb{F}_q$ can be expressed as a power of a primitive element α. Let m be odd. Furthermore, let $q = p^s$, where p is an odd prime and s is an integer. Then

$$a_n = \begin{cases} 0 & : b_n = 0 \\ \frac{1}{\sqrt{E}}(-1)^{u+n} & : b_n = \alpha^u \end{cases} \tag{8}$$

is a perfect ternary sequence with period $N = \frac{q^m-1}{q-1}$, normalized to its resulting energy E.

Perfect Polyphase Sequences: Perfect polyphase sequences are perfect sequences with $a_n = e^{j\beta_n}$. The following construction is due to Zadoff and Chu [5]. Let M be an integer coprime to N. Then, with $0 \leq n < N$,

$$a_n = \begin{cases} \frac{1}{\sqrt{N}} e^{\frac{j\pi M}{N} n^2} & : N \text{ even} \\ \frac{1}{\sqrt{N}} e^{\frac{j\pi M}{N}(n+1)n} & : N \text{ odd} \end{cases} \tag{9}$$

is a normalized perfect sequence of length N.

B. Construction of Deterministic Compression Matrices

Let a_n be a normalized perfect sequence. Construct $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{N-(M-1)} & \cdots & \cdots & a_{N-M-1} & a_{N-M} \end{pmatrix}. \tag{10}$$

A projection with such a compression matrix $\mathbf{A}$ preserves power spectral density. This can be understood by computing eigenvalues and eigenvectors of the projection: any circulant matrix is diagonalized by the Fourier matrix $\mathbf{F}$, and, more specifically [7]

$$\mathbf{A} = \mathbf{F}\,\mathrm{diag}(\mathbf{Fa})\,\mathbf{F}^{-1} \tag{11}$$

holds. The eigenvectors of the projection are hence given by the columns of the Fourier matrix and the eigenvalues are given by the DFT of the first row of $\mathbf{A}$. They have, due to (7), constant amplitude. Therefore, the projection introduces only a phase shift and the PSD is preserved.

Another way to look at this property is incoherence of bases. For $N = M$, $\mathbf{A}$ is a basis in $\mathbb{C}^N$. The sensing basis $\mathbf{A}$ is then maximally incoherent with the discrete Fourier basis $\mathbf{F}$ [1]:

$$\mu(\mathbf{A}, \mathbf{F}) = \sqrt{N} \max_{1 \leq k,j \leq N} |\langle \mathbf{a_k}, \mathbf{f_j} \rangle| = 1 \tag{12}$$

where $\mathbf{a_k}$ and $\mathbf{f_j}$ are the column vectors of $\mathbf{A}$ and $\mathbf{F}$, respectively. With (7) in mind, this is easily verified noting Parseval's identity that the inner product is invariant under a change of basis. Incoherence is a similarity measure of bases. Qualitatively, if the sensing basis $\mathbf{A}$ is maximally incoherent with the representation bases, here the Fourier basis, a coefficient a_n contains an equal amount of information about all frequencies. Hence, if $M < N$, frequency information is discarded democratically.

With respect to implementation, perfect ternary sequences are a good choice due to the small number of phases. Compression using ternary perfect sequences can be done without multiplications, which is a major advantage for implementation in analog hardware or on FPGAs. Using digital signal processing, other sequences can also easily be implemented - perfect polyphase sequences are an obvious choice as they are available for arbitrary N.

IV. Simulation Results

Monte-Carlo Simulation results for compressed spectrum estimation with random and deterministic compression are shown in Figures 3 and 4.

Figure 3 shows a compressed estimate of a smooth spectrum generated from an autoregressive process with random Toeplitz compression. The compression matrix is fixed with $N = 2000$ and $h_i \in \{-1/\sqrt{N}, 1/\sqrt{N}\}$. In Figure 3(b) the compression factor $C = M/N$ is varied from 0.05 to 0.5, which corresponds to 100 and 1000 samples, respectively. As can be seen from the figures, even very low compression factors of 0.05 suffice to approximate the smooth spectrum fairly well.

Figure 4 shows compressed estimates based on a sum of sinusoids. The estimator bias of a random Toeplitz matrix is clearly visible when comparing 4(a) it to Figure 4(b), which shows a spectrum estimate based on the same spectrum with deterministic Zadoff-Chu precompression. Here too, very low compression factors suffice to approximate the spectrum.

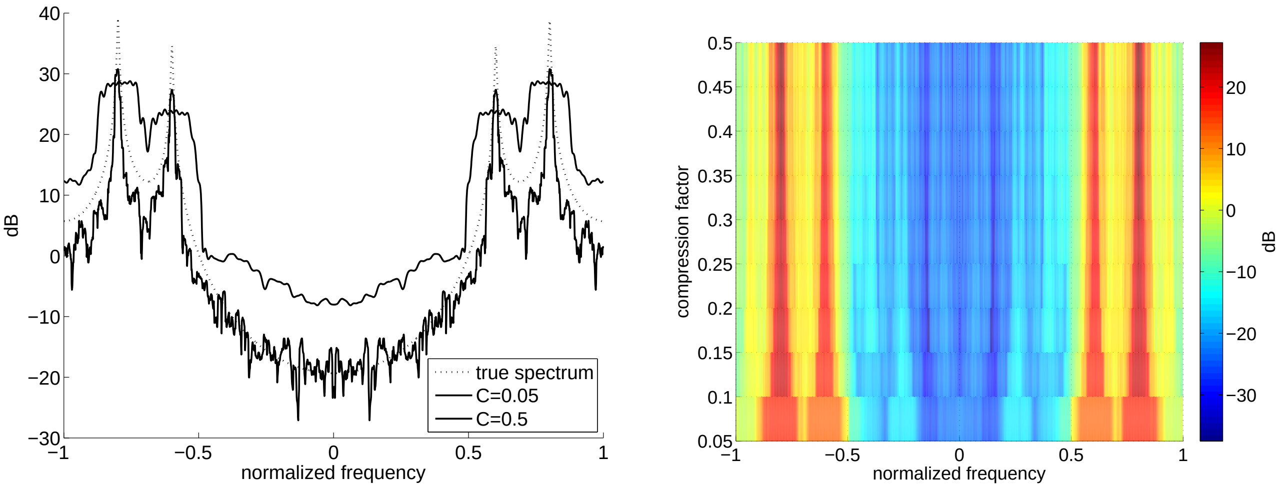
For all simulations, spectrum estimation after compression is based on the Thomson multitaper method with a constant FFT size of 2000. The Thomson multitaper method is close to optimal in the sense that it can be shown to be close to a maximum-likelihood estimate for spectra stemming from correlated Gaussian processes [8].

V. Conclusion

Time-shifted random preintegration can be implemented efficiently both time-continuously in analog hardware with M parallel A/D converters for compressive sensing (Figure 1) and time-discrete and without multiplications on FPGAs. It allows for linear spectrum estimation at a lower data rate, while leaving the possibility to apply the CS methodology for signal detection and reconstruction. The same computational structures needed for time-shifted random preintegration can also be used directly to implement direct sequence spread spectrum demodulation or smashed filtering [2]. This makes preintegration an effective tool to reduce processing bandwidth for signal detection and spectrum estimation in Software Defined Radios.

References

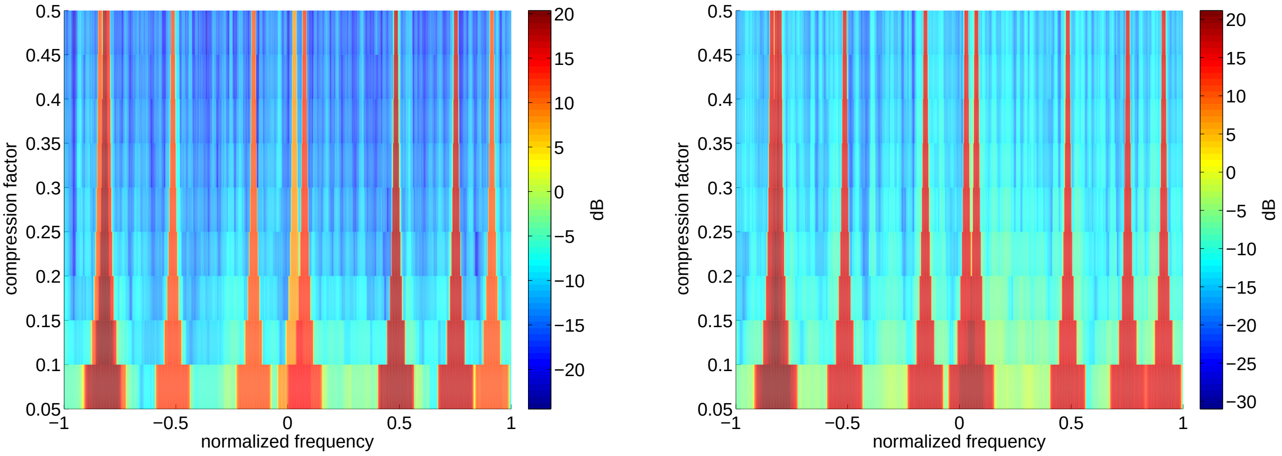
[1] E. J. Candès and M. B. Wakin, "An Introduction To Compressive Sampling," *Signal Processing Magazine, IEEE*, vol. 25, no. 2, pp. 21–30, March 2008.



(a) Comparison of true spectrum with compressed estimates for a random Toeplitz compression matrix. The estimated spectrum stems from an autoregressive process with system function $H(q^{-1}) = 1 - 2.2137q^{-1} + 2.9403q^{-2} - 2.1697q^{-3} + 0.9606q^{-4}$, which exhibits two prominent peaks.

(b) Compression factor versus spectrum estimate. The estimated spectrum is identical to the spectrum in Figure 3(a).

Fig. 3. Compressed estimation of smooth IIR spectra.



(a) Compression factor versus spectrum estimate for a sum of complex harmonics of equal amplitude and additive white Gaussian noise resulting in a signal-to-noise ratio of 20 dB. Compression is based on a random Toeplitz matrix.

(b) Compression based on a deterministic Zadoff-Chu matrix, $N = 2000$. The estimated spectrum is identical to the spectrum in Figure 4(a).

Fig. 4. Compressed estimation of narrow band spectra.

[2] M. Braun, J. Elsner, and F. Jondral, "Signal Detection in Cognitive Radios with Smashed Filtering," in *Proceedings of the IEEE Vehicular Technology Conference*, April 2009, to be published.

[3] G. R. Jarvis Haupt, Waheed U. Bajwa and R. Nowak, "Toeplitz compressed sensing matrices with applications to sparse channel estimation," *submitted to IEEE Transactions on Information Theory*, 2008.

[4] W. U. Bajwa, J. D. Haupt, G. M. Raz, S. J. Wright, and R. D. Nowak, "Toeplitz-structured compressed sensing matrices," *IEEE/SP 14th Workshop on Statistical Signal Processing*, Aug. 2007.

[5] P. Z. Fan and M. Darnell, "The synthesis of perfect sequences," *Lecture notes in Computer Science, Cryptography and Coding, Springer*, 1995.

[6] H. Luke, H. Schotten, and H. Hadinejad-Mahram, "Binary and quadriphase sequences with optimal autocorrelation properties: a survey," *IEEE Transactions on Information Theory*, vol. 49, no. 12, Dec. 2003.

[7] R. Gray, "Toeplitz and circulant matrices: A review," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 3, 2006.

[8] P. Stoica and T. Sundin, "On Nonparametric Spectral Estimation," *Circuits, Systems and Signal processing, Springer*, vol. 18, no. 2, 1999.