AN HISTORICAL ACCOUNT OF SET-THEORETIC ANTINOMIES CAUSED BY THE AXIOM OF ABSTRACTION

JUSTIN T MILLER

1. Introduction

The beginnings of set theory date back to the late nineteenth century, to the work of Cantor on aggregates and trigonometric series, which culminated into his seminal work *Contributions to the Founding of the Theory of Transfinite Numbers* [2]. Although this work was published in 1895 and 1897, in two parts, the theory of sets had been established as an independent branch of mathematics as early as 1890 ([9], p. 1), from Cantor's earlier work. Despite initial reservations against the representations of infinite collections as single entities, Cantor's set theory gained acceptance by the mathematical community once its usefulness was manifested in various areas, including analysis and geometry.

As mathematics was making tremendous strides using the theory of sets, several paradoxes emerged from Cantor's loose definition of a set—around the turn of the century—that threatened to undermine the foundations of mathematics being built upon Cantor's theory. In his *Contributions*, Cantor defined a set to be "any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought" ([2], p. 85). To the naturally finitistic mind this definition appears innocuous and mathematically appropriate, but an application of this definition to infinite sets can be especially pernicious.

Cantor's definition gave rise to a principle asserting that certain formulas define sets, often called the Axiom of Abstraction. This axiom states that for any formula, $\varphi(x)$,there is a set containing precisely the elements a for which $\varphi(a)$ is true. In other words, for every formula $\varphi(x)$ in one free-variable $\exists A \forall x(x \in A \leftrightarrow \varphi(x))$. A form of this axiom was used by Frege in the first volume of *Grundgesetze der Arithmetik*, published in 1893 ([10], p. 10), and later, a paradox caused by the axiom undermined most of his theory immediately before the second volume was published.

This paper will be concerned with a historical description of the prominent paradoxes that resulted from the use of Cantor's definition and the Axiom of Abstraction near the beginning of the twentieth century. The distinction between 'paradox' and 'contradiction' will be used as in [6]. Thus, a contradiction will occur when a statement and its negation can both be proved true, but a paradox will be defined as "an argument which ends in a contradiction although all of its premises and modes of reasoning are *prima facie* acceptable" ([6], p. 321), with the further stipulation that "the one who discovers it give up a premise or mode of reasoning that he has previously accepted as correct" ([6], p. 321). In this case, the Axiom of Abstraction will be the abandoned premise, given up for either Zermelo's Axiom of Separation or Fraenkel's Axiom Schema of Replacement, which implies Zermelo's Axiom.

The three most important paradoxes discovered were the paradoxes of Burali-Forti, Cantor, and Russell, which will be discussed below. The discussion will proceed chronologically, at least to the dates traditionally assigned to each paradox, and consequences of these paradoxes will be assessed.

2. Burali-Forti's Paradox

Scholars traditionally point to Burali-Forti's paradox as the first to discover inherent contradictions in the naive definition of set. The paradox comes from the argument that the set of all ordinals is a set, and results in the contradiction that $\Omega + 1 \leq \Omega$ and $\Omega + 1 > \Omega$ simulateously, for a certain ordinal Ω. In Zermelo-Fraenkel set theory (ZF), this problem can be avoided in several ways, all of which result in showing that Ω (the class of all ordinals) is a proper class and not a set. The most straight-forward shows that if Ω is a set, then Ω is an ordinal, which implies that $\Omega \in \Omega$, which would violate the Axiom of Regularity ([10], p.38).

Another way to avoid this problem in ZF is by the following argument. If Ω is a set then Ω is an ordinal and an initial segment, so Ω is order isomorphic to one of its proper initial segments, which is a contradiction ([9], p. 128). In ZF this contradiction would imply that Ω is not a set, since in that theory sets are only defined by a formula if being defined by that formula does not lead to a contradiction. In naive set theory, however, there is no such restriction on the definition of set, and the Axiom of Abstraction implies that Ω exists by a formula $\varphi(x)$, where $\varphi(x)$ means "x is an ordinal." Thus, in Cantor's set theory this contradiction is not resolved.

There is no general consensus on who first discovered the Burali-Forti paradox, which serves to belie its name, but most attribute the discovery to either Cantor in 1895 ([9],[10]) or to Burali-Forti in 1897, though some give credit to Russell in 1903 ([6]). Nonetheless, the paradox gets its name from a paper written by Burali-Forti in 1897 entitled "A question on transfinite numbers," and from another paper written later in the same year called "On well-ordered classes" ([11], p. 104-112). Burali-Forti's goal was to disprove the trichotomy of ordinals by a *reductio ad absurdum* argument. Since it is, in fact, true that for any ordinals α and β, either $\alpha < \beta$, $\beta < \alpha$, or $\alpha = \beta$, the hypothesis in Burali-Forti's argument was true, so his contradiction demonstrated an inconsistency in set theory, rather than the falsity of his hypothesis. Burali-Forti showed that if the set of ordinals, Ω (assuming it exists), is linearly ordered then Ω and $\Omega + 1$ are both themselves ordinals. This implies that $\Omega + 1 \leq \Omega$, since $\Omega + 1 \in \Omega$, and $\Omega + 1 > \Omega$, since $\alpha + 1 > \alpha$ for all $\alpha \in \Omega$.

Although Burali-Forti's argument was in fact a paradox, he did not recognize it as such, being under the impression, initially, that the ordinals were partially but not linearly ordered. In 1897, just a few months after Burali-Forti published his paper, Cantor published a paper proving the trichotomy of ordinals, indicating an error in Burali-Forti's conclusion. In his paper, Burali-Forti had misused Cantor's definition of a well-ordering, using a much weaker definition, and consequently believed that using the correct definition of well-ordering would eliminate the contradiction ([11], p. 104).

Since Burali-Forti maintained that no paradox existed in his argument, he is often not given credit for its creation, and perhaps justifiably so. Moore and Garciadiego attribute to Russell both the recognition that Burali-Forti's arguments formed a paradox and the naming of the paradox. They contend that the Burali-Forti paradox was "not created by either Burali-Forti or Cantor. It arose gradually and began to take recognizable form only in Russell's *The Principles of Mathematics* of 1903" ([6], p. 319). This contention is dubious for several reasons. In their conclusion, Moore and Garciadiego state that "In 1907 Cantor insisted that during 1883 he had already understood the concept of set in a way that gave rise to no such ordinal" ([6], p. 343). In an 1899 letter to Dedekind, Cantor delineated an argument similar to Burali-Forti's in which he concluded that if the collection of ordinals formed a set, according to the usual definition, then the contradiction $\Omega < \Omega$ would occur. Further, Cantor then stated "The system Ω ($\Omega \setminus 0$ in the notation above) of all numbers is an inconsistent, absolutely infinite multiplicity" ([11], p. 115). Here 'number' is understood to mean 'ordinal' and the term 'inconsistent, absolutely infinite multiplicty' (versus 'consistent') anticipates the distinction between set and class in the modern terminology. Using the definition of paradox given by Moore and Garciadiego, and quoted in the introduction above, Cantor's argument "ends in a contradiction although all of its premises and modes of reasoning are *prima facie* acceptable," and Cantor gives up "a premise or mode of reasoning that he has previously accepted as correct," namely, his definition of set. Thus, according to this definition, Cantor was the creator of the Burali-Forti paradox.

Despite the debate concerning the creator of the Burali-Forti paradox, it remains that this paradox was the beginning of a series of such paradoxes that forced mathematicians to revise their idea of what consituted a set, and to ultimately abandon the Axiom of Abstraction. Another such paradox, the cardinal analogue to Burali-Forti's paradox, emerged at about the same time.

3. Cantor's Paradox

Cantor's paradox, sometimes called the paradox of the greatest cardinal, expresses what its second name would imply—that there is no cardinal larger than every other cardinal. There seems to be close consensus that Cantor discovered this paradox in 1899 or between 1895 and 1897 ([1], p. 34), but there are some, including the authors who attribute the Burali-Forti paradox to Russell, who give credit to Russell in 1899 or 1901 ([6], p. 343).

The crux of Cantor's paradox is Cantor's Theorem, which states that for any set A, $|\mathcal{P}(A)| > |A|$, where $\mathcal{P}(X)$ is the power set of X and $|X|$ is the cardinality of X. The typical, modern proof for this theorem is as follows, and can be found, among others, in [3],[9], and [10]. Let A be a fixed set. Then $f : A \to \mathcal{P}(A)$ defined by $f(a) = \{a\}$, is an injection, so $|A| \leq |\mathcal{P}(A)|$. It remains to show that $|A| \neq |\mathcal{P}(A)|$, so by way of contradiction assume that $f : A \to \mathcal{P}(A)$ is a surjection. Then $B = \{x \in A | x \notin f(x)\} \subset A$, so there exists a $z \in A$ with $f(z) = B$. Now $z \in B$ implies $z \notin B$ and $z \notin B$ implies $z \in B$, so a contradiction has been reached. Thus, $|A| \neq |\mathcal{P}(A)|$, so $|A| < |\mathcal{P}(A)|$.

The conclusion in the preceding proof that $z \in B \Leftrightarrow z \notin B$ looks almost identical to the contradiction reached in Russell's paradox, and indeed, the most prominent theories on the origins of Russell's paradox suggest that his paradox was derived from Cantor's paradox alone or from a combination of Cantor's paradox and the proof of Cantor's Theorem.

Given Cantor's Theorem, Cantor's paradox following almost immediately. Suppose that V is the set of all sets, defined by the formula $\varphi(x)$, where $\varphi(x)$ means "x is a set." By Cantor's Theorem $|\mathcal{P}(V)| > |V|$, but $\mathcal{P}(V)$ is a set (containing other sets), so $\mathcal{P}(V) \subset V$, which implies that $|\mathcal{P}(V)| \leq |V|$, a contradiction. Cantor proves his theorem in two ways, function-theoretically and by using a diagonal argument. From the function-theoretic proof of his theorem he derives his paradox, which is very similar to the argument above, except Cantor came to the conclusion that $|\mathcal{P}(V)| > |V|$ and $|\mathcal{P}(V)| = |V|$ ([1], p. 39).

Historians typically cite Cantor's July 28, 1899 letter to Dedekind as the origin of this paradox. These conclusions are sound, given that Cantor stated in his letter that the "totality of everything thinkable" ([11], p. 114) is an inconsistent multiplicity, or absolutely infinite. He also stated that the set of all cardinals was an inconsistent multiplicity, which follows from conclusion B of his letter ([11], p. 116), which says:

"B. *The system* $\beth$[1] *of all alephs, when ordered according to magnitude,*

$$\aleph_0, \aleph_1, \ldots, \aleph_{\omega_0}, \aleph_{\omega_0+1}, \ldots, \aleph_{\omega_1}, \ldots,$$

forms a sequence that is similar to the system Ω *and therefore likewise inconsistent, or absolutely infinite.*"

The Ω to which Cantor referred is the collection of all ordinals without 0. As with the Burali-Forti paradox, the definition given by Moore and Garciadiego would imply that Cantor was the creator of the paradox that bears his name. However, Moore and Garciadiego claim "the first paradox he (Russell) created was the paradox of the largest cardinal, which began to take form in 1899 but only became definitive in 1901" ([6], p. 343). Regardless of who first recognized the paradox of the largest cardinal as a paradox, it is true that Russell arrived at the paradox in an almost independent way, and that this paradox played a major role in the creation of his own antinomy.

4. Russell's Paradox

Russell's paradox, also referred to as Russel's antinomy, Russell's problem, Russell's argument, and Zermelo-Russellsches paradoxon ([7], p. 21), is by far the most famous of the classical paradoxes of set theory, owing much of its fame to its simplicity and far-reaching implications. Russell is usually credited with its discovery in his *The Principles of Mathematics* (1903) ([4], [3]), or during June of 1901 ([11]); however, it is clear that Zermelo arrived at the paradox independently one or two years earlier ([7]). The paradox is more simple than the paradoxes of Burali-Forti and Cantor because it

[1] $\beth$ was the Hebrew letter tav in Cantor's letter.

relies only on the most elementary ideas of set theory—the notion of set, set membership, and the Axiom of Abstraction.

The paradox can be formulated in the following way. Suppose that the the collection R defined by the formula $x \notin x$ is a set. Then if $R \in R$, $R \notin R$, and if $R \notin R$, $R \in R$. Russell also used a statement about a barber to illustrate this principle: If a barber cuts the hair of exactly those who do not cut their own hair, does the barber cut his own hair? ([4], p. 809).

The first clear mention of this paradox occurred in Russell's letter to Frege, written on June 16, 1902. Russell realized that his paradox undermined Frege's theory set forth in *Grundgesetze der Arithmetik*, but with humility and marked deference for Frege and his work Russell explained how Frege's Rule V lead to a contradiction in set theory. Russell stated his paradox to Frege in the following way ([11],p. 125):

> Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.

Frege was surprised and somewhat dismayed by Russell's revelation, understanding its implications, one of which was the falsity of his fifth rule. He replied to Russell in less than a week, on June 22, saying "with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish" ([11], p. 127-128). Much to Frege's consternation, the second volume of his *Grundgesetze der Arithmetik* was about to be published when Frege received Russell's letter, and all he could do in an attempt to salvage part of his theory was to add an appendix describing the derivation of Russell's paradox from Rule V. Although the foundation of his theory had been destroyed, Russell described Frege's reaction as "superhuman," writing to van Heijenoort that Frege "responded with intellectual pleasure clearly submerging any feelings of personal disappointment" ([11],p. 127).

The origins of Russell's paradox are even more controversial than the origins of Burali-Forti and Cantor's paradoxes. There are five main theories on the origins of Russell discovery, two of which have already been mentioned. Two of the other three hold that Russell derived his paradox from the Burali-Forti paradox or from an analysis of its proof, while the last maintains that "set-theoretic paradoxes share a common structure, which is exposed by the Russell paradox" ([1], p. 36). Although these paradoxes do share a common structure and Russell was interested in the Burali-Forti paradox, the theories given in the previous section seem most plausible, since Russell himself recounted, "When I first came upon this contradiction, in the year 1901, I attempted to discover some flaw in Cantor's proof that there is no greatest cardinal," after which he was "led to a new and simpler contradiction" ([8], p. 136).

Given any of those five theories, however, Russell's motivation remains the same. From 1896-1900 Russell harbored the same reservations for Cantor's set theory as many mathematicians had a decade or so before, that is, reservations against the use of the infinite as actual ([5], p. 217). He was persuaded by Peano to accept this use of infinity, but once again became skeptical of some foundations of set theory once he saw a similarity between Cantor's paradox and his own antinomy of infinite number. Russell's philosophical traditions led him to the following view of infinity which appeared in a draft of his *The Principles of Mathematics* in 1900. From this draft Moore quotes ([5], p. 224):

> Mathematical ideas are almost all infected with one great contradiction. This is the contradiction of infinity. All antinomies, I believe, so far as they are valid at all, will be found reducible to the antinomy of infinite number.

Russell's views of the self-contradictory nature of infinity changed slightly, allowing him to believe that its contradictions were just difficulties that could be fixed, rather than chronic defects, but he still held that all antinomies were reducible to the antinomy of infinite number. After he understood the paradox of the greatest cardinal in the context of the antinomy of infinite number, Russell was able to create his own paradox that in essence, reduced both the Burali-Forti and the Cantor paradoxes to the antinomy of infinite number.

Many point to the 1903 publication of *The Principles of Mathematics* as Russell's discovery of his paradox, but in light of Russell's letter to Frege, the date can be no later than June 16, 1902. Russell claims to have discovered the paradox in June 1901 ([11], p. 124), and some even contend that he had discovered the paradox as early at January 1901 ([1], p. 35). Anellis claims a slightly earlier date, saying that Russell had a "*primitive version of the Russell paradox*, that is, the *first version of the Russell Paradox*, on 8 December 1900" ([1], p. 40), based on a letter that Russell wrote to Couturat.

Zermelo independently discovered the paradox, which evidence has shown probably occurred one or two years before Russell's discovery, and possibly as early as 1899 ([7], p. 18). Zermelo's arrived at the paradox in a similar, but slightly more complicated manner. Zermelo proceeded as follows. Suppose that there exists a set M with the property that $m \subset M$ implies $m \in M$. Then Zermelo claimed that such a set must be inconsistent, since if it were consistent $M_0 = \{m \in M | m \notin m\}$ would lead to a contradiction. That is $M_0 \subset M$ so $M_0 \in M$, which implies that $M_0 \in M_0 \Leftrightarrow M_0 \notin M_0$. It has been suggested that Zermelo did not publish his proof because he was unaware of Frege's work and of the logical implications of the paradox. He probably was also aware of the Burali-Forti paradox and did not consider his own discovery to be fundamentally new or important ([7], p. 20), but rather just another paradox exposing a weakness in set theory. Zermelo, however, did not remain troubled by this and the other paradoxes, but in 1908 axiomatized set theory in such a way that these paradoxes could be avoided.

5. Conclusion

Though the paradoxes that emerged around the turn of the century initially weakened the foundations of Cantorian set theory and Frege's theory, they were the catalyst that caused both the foundations of logic and set theory to be fortified with better and more consistent axioms. It became clear to mathematicians, including Cantor, Zermelo, and Russell, that the axioms of set theory and logic needed to be refined in order to avoid these paradoxes and achieve consistency. The first attempt was by Russell with his theory of types in *Principia Mathematica*, although this never became popular. Shortly after that, in 1908, Zermelo published his axiomatization of set theory, which largely remains intact today. He replaced the Axiom of Abstraction by his Axiom Schema of Separation, doing away with the Burali-Forti, Cantor, and Russell paradoxes all at once. Zermelo's Axiom of Separation was replaced by Fraenkel's Axiom Schema of Replacement, which implies the Axiom of Separation, and Fraenkel's Axiom is used today in the most widely accepted version of set theory, ZFC. Had these paradoxes not been discovered so early and close to one another, the foundations of mathematics would certainly not be as strong as they are today.

References

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