Multidimensional global extremum seeking via the DIRECT method

Sei Zhen Khong, Chris Manzie, Dragan Nešić, and Ying Tan

***Abstract*— This paper adapts the DIRECT method with a modified termination criterion for global extremum seeking control of multivariable dynamical plants — DIRECT is a sampling type global optimisation method for Lipschitz-continuous functions defined over compact multidimensional domains. Finite-time semi-global practical convergence is established based on a deadbeat sampled-data control law, whose sampling period is a parameter which determines the region and accuracy of convergence. A crucial part of the development is dedicated to a robustness analysis of the DIRECT method against bounded additive perturbations on the objective function. A numerical example of global extremum seeking in the presence of local extrema based on DIRECT is presented at the end.**

***Index Terms*— Extremum seeking control, multidimensional global optimisation, DIRECT method, robustness analysis.**

I. INTRODUCTION

Extremum seeking is a control scheme which optimises the *steady-state* input-output behaviour of a dynamical system, for which a precise mathematical model may *not* be readily available [1]. By prescribing certain generic stability, attractivity, and robustness properties of a class of continuous-time dynamical systems and a class of discrete-time nonlinear programming methods, [2] proposes a unifying framework in which extremum seeking can be achieved with a sampled-data type controller. There, convergence of the method is established via Lyapunov-based arguments. The underlying idea of being able to apply a large class of optimisation algorithms in extremum seeking control inspired the recent paper [3], which provides a similar unifying framework based on continuous-time optimisation methods whose convergence is proven using averaging and singular perturbation techniques described in [4]. Despite the generality of these proposed frameworks, two of their downsides may be identified. First of all, they require online estimation of the derivatives of the steady-state input-output map, which is often inaccurate due the presence of system dynamics. Certain regularity conditions are stated in [2], [4] to guarantee robustness to such inaccuracy, but they may be difficult to verify. Second, a particular state-update form to describe the optimisation algorithms (for e.g., gradient descent, Newton-Raphson [5]) is stipulated, along with their asymptotic stability. Often outputs from algorithms of this form tend to be entrapped at *local* extrema. Furthermore, certain optimisation methods cannot be written in a state-update form with the required stability property; see, for instance, [6].

Several sampling-based discrete-time Lipschitzian optimisation methods for a static map, which do not rely on derivatives in their formulation, can be found in the literature [7]. These deterministic methods are capable of locating a *global* extremum of a Lipschitz function in the multiextremal case within a *compact* domain of search. Of them, the so-called Piyavskij-Shubert algorithm [8], [9] has been adapted for extremum seeking control of dynamical plants in [10], where periodic sampled-data controllers are used along the lines of [2]. It is worth noting that the algorithm does not fit in the aforementioned frameworks of [2], [3]. The extremum seeking convergence proof therein is carried out through careful robustness analysis of the specific algorithm itself. However, for general application to multivariate function optimisations the Piyavskij-Shubert algorithm suffers a serious drawback: an exponentially increasing running time leads to computational intractability.

Towards addressing the above issue, this paper considers another Lipschitzian optimisation method, the so-called DIRECT (DIviding RECTangles) [11]. DIRECT searches through all ‘possible’ Lipschitz bounds on the objective function, and therefore operates intelligently between the local and global levels, the former of which is important in speeding up the rate of convergence once a neighbourhood around a global extremum point is found. The search space of both the Piyavskij-Shubert and DIRECT methods is composed of rectangles. By contrast with sampling all the vertices as in Piyavskij-Shubert, DIRECT only requires the sample of a rectangle’s midpoint, irrespective of the dimensionality. As such, global optimisation of functions of several variables is computationally feasible with DIRECT. The convergence of DIRECT, however, relies on the fact that the input samples form a dense subset of the domain of search. This property is undesirable in extremum seeking control because even after DIRECT locates the basin of convergence of a global extremum at any point of time, there always exists a future time at which DIRECT samples outside this basin, which amounts to leaving the neighbourhood of the global extremum point and probing the system elsewhere. This process does *not* eventually relinquish, as is necessary for dense sampling. Therefore, a sensible termination procedure for the algorithm, which guarantees a point within a certain neighbourhood of a global optimum has been sampled, is needed from an extremum seeking control perspective.

This paper adopts a *modified* DIRECT method for global extremum seeking control of asymptotically stable time-

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invariant finite-dimensional dynamical systems with multiple inputs belonging to a compact set, in a similar spirit to that of [10]. A deadbeat sampled-data control scheme is proposed to achieve semi-global practical convergence in finite time, after which DIRECT is terminated with the input resulting in the closest sample to a global extremum selected onwards as a constant reference to the system. Tuning guidelines are provided for two design parameters, namely the sampling/waiting period and the accuracy of DIRECT's estimate before termination. Knowledge of the Lipschitz constant of the steady-state input-output map is assumed and exploited in deciding on when to terminate DIRECT[1]. An important part of the convergence proof extends that of the original DIRECT method for a static objective function to one which is perturbed by the dynamics of the plant, in which bounds on the perturbations' magnitude can be controlled by the waiting time[2].

The paper is organised as follows. The next section contains a description of the DIRECT method from [11]. Analysis of the robustness of DIRECT to small additive perturbations on the objective function is performed in Section III. Subsequently in Section IV, DIRECT is adapted into extremum seeking of multi-input single-output (MISO) dynamic systems. To illustrate the result, a simulation example is provided in Section V.

II. The DIRECT method

The natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Consider the following bound-constrained optimisation problem:

$$y^* := \min_{u\in\Omega} Q(u), \tag{1}$$

where

$$\Omega := \{u \in \mathbb{R}^m \mid u_i \in [a_i, b_i] \subset \mathbb{R}, i = 1, 2, \dots, m\} \tag{2}$$

and the following assumption holds.

Assumption 2.1: $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ is a globally Lipschitz continuous function [12], i.e. there exists a known $L > 0$ such that

$$|Q(u) - Q(u')| \le L\|u - u'\|_2 := L\left(\sum_{i=1}^{m} |u_i - u_i'|^2\right)^{\frac{1}{2}}$$

for all $u, u' \in \Omega$.

Assume without loss of generality that $a_i = 0$ and $b_i = 1$ for all $i = 1, \dots m$, i.e. Ω is a unit hypercube, which can be obtained by appropriately normalising (1). Note that Q does possess a global minimum by the Extreme Value Theorem [13, Thm. 4.16] since it is continuous on a compact domain. The map Q can be viewed as a MISO *static* system (i.e. without dynamics) and DIRECT [11] is a deterministic sampling method which solves (1). The only assumption that DIRECT makes is the Lipschitz continuity of Q; no knowledge of the system model is needed. This section gives a brief review of the DIRECT optimisation method [11] and Section IV demonstrates how it can be applied for extremum seeking control of MISO *dynamical* systems.

Algorithm 2.2: The DIRECT algorithm [11].
Given: A Lipschitz function $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$.
Notation: q denotes the iteration number of DIRECT and k the total number of samples. Input samples taken by DIRECT are denoted $u_i \in \Omega$, $i = 0, 1, \dots, k-1$ and the corresponding outputs $y_{i+1} := Q(u_i)$.

(i) Initialise $q := 1$ and $k := 0$.
(ii) Evaluate $Q(u_0)$, where $u_0 \in \mathbb{R}^m$ denotes the centre point of Ω. Set $\hat{y}_1 := Q(u_0)$ and increment k, i.e. $k^+ := k + 1$.
(iii) Identify the set of indices of potentially optimal hyper-rectangles S, i.e. all $j \in \{0, \dots, k-1\}$ for which there exists a positive $\tilde{L} \in \mathbb{R}$ such that[3]

$$\begin{aligned} Q(u_j) - \tilde{L}d_j &\le Q(u_i) - \tilde{L}d_i \;\forall i = 0, \dots, k-1 \\ Q(u_j) - \tilde{L}d_j &\le \hat{y}_k - \epsilon|\hat{y}_k| \text{ for some } \epsilon > 0, \end{aligned} \tag{3}$$

where d_i denotes the distance from the centre point to the vertices of the i^{th} hyper-rectangle.
(iv) For every $j \in S$, subdivide the j^{th} hyper-rectangle with centre u_j according to the following rule. First identify the set I of dimensions $i \in \{1, \dots, m\}$ in which the j^{th} hyper-rectangle has the maximum side length and let δ be one-third of this value. Sample Q at the points $u_j \pm \delta e_i$ for all $i \in I$, where e_i denotes the i^{th} unit vector in $\mathbb{R}^m$. Sequentially divide the j^{th} hyper-rectangle into thirds along the dimension $i \in I$ in an ascending order of $\min\{Q(u_j + \delta e_i), Q(u_j - \delta e_i)\}$. Set $k^+ := k + \Delta k$, where Δk is the number of *new* points sampled during the q^{th} iteration.
(v) Set $q^+ := q + 1$ and the minimal estimate

$$\hat{y}_q := \min_{i=1,\dots k} y_i. \tag{4}$$

(vi) Loop from (iii).

Remark 2.3: The ϵ in (3) is a balance parameter which serves to prevent DIRECT from entrenching itself in the local search. Robustness of the algorithm to this parameter is analysed in detail in [15]. To alleviate the sensitivity to this parameter, which when inappropriately set may compromise the efficiency of DIRECT for certain problems, a modification with a time-varying ϵ is also proposed therein.

Remark 2.4: Since DIRECT identifies all potentially optimal hyper-rectangles via (3) for subdivision, it is well-balanced between local and global searches. Indeed, once DIRECT locates the basin of convergence of a global optimum, the local part of the algorithm automatically exploits it to accelerate the search [11]. A modification of DIRECT which is strongly biased towards the local search can be found in [16], [17].

[1]The original DIRECT method for a static mapping requires no knowledge of its Lipschitz constant to work.

[2]The waiting time is equal to the sampling period of the control law. These two terms are used interchangeably throughout.

[3]As noted in [11], the set of potentially optimal hyper-rectangles can be found using the Graham's scan [14], which is an algorithm for determining the convex hull of a finite set.

Proposition 2.5 ([11]): As the number of iterations approaches infinity, the points sampled by DIRECT form a dense subset of Ω. Because Q is Lipschitz continuous and hence continuous, this implies that the estimate by DIRECT eventually converges to y^* in (1). Mathematically, it holds that

$$\lim_{q\to\infty} \hat{y}_q = y^* := \min_{u\in\Omega} Q(u).$$

The denseness in domain-sampling property of DIRECT mentioned in the proposition above is critical in the proof of its convergence. It holds by the way the algorithm is set up, which subdivides all potentially optimal rectangles in the search space iteratively. The Piyavskij-Shubert algorithm does not share this feature and is thus based on a different convergence analysis [8]–[10].

Remark 2.6: In the generalised extremum-seeking framework of [2], nonlinear optimisation algorithms are assumed to take a difference inclusion form:

$$u^+ \in F(u, G(u)), \tag{5}$$

where $u_k \in \Omega$ denotes the sample point at 'time' k, F is a 'state-update' set-valued map and G is a function that carries information regarding the estimate of the gradient of Q around u. Furthermore, (5) is required to satisfy a type of asymptotic stability property as guaranteed by the existence of a corresponding Lyapunov-like function. The DIRECT Algorithm 2.2 cannot be classified as being in this class of optimisation algorithms, since expressing it in the form of (5) does not seem possible and by virtue of the fact that u_k for $k = 1, 2, \ldots$ will form a dense subset of Ω, the asymptotic stability property stipulated in [2] does not hold.

III. Robustness analysis of DIRECT

A. Termination of DIRECT

The standard termination criterion of the DIRECT algorithm is a pre-specified number of iterations. Using knowledge of the Lipschitz constant L of Q, a lower bound on Q can be computed and progressively improved based on each new sample y_i. A less heuristic stopping criterion for DIRECT can then be y_i being within some tolerance of the bound, as is the case for Piyavskij-Shubert algorithm [8], [9].

To be specific, suppose after a number of DIRECT iterations, there are J number of hyper-rectangular subblocks in the input space Ω. Consider the j^{th} rectangle with centre u_j, and let the distance between its vertices and u_j be d_j. Then by the Lipschitz continuity Assumption 2.1, the lowest possible value Q can attain on this rectangle is given by $Q(u_j) - \eta$ with $\eta := Ld_j$. It follows that

$$E := \min_{i=1,\ldots,J} Q(u_j) - \eta \le \min_{u\in\Omega} Q(u), \tag{6}$$

i.e. E is an estimate of the lower bound on Q, which improves with each new sample DIRECT takes. Indeed, DIRECT can be programmed to terminate when a point within some η-tolerance from this lower bound is sampled. The following lemma gives an estimate of the maximum number of iterations DIRECT needs to locate such a point.

Lemma 3.1: Suppose Q in problem (1) satisfies the global Lipschitz condition in Assumption 2.1 with Lipschitz constant L. Given any $\eta > 0$, let

$$N := 3^{m-1}\left(\frac{3^{m(i+1)} - 1}{3^m - 1}\right), \tag{7}$$

where m is the dimension of the input space as in (2) and $i \in \mathbb{N}$ is such that

$$\frac{(m3^{-2i})^{\frac{1}{2}}}{2} \le \frac{\eta}{L}. \tag{8}$$

Then

$$\hat{y}_N - y^* \le \eta,$$

where $\{\hat{y}_k\}_{k=1}^N$ is the sequence of estimates from applying DIRECT to problem (1).

Proof: The proof is established by combining aspects of [11] and [17]. In particular, because hyper-rectangles in DIRECT's search space are constructed by dividing existing ones into thirds along selected dimensions in $\mathbb{R}^m$, the only possible side length a rectangle can have is 3^{-i} for $i = 0, 1, 2, \ldots$. Furthermore, since DIRECT always divides a rectangle on its largest side, it follows that after r divisions, a hyper-rectangle will have $j := r \mod m$ sides of length $3^{-(i+1)}$ and $m-j$ sides of length 3^{-i}, where $i := (r-j)/m$. Therefore, the distance from its centre to vertices is

$$d := (j3^{-2(i+1)} + (m-j)3^{-2i})^{\frac{1}{2}}/2, \tag{9}$$

which decreases with the increase in $i \in \mathbb{N}$. It is established in [17, Thm. 4.2] that given any $i \in \mathbb{N}$, after $N \in \mathbb{N}$ iterations, DIRECT will leave no rectangle of side length greater than or equal to 3^{-i} in Ω, where N is as defined in (7). When this is the case, in view of (9), no rectangle has a centre-to-vertex distance greater than $(m3^{-2i})^{\frac{1}{2}}/2$. Combining this with (8) and the argument preceding (6) implies at least one point on which Q evaluates to a value no greater than $y^* + \eta$ has been sampled by DIRECT. ■

Remark 3.2: The estimate (7) derived in [17, Thm. 4.2] is based on the worst-case analysis which assumes that only one hyper-rectangle is subdivided per DIRECT iteration. In general, N in (7) is an upper bound on the number of iterations needed before all rectangles' side lengths are less than 3^{-i}. The bound is tight, for example, when the objective function is constant.

B. Imprecise sampling

Frequently the output samples of the function Q are corrupted by some ν-bounded additive noise, i.e. given any $u \in \Omega$, the sample of Q evaluated at u can take any value in

$$Q(u) + [-\nu, \nu] := \{y \in \mathbb{R} \mid y = Q(u) + \delta;\ |\delta| \le \nu\},$$

for some $\nu \ge 0$. Towards characterising such a perturbed function, given any $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$, let

$$\|Q\|_\infty := \sup_{u\in\Omega} |Q(u)|$$

and

$$\mathbb{Q}(Q, \nu) := \left\{\tilde{Q} : \Omega \to \mathbb{R} \;\middle|\; \|\tilde{Q} - Q\|_\infty \le \nu\right\}. \tag{10}$$

As such, given any $u \in \Omega$ and $\tilde{Q} \in \mathbb{Q}(Q, \nu)$, we have that

$$|\tilde{Q}(u) - Q(u)| \leq \nu,$$

i.e. $\tilde{Q}$ is a perturbed version of Q by some additive noise whose magnitude is bounded above by ν.

Theorem 3.3: Given a globally Lipschitz continuous function $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ with Lipschitz constant L and any $\tilde{Q} \in \mathbb{Q}(Q, \nu)$ for some $\nu \geq 0$, let

$$y^* := \min_{u \in \Omega} Q(u).$$

Suppose for an $\eta > 0$, N and i are chosen such that (7) and (8) are satisfied. Denote by $\{\hat{y}_k\}_{k=1}^{\infty}$ the estimate sequence from DIRECT when applied to optimising $\tilde{Q}$. It holds that

$$|\hat{y}_N - y^*| \leq \eta + \nu.$$

Proof: After N number of DIRECT iterations, let u be the centre point of any hyper-rectangle $H \subset \Omega$. We know by Lemma 3.1 that the minimum value Q can take on H is $Q(u) - \eta$. Furthermore, note that

$$\sup_{\tilde{Q} \in \mathbb{Q}(Q,\nu)} \left| \tilde{Q}(u) - (Q(u) - \eta) \right| \leq \nu + \eta.$$

The claimed result then follows from the inequality above, in light of the estimate update of DIRECT described in (4), which is set to be the minimum of $\tilde{Q}$ evaluated at the centres of all existing hyper-rectangles in Ω. ■

We once again emphasise that the N in Theorem 3.3 gives only an upper bound on the number of runs DIRECT needs before a sample of at most $\eta + \nu$ distance from y^* is taken. A criterion for when such a point is sampled is given by the next result.

Lemma 3.4: Following Theorem 3.3, if for a $q \leq N$ the minimising input u_b which results in the current best estimate $\hat{y}_q = \tilde{Q}(u_b)$ is the centre point of a hyper-rectangle with centre-to-vertex distance d satisfying $Ld \leq \eta$, then it holds that

$$|\hat{y}_q - y^*| \leq \eta + \nu,$$

whereby

$$|Q(u_b) - y^*| \leq \eta + 2\nu$$

because $|\hat{y}_q - Q(u_b)| \leq \nu$ by the definition of $\tilde{Q}$.

Proof: Let $Q_{max} : \Omega \to \mathbb{R}$ be defined by

$$Q_{max}(u) := Q(u) + \nu,$$

which is Lipschitz with constant L. In effect, for any $u \in \Omega$, $Q_{max}(u)$ is the upper bound on the perturbed value of $Q(u)$ by ν-bounded noise. It follows by the same argument before (6) that $\hat{y}_q - Ld$ is a lower bound on $\min_{u \in \Omega} Q(u) = y^* + \nu$, i.e. $\hat{y}_q - Ld \leq y^* + \nu$. This implies that

$$|\hat{y}_q - y^*| \leq \eta + Ld \leq \eta + \nu,$$

as required. ■

Remark 3.5: It can be seen from the proof above that the lemma would still hold with the global Lipschitz constant L replaced with a local Lipschitz bound for a neighbourhood encompassing the hyper-rectangle in which the current best estimate $\hat{y}_q$ lies.

Remark 3.6: In the case where the Lipschitz constant L of the objective function Q is known, the DIRECT Algorithm 2.2 for a static map would normally be modified for efficiency by restricting the search for $\tilde{L}$ in (3) to those which satisfy $\tilde{L} \leq L$; this is not necessary for convergence. In the face of imprecise sampling of Q, i.e. sampling from a member $\tilde{Q}$ of $\mathbb{Q}(Q, \nu)$, this modification should *not* be enforced since there is no guarantee that $\tilde{Q}$ would be Lipschitz continuous. Otherwise, it may result in certain potentially optimal hyper-rectangles not being selected for subdivision and thus possible failure in locating a global extremum.

IV. Extremum seeking via the DIRECT method

A. The plant

This section demonstrates the application of the DIRECT method to seeking a global extremum of the steady-state behaviour of a MISO dynamical system with a compact input set through the use of periodic sampled-data components. In particular, consider

$$\begin{aligned} \dot{x} &= f(x, u) \quad x(0) = x_0; \\ y &= h(x), \end{aligned} \tag{11}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions in each argument. Assume that $u(t) \in \Omega$, the unit hypercube as in (2), for all $t \geq 0$. Such an assumption arises most naturally from actuator constraints on the inputs in the form of saturation nonlinearity [12].

In general, some form of Lyapunov stability property is assumed of the dynamical system within an extremum seeking framework; see [1]–[4], [10]. Here, we adopt a similar approach and assume for each constant $u \in \Omega \subset \mathbb{R}^m$, there exists a globally asymptotically stable equilibrium point for (2) in the following sense.

Assumption 4.1: Given a system described by (11), there exists a globally Lipschitz function $\ell : \Omega \to \mathbb{R}^n$ such that

$$f(\ell(u), u) = 0 \quad \forall u \in \Omega.$$

Furthermore, there exists a $\mathcal{KL}$ function[4] β such that for any $u \in \Omega$ and $x_0 \in \mathbb{R}^n$,

$$\|x(t, x_0, u) - \ell(u)\|_2 \leq \beta(\|x_0 - \ell(u)\|_2, t) \quad \forall t \geq 0,$$

where $x(\cdot, x_0, u)$ denotes the solution to (11) with respect to the initial condition x_0 and input u.

Definition 4.2: Let

$$Q(\cdot) := h \circ \ell(\cdot) : \Omega \to \mathbb{R}$$

be the steady-state input-output map of system (11). Note that Q is globally Lipschitz on Ω as per Assumption 2.1 because h is locally Lipschitz on $\mathbb{R}^n$ and ℓ is globally Lipschitz on Ω.

Let $\{u_k\}_{k=0}^{\infty}$ be a sequence of vectors in Ω and define the zero-order hold (ZOH) operation

$$u(t) := u_k \quad \text{for all } t \in [kT, (k+1)T) \tag{12}$$

[4]A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed t, $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and for each fixed s, $\beta(s, \cdot)$ is decreasing to zero.

and $k = 0, 1, 2, \ldots$, where $T > 0$ denotes the period or waiting time. Furthermore, let the state and output of the system (11) with respect to the input u be respectively x and y and define the periodic sampling operation $x_k := x(kT)$;

$$y_k := y(kT) \quad \text{for all } k = 1, 2, \ldots. \tag{13}$$

The global asymptotic stability of the system (11) gives rise to the following proposition.

Proposition 4.3: Suppose Assumption 4.1 holds, then given any $\Delta > 0$ and $\nu > 0$, there exists a $T > 0$ such that for any $\|x_0\|_2 \leq \Delta$ and $\{u_k\}_{k=0}^{\infty} \subset \Omega$,

$$|y_k - Q(u_{k-1})| \leq \nu \quad \text{for all } k = 1, 2, \ldots,$$

where y_k and u_k are related via (11) as above. Furthermore, given any $t_0 \geq 0$ and $u_b \in \Omega$, suppose $u(t) := u_b \, \forall t \geq t_0$, then for any $\|x_0\|_2 \leq \Delta$,

$$|y(t) - Q(u_b)| \leq \nu \quad \text{for all } t \geq t_0 + T.$$

where $T > 0$ is the same waiting time as above.

Proof: The first part of the result is established in [10, Prop. 1]. The second part can be shown using the same arguments therein. ■

Remark 4.4: The proof of Proposition 4.3 is based on the fact that the global asymptotic stability of (11) stated in Assumption 4.1 guarantees that after a sufficiently long waiting period T, the dynamics of the time-invariant system (11), initialised at any point in time with respect to any initial condition, will vanish to within a small perturbation on its steady-state input-output behaviour. Moreover, the magnitude of the perturbation reduces with the elongation of the waiting period T.

B. Extremum seeking

Consider the modified-DIRECT-based extremum seeking scheme illustrated in Figure 1. The feedback system is interconnected through an ideal sampler of period $T > 0$ (cf. (13)) and a synchronised zero-order hold (ZOH) (cf. (12)) device. There, the modified DIRECT algorithm is given by the following.

Algorithm 4.5: The modified DIRECT for the extremum seeking setup in Figure 1 is as described in Algorithm 2.2, but with an additional input $\eta > 0$ (the error margin) and an amendment to its termination criterion. In particular, Step (vi) of Algorithm 2.2 is revised to be the following.

(vi) Let d be the centre-to-vertex distance of the hyper-rectangle within which u_b satisfying $\hat{y}_q = Q(u_b)$ lies (cf. (4)). If $Ld \leq \eta$, set the subsequent sample points to be the input which results in $\hat{y}_q$:

$$u_{k+j} := u_b \quad \text{for all } j = 1, 2, \ldots.$$

Otherwise, loop from (iii).

Theorem 4.6: The closed-loop system depicted in Figure 1, consisting of the plant (11) satisfying Assumption 4.1, periodic sampler (13), zero-order hold (12), and modified DIRECT Algorithm 4.5 has the following convergence property: Given any $\Delta > 0$ and $\mu > 0$, let the parameter η of Algorithm 4.5 be any number less than μ, then there exist a sampling/waiting period $T > 0$ and μ-convergence time $\tilde{T} > 0$ such that for any $\|x_0\|_2 \leq \Delta$,

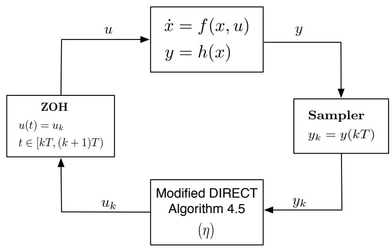

Fig. 1. Extremum seeking based on modified DIRECT

$$|y(t) - y^*| \leq \mu \quad \text{for all } t \geq \tilde{T},$$

where $y^* := \min_{u \in \Omega} Q(u)$ and $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the steady-state input-output map with Lipschitz constant L as in Definition 4.2.

Proof: Let $\nu := (\mu - \eta)/3$. Application of Proposition 4.3 to the plant with respect to ν yields a sampling/waiting period $T > 0$ such that for any $\|x_0\|_2 \leq \Delta$,

$$|y_k - Q(u_{k-1})| \leq \nu \text{ for all } k = 1, 2, \ldots.$$

Correspondingly, there exists a $\tilde{Q} \in \mathbb{Q}(Q, \nu)$ for which the samples y_k are the evaluations of $\tilde{Q}$ on u_k; see (10). Theorem 3.3 then gives an upper bound $N \in \mathbb{N}$ such that there exists a $q \leq N$ for which the input u_b resulting in the current best estimate $\hat{y}_q$ of DIRECT Algorithm 4.5 lies in a hyper-rectangle of centre-to-vertex distance d that satisfies $Ld \leq \eta$, whereby invocation of Lemma 3.4 yields

$$|\hat{y}_q - y^*| \leq \nu + \eta$$

and

$$|Q(u_b) - y^*| \leq 2\nu + \eta. \tag{14}$$

Denote by $t_0 \geq 0$ the time at which the DIRECT iteration within which u_b is sampled ends. Note that by Step (vi) of DIRECT Algorithm 4.5,

$$u(t) := u_b \quad \text{for all } t \geq t_0.$$

Let $\tilde{T} := t_0 + T$. By the second part of Proposition 4.3, we have that

$$|y(t) - Q(u_b)| \leq \nu \quad \text{for all } t \geq \tilde{T}.$$

Together with (14), this implies that

$$|y(t) - y^*| \leq 3\nu + \eta = \mu \quad \text{for all } t \geq \tilde{T},$$

as required. ■

Remark 4.7: Theorem 4.6 provides a convergence proof for a deadbeat type extremum seeking of a general dynamical plant based on the DIRECT method (cf. Algorithm 4.5). The control scheme takes a finite amount of time to locate a reference input to the system which drives the steady-state behaviour into a neighbourhood of a global extremum. The region and accuracy of convergence can be improved at the expense of convergence rate, which corresponds to an increase in the waiting period T and/or number of iterations of DIRECT.

Remark 4.8: In the case where the dynamical system is slowly time-varying, the deadbeat extremum seeking scheme depicted in Figure 1 can be 'restarted' after a period of idle time (during which Algorithm 4.5 persistently outputs the previously determined minimising command) to recalibrate the system and locate a (possibly different) global extremum of the steady-state. A sampled-data controller which cycles through several basic 'modes' of operation in a similar fashion described may be found, for e.g., in [18], where input-to-state stability in the presence of signed measurement disturbance is investigated.

V. Numerical simulation

The simulation results in this section are generated with the help of the MATLAB source code from [19]. We consider the following two-dimensional dynamical system with two inputs which is of the differential form (11):

$$\dot{x} = Ax + Bu, \quad x(0) := \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]; \qquad y = h(x),$$

where $A := \left[\begin{smallmatrix} -5 & 1 \\ 0 & -2 \end{smallmatrix}\right]$, $B := \left[\begin{smallmatrix} 5 & -1 \\ 0 & 2 \end{smallmatrix}\right]$,

$$u(t) \in \Omega := \left\{ u \in \mathbb{R}^2 : -3 \le u_1 \le 2,\ -3 \le u_2 \le 2 \right\}$$

for all $t \ge 0$, and

$$h(u_1, u_2) := (4 - 2.1u_1^2 + u_1^4/3)u_1^2 + u_1u_2 + (-4 + 4u_2^2)u_2^2.$$

It is clear that for any $u \in \Omega$, $x = -A^{-1}Bu = u$ is a globally exponentially stable equilibrium. Thus, the steady-state map (cf. Definition 4.2) of the above dynamical system is given by $Q(u) := h(u)$, which is the so-called six hump camelback test function for bound-constrained optimisation [11]. Q has 4 local minima and its global minimum value is -1.032.

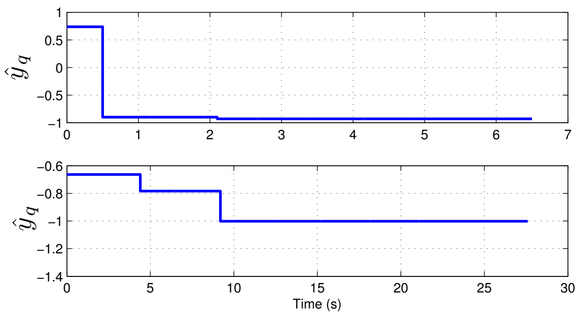

Fig. 2. The estimates by DIRECT over time for waiting periods $T = 0.1$s (top) and $T = 0.4$s (bottom).

Setting $L := 2$ and $\eta := 0.01$, Figure 2 shows the current estimate of the minimum of Q found by DIRECT over time within the sampled-data based extremum seeking framework of Figure 1 for two different sampling/waiting periods, namely $T = 0.1$s and $T = 0.4$s. Recall from Proposition 4.3 that the use of a smaller waiting time results in lesser estimation accuracy. This fact is summarised in Table I, in which for the case $T = 0.4$s the final estimate is fairly close to the global minimum value at the expense of longer convergence duration.

T	Est. min	Duration	Iter. no.	No. of samples
0.1s	-0.928	5.5s	8	55
0.4s	-1.001	23.6s	7	59

TABLE I

Performance characteristics

VI. Conclusion

A deadbeat sampled-data extremum seeking control of dynamical systems is proposed based on a modified DIRECT global optimisation method. We establish semi-global practical convergence of the proposed scheme with respect to two design parameters, the waiting/sampling period T and DIRECT-termination accuracy parameter η. Future research directions involve adapting other sampling-based optimisation algorithms in [6], for instance, within the context of extremum seeking control.

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