

Laws of Trigonometry in Symmetric Spaces

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Abstract. This paper consists of two parts. In the first part, we reformulate the work of E. Leuzinger on trigonometry in noncompact symmetric spaces. In the second part, we outline an alternative method using invariants of the isotropy group representation. Appropriately formulated, these methods apply to both compact and noncompact symmetric spaces. This work is contained in the Ph.D. dissertation of H.-L. Huynh.

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1. What is trigonometry?

Trigonometry is fundamental to the study of the classical geometries. First of all, it gives congruence conditions for triangles. In Euclidean, hyperbolic and spherical spaces, two triangles are congruent if and only if they satisfy the side-angle-side (SAS) or side-side-side (SSS) condition.

Given a triangle, we also want to associate to it metric quantities that are invariant under the group of isometries. The laws of trigonometry can then be interpreted as relations between such quantities. Given two sides and their subtended angle, the law of cosines determines the third side, and given two sides

and one angle adjacent to the third side, the law of sines determines the second adjacent angle.

In a general Riemannian manifold, the law of cosines can be generalized as a comparison theorem for geodesic triangles in the form of an inequality. On the other hand, it is clear that congruence conditions among configurations of points can only be found in spaces with a high degree of symmetry. Only in such spaces can we obtain exact formulas relating invariant geometric quantities.

We will hence consider a globally symmetric space X . We will also be interested in configurations of n points, where n may be greater than 3, and we will refer to this as trigonometry, too. Let X^n be the space of ordered subsets of n points. Given a group of isometries G of X , we let G act diagonally on X^n . That is, $(x_1, \dots, x_n) \rightarrow (gx_1, \dots, gx_n)$ for $g \in G$. Let U be a G -invariant open dense subset of X^n . Once U is specified, an n -tuple $(x_1, \dots, x_n) \in U$ is said to be in general position.

A trigonometric quantity with respect to G is a G -invariant function $\alpha: U \rightarrow \mathcal{F}$, where \mathcal{F} is a set on which G acts trivially. We will also call the induced function $\alpha: U/G \rightarrow \mathcal{F}$ a trigonometric quantity. Let R be a subset of \mathcal{F} . We say that α satisfies the trigonometric relation R if $\alpha(U) \subseteq R$. Finally, α is said to be a congruence condition with respect to G if it is an injective function into \mathcal{F} .

For example, let X be the Euclidean plane and let $U \subset X^3$ consist of the 3-tuples of distinct points. Then $\alpha: U \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S^1$, which maps an ordered triangle to its three sides and the angle subtended by the first two sides, is a trigonometric quantity with respect to the group of orientation-preserving isometries. It satisfies the trigonometric relation

$$\{(a, b, c, \theta) \mid a^2 + b^2 - c^2 - 2ab \cos \theta = 0\}.$$

It is also a congruence condition with respect to the same group.

2. The moduli space of n points in symmetric spaces

By definition, a Riemannian (globally) symmetric space X is a Riemannian manifold with central symmetry through every point $x \in X$, i.e., the diffeomorphism that reverses all the geodesics through x is a global isometry. We will make use of some basic facts about symmetric spaces. Proofs and details can be found in [He], [Mo] or [Wo].

Let $I(X)$ be the full isometry group of X , and let $I_0(X)$ be the component of the identity. Then $I_0(X)$ acts transitively on X . Let G be a Lie group such that $I_0(X) \subseteq G \subseteq I(X)$ and pick a basepoint $x_0 \in X$. Let K be the subgroup of G that fixes x_0 and let σ be the geodesic symmetry at x_0 . Then $X = G/K$, and $(G_\sigma)_0 \subseteq K \subseteq G_\sigma$, where G_σ is the set of fixed points of σ and $(G_\sigma)_0$ is the

identity component of G_σ . If X is of noncompact type, then K will be a maximal compact subgroup of G .

We would like to determine X^n/G or U/G , where U is a suitably chosen set of points in general position. Let \mathfrak{p} be the tangent space of X at x_0 . Under the exponential map $\text{Exp}: \mathfrak{p} \rightarrow X$, we have a correspondence $X^n/G \leftrightarrow (\bigoplus^n \mathfrak{p})/K$ in the noncompact case, and $U/G \leftrightarrow (\bigoplus^n \mathfrak{p})/K$ in the compact case. The ring of invariant polynomial functions, $\mathbb{R}[\bigoplus^n \mathfrak{p}]^K$, is the coordinate ring of the space $(\bigoplus^n \mathfrak{p})/K$, and we wish to describe the moduli space of n points by giving a set of generators and relations for this ring. Note that the generators will then be trigonometric quantities, and the relations will be trigonometric relations.

For the classical geometries, the full isotropy group is $O(m)$. The isotropy representation is the standard action on \mathbb{R}^m , and the generators and relations of the ring $\mathbb{R}[\bigoplus^n \mathfrak{p}]^K$ are given by the First and Second Fundamental Theorems of invariant theory of H. Weyl ([We]). For symmetric spaces of rank greater than one, our knowledge of $\mathbb{R}[\bigoplus^n \mathfrak{p}]^K$ is less complete.

On the other hand, the geometry of two points in a symmetric space is well understood. Consider the set of flats in X . These are the maximal, flat, totally geodesic subspaces. They are all of the same dimension r , which is the rank of the symmetric space X . Each flat is isometric to a Euclidean space in case X is of noncompact type, and a flat torus in case X is compact. Every point of X is contained in a flat, G permutes the flats transitively and K permutes transitively all those that contain the point x_0 . Given a flat F , there is an Abelian subgroup A of G that restricts to the full group of Euclidean translations on F . Let us choose a base flat F_0 containing x_0 . Then each $x \in F_0$ is ax_0 for some $a \in A$, and each $x \in X$ is ky for some $k \in K$ and $y \in F_0$.

It follows that if we have two points in X , then we may assume that the first point is x_0 and the second point is $x_1 \in F_0$. Hence $X^2/G \cong F_0/M'$, where M' is the subgroup of K that preserves the set F_0 . Under the correspondence $a \rightarrow ax_0$, the action of M' on F_0 is equivalent to the action of M' on A given by $a \rightarrow mam^{-1}$, and M' is the normalizer of A in K . Furthermore, $M = Z_K(A)$, the centralizer of A in K , is the subgroup of K that fixes F_0 pointwise. The Weyl group $W = M'/M$ decomposes F_0 into a union of Weyl chambers. For each chamber \mathcal{C} , every W -orbit in F_0 intersect \mathcal{C} exactly once. And W permutes the set of chambers simply and transitively. Now let us choose among the chambers a base chamber \mathcal{C}_0 , so that it contains the point x_1 . Collecting all these facts together, we obtain the following description of the moduli of 2 points.

$$X^2/G \cong K \backslash G / K \cong A / W \cong \mathcal{C}_0.$$

To specify a pair of points (y, y') up to congruence with respect to the full group of isometries $I(X)$, we first move them by an element of $g \in G$ to a pair $(gy, gy') = (x_0, x_1) = (x_0, ax_0)$, where x_0 is the base point, x_1 lies in the base chamber and $a \in A$. Define $M^* \subseteq I(X)$ to be the subgroup that fixes the point x_0 and preserves the set \mathcal{C}_0 . Then

$$X^2/I(X) \cong M^* \backslash A / W.$$

The coset M^*a is called the interval determined by the ordered pair of points (y, y') .

The structure of $\mathbb{R}[\mathfrak{p}]^K$ is completely determined by a theorem of Chevalley. Let \mathfrak{A} be the linear subspace of \mathfrak{p} that exponentiates to F_0 . The adjoint action of K on \mathfrak{p} restricts to an action of M' on \mathfrak{A} , with kernel M . Hence the group W acts on \mathfrak{A} , and we can state the following theorem.

Theorem 1.

$$\mathbb{R}[\mathfrak{p}]^K \cong \mathbb{R}[\mathfrak{A}]^W \cong \mathbb{R}[\sigma_1, \dots, \sigma_r],$$

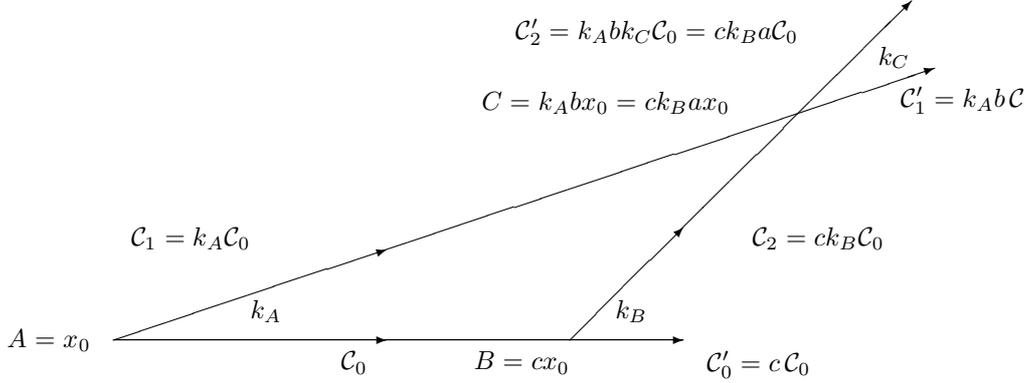
where $\mathbb{R}[\sigma_1, \dots, \sigma_r]$ is a free polynomial algebra on the generators $\sigma_1, \dots, \sigma_r$.

3. The laws of sines and cosines for noncompact symmetric spaces

We will now consider triangles, i.e., the case of three points. The symmetric spaces of rank one turn out to be the so-called two-point homogeneous spaces, and consist of the classical geometries, the complex and quaternionic projective and hyperbolic spaces and the Cayley projective and hyperbolic plane. The trigonometry of the classical geometries can be summed up in the well-known laws of sines and cosines. Subsequently, the trigonometry of $\mathbb{P}^n(\mathbb{C})$ was studied by W. Blaschke and H. Terheggen ([BT]) in 1939, after partial results by J. L. Coolidge ([Co]) in 1921. A different approach was taken by P. A. Sirokov ([Sir]), who obtained more complete results. Sirokov died in 1944, but the results were found among his papers, and published under his name in 1957 by A. P. Sirokov, A. Z. Petrov and B. A. Rozenfeld ([Sir]). B. A. Rozenfeld ([Ro]) later generalized Sirokov's results to the other compact two-point homogeneous spaces, i.e., quaternionic projective space and the Cayley projective plane. In 1986, W.-Y. Hsiang ([Hs]) independently developed the trigonometry of all the two-point homogeneous spaces, giving a unified and differential-geometric treatment that applies equally well to the compact and noncompact cases. In 1987, U. Brehm ([Br]) modified Hsiang's results, using an approach similar to [BT].

The first trigonometric relations for spaces of rank greater than one were given in 1988 in the Ph.D. thesis of H. Aslaksen ([As]) under the supervision of W.-Y. Hsiang. He obtained trigonometric formulas for $SU(3)$ by using Lie theory and direct manipulation of matrices. In his 1990 Ph.D. thesis, E. Leuzinger ([Le]) found the laws of sines and cosines for symmetric spaces of noncompact type. The essential idea is to consider the motions of Weyl chambers, rather than points or geodesic segments. In this section we will describe a reformulation of Leuzinger's method.

Fix a symmetric space of noncompact type X . Recall that a geodesic is said to be regular if it is contained in exactly one flat. We will say that a triple of

Figure 1: A triangle in general position in X

points (A, B, C) is in general position if the geodesic segments AB, BC, CA are all regular, and they lie in three distinct flats. (Note that if A, B, C lie in the same flat, then Euclidean trigonometry applies, but the Euclidean rotations in the flat are generally not elements of $I(X)$.)

Suppose we have two chambers \mathcal{A} and \mathcal{B} with a common apex x but lying in two distinct flats. By an element of $g \in G$ we can move x to the base point x_0 , and \mathcal{A} to the base chamber \mathcal{C}_0 . Then $g\mathcal{B} = k g\mathcal{A} = k\mathcal{C}_0$ for some $k \in K$. The element k is determined only up to right multiplication by elements of the subgroup M . Furthermore, left multiplication by M^* does not affect the congruence class of the pair of chambers with respect to $I(X)$. The double coset M^*kM is called the angle between the ordered pair of chambers $(\mathcal{A}, \mathcal{B})$ with a common apex.

Given a triple (A, B, C) in general position, we may assume that A is the base point x_0 and B lies in the base chamber \mathcal{C}_0 . In Figure 1, the intervals BC, AC, AB are represented by the elements M^*a, M^*b, M^*c with $a, b, c \in A$.

\mathcal{C}'_0 is the translate of \mathcal{C}_0 along the supporting flat of AB to the apex B , \mathcal{C}_2 is the chamber with apex B containing BC , \mathcal{C}'_2 is the translate of \mathcal{C}_2 along the supporting flat of BC to the apex C , \mathcal{C}_1 is the chamber with apex A containing AC , and \mathcal{C}'_1 is the translate of \mathcal{C}_1 along the supporting flat of AC to the apex C .

The angle at $(\mathcal{C}_0, \mathcal{C}_1)$ is $M^*k_A M$, at $(\mathcal{C}'_0, \mathcal{C}_2)$ is $M^*k_B M$, and at $(\mathcal{C}'_1, \mathcal{C}'_2)$ is $M^*k_C M$. The function

$$(A, B, C) \rightarrow (M^*a, M^*b, M^*c, M^*k_A M, M^*k_B M, M^*k_C M)$$

is a trigonometric quantity with respect to $I(X)$. Our aim is to find relations among its components. For suitable representatives of intervals and angles, the

following equations relate the various chambers.

$$\begin{aligned}
\mathcal{C}'_0 &= c\mathcal{C}_0 \\
\mathcal{C}_2 &= ck_Bc^{-1}\mathcal{C}'_0 = ck_B\mathcal{C}_0 \\
\mathcal{C}'_2 &= (ck_B)a(ck_B)^{-1}\mathcal{C}_2 = ck_Ba\mathcal{C}_0 \\
\mathcal{C}_1 &= k_A\mathcal{C}_0 \\
\mathcal{C}'_1 &= k_Ab\mathcal{C}_0 \\
\mathcal{C}'_2 &= k_Abk_C(k_Ab)^{-1}\mathcal{C}'_1 = k_Abk_C\mathcal{C}_0
\end{aligned} \tag{1}$$

Equating the two expressions for \mathcal{C}'_2 , we obtain the following important relation.

$$(k_Abk_C)\mathcal{C}_0 = (ck_Ba)\mathcal{C}_0. \tag{2}$$

This gives the following fundamental relation at group level.

Theorem 2. *It is possible to choose representatives for the cosets corresponding to the sides and angles so that*

$$(k_Abk_C)M = (ck_Ba)M.$$

In general, the quantities appearing on the two sides of this relation are not determined by the intervals and angles alone. In other words, the representatives for the sides and angles cannot be chosen arbitrarily from the cosets they represent. The following theorem from [Le] clarifies this matter. It is essentially the SAS condition.

Theorem 3. *The set of triples (c, k_B, a) that satisfy Theorem 2 consists of all the representatives of a diagonal double coset $M^*(c, k_B M, a)$. The function*

$$(A, B, C) \rightarrow M^*(c, k_B M, a)$$

is a congruence condition.

We will now show how we can derive the laws of trigonometry from Theorem 2, using the KAK and KNA decompositions of G . For any group H and subgroups E, F of H , a function on H is said to be (E, F) -invariant if it is invariant under left multiplication by E and right multiplication by F . Note that M normalizes K and centralizes A , so that $KM = MK$ and $AM = MA$ are subgroups of G . The next two theorems follow easily from Theorem 2.

Theorem 4 (Law of cosines). *Suppose $f: G \rightarrow \mathbb{R}$ is a (K, KM) -invariant function. Then*

$$f(b) = f(ck_Ba).$$

*If there are r such f 's that are functionally independent, then M^*b can be uniquely computed from the data $M^*(c, k_B, a)$.*

Theorem 5 (Law of sines). *Suppose $g: G \rightarrow \mathbb{R}$ is a (K, AM) -invariant function. Then*

$$g(bk_C) = g(ck_B).$$

Thus the problem of finding trigonometric formulas is reduced to the search for functions on the group G that are (K, K) - and (K, AM) -invariant. We will briefly indicate how that can be done.

We have $K \backslash G / K \cong A / W \cong \mathfrak{A} / W$. By Theorem 1, $\sigma_1, \dots, \sigma_r$ induce r independent (K, K) -invariant functions on G via the exponential map.

Recall the Iwasawa Decomposition $G = KNA$, where N is a unipotent subgroup and the map $K \times N \times A \rightarrow G$ is a diffeomorphism. Now M centralizes A and normalizes N . It is easy to check that a function \bar{g} on N that invariant under conjugation by M can be extended to a function g on G that is (K, AM) -invariant. (This is the reason why we write the Iwasawa decomposition as KNA rather than KAN .)

Finally, we say that a set of sine laws is complete if, by using these laws, $M^*k_B M$ can be determined uniquely from $M^*k_C M$, M^*c and M^*b .

4. The Space $SL(m, \mathbb{R})/SO(m)$

Let $X = \mathcal{P}(m, \mathbb{R})$ be the space of $m \times m$ positive definite symmetric matrices with determinant 1. The group $G = SL(m, \mathbb{R})$ acts on X by $(g, p) \rightarrow gpg^t$. The stabilizer K of the base point $x_0 = I_m$ is $SO(m)$. X is an irreducible symmetric space of noncompact type, whose central symmetry at I is given $p \rightarrow p^{-1}$. The tangent space of $\mathcal{P}(m, \mathbb{R})$ at I is the space \mathfrak{p} of symmetric matrices with trace equal to 0. The exponential map $\text{Exp}: \mathfrak{p} \rightarrow \mathcal{P}(m, \mathbb{R})$ is defined by the usual matrix exponential.

The base chamber can be chosen to be A^+ , the set of diagonal matrices $\text{diag}(a_1, \dots, a_m)$ with $a_1 \dots a_m = 1$ and $a_1 > \dots > a_m > 0$. The rank of X is $m - 1$. The Abelian subgroup $A \subset G$ that induces Euclidean translations on this choice of base flat is the group $A(m)$ of diagonal matrices with determinant 1. $M = Z_K(A)$ is the finite group consisting of $\text{diag}(\epsilon_1, \dots, \epsilon_m)$, with $\epsilon_i = \pm 1$ and $\epsilon_1 \dots \epsilon_m = 1$. The laws of sines and cosines can now be stated as follows ([Le]).

Theorem 6. Consider $\mathcal{P}(m, \mathbb{R}) = SL(m, \mathbb{R})/SO(m)$. Define $p: SL(m, \mathbb{R}) \rightarrow \mathcal{P}(m, \mathbb{R})$ by $p(g) = g^t g$. The functions $F_i(g) = \text{tr}(p(g)^i)$, $i = 1, \dots, m - 1$, form a complete and independent set of (K, K) -invariant functions on $SL(m, \mathbb{R})$. From these functions a complete set of cosine laws can be obtained by Theorem 4.

Let \vec{g}_i be the column vectors of g . The functions

$$G_{ij}(g) = (\vec{g}_i, \vec{g}_j)^2 |\vec{g}_i|^{-2} |\vec{g}_j|^{-2},$$

for $i, j = 1, \dots, m$, form a complete set of (K, AM) -invariant functions on $SL(m, \mathbb{R})$. From these functions a complete set of sine laws can be obtained by Theorem 5.

The (K, K) -invariant functions given by Chevalley's theorem can be taken to be the elementary symmetric functions in the eigenvalues of $p(g)$, which are related to $\text{tr}(p(g)^i)$ by Newton's formulas. The functions G_{ij} are associated with the Gram-

Schmidt process, which gives the KNA -decomposition for $SL(m, \mathbb{R})$. Note that N is the subgroup of upper triangular matrices with unit diagonal.

For $m = 2$, the space X is the hyperbolic plane, and Theorem 6 reduces to the hyperbolic sine and cosine laws

$$\begin{aligned} \cosh a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \\ \frac{\sin A}{\sinh a} &= \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}. \end{aligned}$$

For these and other explicit formulas that can be derived from Theorem 6, we refer to [Le].

The symmetric space $\mathcal{P}(m, \mathbb{R})$ is universal in a certain sense. Given a symmetric space $X = G/K$ of noncompact type, we can find an irreducible, faithful representation $\rho: G \rightarrow SL(m, \mathbb{R})$, which in addition is self-adjoint, i.e., $\rho(G)^t \subseteq \rho(G)$. The representation ρ induces an embedding of X into $\mathcal{P}(m, \mathbb{R})$ as a totally geodesic subspace, and the subgroup of $SL(m)$ that preserves X acts by isometries on X . The base flat and base chamber in X are the restrictions of those in $\mathcal{P}(m, \mathbb{R})$. The subgroup $A \subset G$ that induces translations on the base flat is the restriction of the group $A(m)$. (But note that a regular geodesic in X need not be regular in $\mathcal{P}(m, \mathbb{R})$.) Furthermore, the maximal compact subgroup $K = O(m) \cap G$, after we have identified G with its image under ρ . (See [Mo].)

From these facts it follows that the restrictions of $(O(m), O(m))$ -invariant functions to G are (K, K) -invariant and the restrictions of $(O(m), A(m)M(m))$ -invariant functions to G are (K, AM) -invariant. In fact, complete sets of invariant functions can be obtained in this way. With Theorem 6, this gives us an alternative method for getting the sine and cosine laws for a general symmetric space of noncompact type.

We now turn to a discussion of congruence conditions for the space $\mathcal{P}(m, \mathbb{R})$.

Theorem 7. *The SSS condition fails to be a congruence condition for $\mathcal{P}(m, \mathbb{R})$ when $m \geq 3$.*

Proof. From Theorem 3 we know that the space of triangles has the form $M^* \setminus (A^+ \times K/M \times A^+)$. Since the group M^* is finite for the space $\mathcal{P}(m, \mathbb{R})$, the dimension of the space of triangles is

$$2 \dim(A^+) + \dim(K) = 2(m-1) + (m-1)m/2,$$

while the dimension of the space of three pairs is $3(m-1)$. □

Theorem 8. *Consider two n -tuples of points, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, in general position in $\mathcal{P}(m, \mathbb{R})$. If all the corresponding triples $(x_{i_1}, x_{i_2}, x_{i_3})$ and $(y_{i_1}, y_{i_2}, y_{i_3})$ are congruent with respect to $SL(m, \mathbb{R})$, then x and y are congruent.*

Proof. We may assume that the first three points of the two triples coincide, and that $x_1 = y_1 = I$, $x_2 = y_2 = a \in A(m)$, and $x_3 = y_3 = p$. For some given k , let $x_k = q$ and $y_k = q'$. We wish to show that $q = q'$.

Now by assumption we have $(I, a, q) \cong (I, a, q')$ mod $SL(m)$. Hence there is an $\epsilon = \text{diag}(\epsilon_1, \dots, \epsilon_m) \in M$ such that $q' = \epsilon q \epsilon$. Since M is finite, for each ϵ there must be an open set of q for which this equation holds.

By assumption we also have $(I, p, q) \cong (I, p, q')$. Hence $\text{tr}(pq) = \text{tr}(pq') = \text{tr}(p\epsilon q \epsilon)$ for an open set of (p, q) . Rewriting this in terms of matrix entries, we have $\sum p_{ij}q_{ij} = \sum p_{ij}q_{ij}\epsilon_i\epsilon_j$. But this is only possible if $\epsilon_i = \epsilon_j$ for all i, j . In other words, $\epsilon = 1$ and $q = q'$, as desired. \square

5. Compact symmetric spaces

We will now discuss how to approach the compact case. We must restrict our attention to configurations of points that are “small”. For example, we may require that the whole configuration stays within the conjugate locus of each one of its points. The flats then become flat tori, and the chambers are closed linear simplexes.

We will consider $SU(m)/SO(m)$, the dual of $SL(m, \mathbb{R})/SO(m)$.

Theorem 9. *Consider $SU(m)/SO(m)$. Let A be the set of diagonal matrices in $SU(m)$, and let M be the finite group consisting of $\text{diag}(\epsilon_1, \dots, \epsilon_m)$, with $\epsilon_i = \pm 1$ and $\epsilon_1 \dots \epsilon_m = 1$. Let (u, v) be the symmetric bilinear form $u_1v_1 + \dots + u_mv_m$ on \mathbb{C}^m .*

1. *The functions $f_i(u) = \text{tr}(uu^t)^i$, $i = 1, \dots, m$, form a complete and independent set of (K, K) -invariant functions on $SU(m)$. From the real and imaginary parts of these functions we can obtain a complete set of cosine laws for $SU(m)/SO(m)$ by Theorem 4.*

2. *Let \vec{u}_i be the column vectors of u . The functions*

$$g_{ij}(u) = (\vec{u}_i, \vec{u}_j)^2 (\vec{u}_i, \vec{u}_i)^{-1} (\vec{u}_j, \vec{u}_j)^{-1},$$

for $i, j = 1, \dots, m$, form a complete set of (K, AM) -invariant functions on $SU(m)$. From the real and imaginary parts of these functions we can obtain a complete set of sine laws for $SU(m)/SO(m)$ by Theorem 5.

The symmetric spaces $SU(m)/SO(m)$ and $SL(m, \mathbb{R})/SO(m)$ are dual to one another. In particular, their Lie algebras $\mathfrak{su}(m)$ and $\mathfrak{sl}(m, \mathbb{R})$ are real forms of $\mathfrak{sl}(m, \mathbb{C})$. They also have the same isotropy group $SO(m)$. Comparing Theorems 6 and 9, we see that we have found a set of functions on $SL(m, \mathbb{C})$ that restrict to (K, K) - and (K, AM) -invariant functions on the groups of both symmetric spaces.

This is a general phenomenon, and gives rise to a correspondence between the trigonometric laws for dual pairs of symmetric spaces. The trigonometric laws described above are expressed in terms of the coordinates of the corners of the triangles, i.e., matrices in $\mathcal{P}(m, \mathbb{C})$ or $SU(m)$. But we can also write $x_i = \exp V_i$ where

V_i is a tangent vector. This will give relations between the tangent vectors describing the sides of the triangle. We will call such relations tangential trigonometric relations. Such relations have many advantages, and they will be discussed in detail later on. In particular, it is easier to describe the relationship between the laws for compact and noncompact spaces using tangential relations. Let $\mathfrak{p}(m, \mathbb{C})$ denote the set of symmetric matrices with complex entries and trace 0. Let $SO(m)$ act on this space by conjugation. The restriction of this representation to the symmetric matrices with real entries gives us the isotropy representation of $SL(m, \mathbb{R})/SO(m)$, while its restriction to those with purely imaginary entries yields the isotropy representation of $SU(m)/SO(m)$. The complex exponential map restricts to the exponential maps of each of the two spaces. Thus the substitution $V \rightarrow iV$ establishes a correspondence between the tangential trigonometric relations for the two spaces. The substitutions $\lambda \rightarrow i\lambda$, $\text{tr}(V_{i_1} \dots V_{i_d}) \rightarrow i^d \text{tr}(V_{i_1} \dots V_{i_d})$ transform a tangential trigonometric formula on X to such a formula on its dual symmetric space X' .

An obvious question is now how the results of this section are related to the laws of trigonometry for $SU(3)$ derived in [As]. If we consider $SU(3)$ as a compact Lie group and try to study the geometry of Cartan cells, we can easily derive the cosine laws, but it is not clear how to get the sine laws. In order for this method to work, we have to view $SU(3)$ as $SU(3) \times SU(3)/SU(3)$, or imbed it in $SL(6, \mathbb{R})/SO(6)$. But in Section 6 we will discuss a method which is more comparable to the results in [As], and which also applies to both the compact and the noncompact case.

6. Invariants of n points and their relations

Let $X = G/K$ be a symmetric space of compact or noncompact type. Consider an n -tuple of points $x = (x_1, \dots, x_n)$. By Theorem 8, its congruence class is determined once we know the congruence class of all its subtriples. Hence all the congruence invariants of x and their relations can be expressed in terms of the invariants and relations associated to the subtriples. Nevertheless, it is of interest to discuss the invariants of n points and the trigonometric relations among such invariants directly. This will also give us an alternative method for deriving the laws of trigonometry for both compact and noncompact spaces. It turns out that this method is also more suitable for computation.

Suppose there are two observers, A and B , attached to the points x_1 and x_2 , respectively. Let $g \in G$ be an isometry that takes x_1 to the base point I and $h \in G$ an isometry that takes x_2 to I . (We can assume that the basepoint is I since any noncompact space can be imbedded in $\mathcal{P}(m, \mathbb{R})$ and any compact space can be imbedded in $SO(m)$.) Then

$$\begin{aligned} (x_1, x_2, \dots, x_n) &\cong (gx_1, gx_2, \dots, gx_n) = (I, p_2, p_3, \dots, p_n) \\ &\cong (hx_1, hx_2, \dots, hx_n) = (q_1, I, q_3, \dots, q_n). \end{aligned} \tag{3}$$

Let $\mathcal{T} = \{\tau\}$ be a complete set of K -invariant functions in the n variables x_1, \dots, x_n . (Such a set exists since K is compact.) We define the trigonometric data observed by A to be the set of values

$$\tau(gx_1, gx_2, \dots, gx_n) = \tau(I, p_2, p_3, \dots, p_n). \quad (4)$$

This is independent of the choice of g , and it is a trigonometric quantity associated to the n -tuple x .

Similarly, we define the trigonometric data observed by B to be the set of values

$$\tau(hx_1, hx_2, \dots, hx_n) = \tau(q_1, I, q_3, \dots, q_n). \quad (5)$$

If we set $g = p_2^{-1/2}$, then $g = g^t$ and $z \rightarrow gzg^t = gzg$ is an isometry which moves p_2 to I . Hence there is an isometry $k \in K$ such that

$$(gIg, gp_2g, gp_3g, \dots, gp_n g) = (p_2^{-1}, I, gp_3g, \dots, gp_n g) = k(q_1, I, q_3, \dots, q_n).$$

(Notice that q_1 and p_2^{-1} are conjugate. This corresponds to the side joining x_1 and x_2 .) But then the data observed by B is of the form

$$\tau(q_1, I, q_3, \dots, q_n) = \tau(p_2^{-1}, I, p_2^{-1/2}p_3p_2^{-1/2}, \dots, p_2^{-1/2}p_n p_2^{-1/2}), \quad (6)$$

and since the right hand side only involves the p_i , we might be able to express it in terms of the trigonometric data observed by A, i.e., $\tau(I, p_2, \dots, p_n)$. This will give us the trigonometric relations we desire. In particular, for $n = 2$ we get the laws of trigonometry for triangles.

Since any noncompact symmetric space can be imbedded in $\mathcal{P}(m, \mathbb{R})$ and any compact symmetric space can be imbedded in $SO(m)$, we need to consider orthogonal invariants of symmetric or skewsymmetric matrices. In either case we can take \mathcal{T} to be a set of functions of the form $\tau(x) = \text{tr}(x_{i_1} \cdots x_{i_k})$ ([Sib]). Then the trigonometric relations we require are of the form

$$\text{tr}(q_{i_1} q_{i_2} \cdots q_{i_k}) = \text{tr}(p_2^{-1/2} p_{i_1} p_2^{-1} p_{i_2} \cdots p_2^{-1} p_{i_k} p_2^{1/2}) = \text{tr}(p_2^{-1} p_{i_1} p_2^{-1} p_{i_2} \cdots p_2^{-1} p_{i_k}), \quad (7)$$

where p_1 and q_2 are taken to be I . In other word, for $f(x_1, \dots, x_n)$ we get

$$f(q_1, q_3, q_4 \dots, q_n) = f(p_2^{-1}, p_2^{-1} p_3, p_2^{-1} p_4 \dots, p_2^{-1} p_n). \quad (8)$$

See [AH] for details on how to reduce such expressions. We can also express this in terms of tangential data. We can write $p_i = e^{V_i}$ and $q_i = e^{W_i}$, where $V_i, W_i \in \mathfrak{p}$ and try to find relations between the tangent vectors V_i and W_i . Notice that W_2 and $-V_1$ are conjugate, since they represent the side joining x_1 and x_2 (see Figure 2).

Let $\mathcal{T}' = \{\tau'\}$ be a complete set of K -invariant functions in the $n - 1$ variables U_1, \dots, U_{n-1} . (Such a set exists since K is compact.) We define the tangential data observed by A to be the set of values

$$\tau'(V_2, \dots, V_n), \quad (9)$$

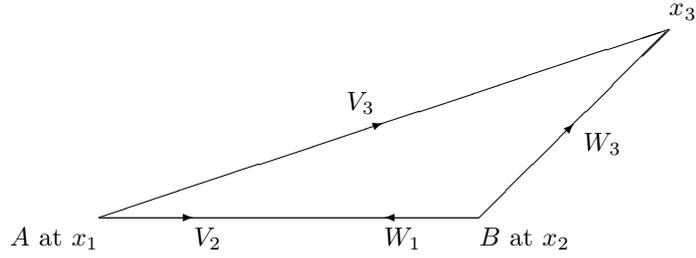


Figure 2: Relating the tangential trigonometric data of two observers

and the tangential data observed by B to be the set of values

$$\tau'(W_1, W_3, \dots, W_n). \quad (10)$$

Since we can imbed our space into $\mathcal{P}(m, \mathbb{R})$ or $SO(m)$, the tangential trigonometric data observed by A is of the form

$$\text{tr}(V_{i_1} \cdots V_{i_k}) \quad (11)$$

and the tangential trigonometric data observed by B is of the form

$$\text{tr}(W_{i_1} \cdots W_{i_k}). \quad (12)$$

These are the corner invariants of Hsiang ([Hs]) and Aslaksen ([As]).

Symmetric functions in the eigenvalues $\lambda_i(V_j)$ of V_j can easily be determined from $\text{tr}(V_i^k)$, so we will also include them as part of the tangential trigonometric data. Geometrically, the λ_i measure the interval between the observer and the observed point.

We call relations between the tangential data at the different corners for tangential trigonometric relations. Relation (8) now becomes

$$f(e^{W_1}, e^{W_3}, e^{W_4}, \dots, e^{W_n}) = f(e^{-V_2}, e^{-V_2}e^{V_3}, e^{-V_2}e^{V_4}, \dots, e^{-V_2}e^{V_n}). \quad (13)$$

This will give us tangential trigonometric relations once we are able to express the trigonometric data observed at the group level,

$$\text{tr}(e^{V_{i_1}} \cdots e^{V_{i_k}}), \quad (14)$$

in terms of the tangential trigonometric data

$$\text{tr}(V_{i_1} \cdots V_{i_k}). \quad (15)$$

This is accomplished by the following identity.

Theorem 10. *Let $M, P \in M(m, \mathbb{C})$ and let f be an analytic function in one variable. Let $\lambda(M) = (\lambda_1(M), \dots, \lambda_m(M))$ be the eigenvalues of M . Then*

$$\text{tr}(f(M)P) = \sum_{k=0}^{m-1} h_k(\lambda(M)) \text{tr}(M^k P),$$

where the h_k are of the form

$$h_k(\lambda(M)) = \sum_i f(\lambda_i(M)) R_{ik}(\lambda(M)),$$

and the R_{ik} are symmetric, rational functions in $\lambda(M)$.

Proof. We give the proof in the case where M is diagonalizable with distinct eigenvalues, and P is arbitrary. This forms an open dense subset of matrices, and the remaining cases will follow by proceeding to the limit.

We can write $M = gdg^{-1}$ where d is diagonal. Then $gf(d)g^{-1} = f(M)$ also. Furthermore, since the eigenvalues of d are distinct, the set $1, d, d^2, \dots, d^{m-1}$ form a basis for the set of diagonal matrices, \mathfrak{A} .

There is another basis e_1, e_2, \dots, e_m for \mathfrak{A} , where e_i is the matrix with 1 at the ii -position and 0 elsewhere. These two bases are related as follows

$$(1 \ d \ d^2 \ \dots \ d^{m-1}) = (e_1 \ e_2 \ \dots \ e_m)V(\lambda),$$

where $V(\lambda)$ is the Vandermonde matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{m-1} \\ & & & \vdots & \\ 1 & \lambda_m & \lambda_m^2 & \dots & \lambda_m^{m-1} \end{pmatrix}.$$

We let $R_{ik}(\lambda)$ be the ki -th entry of $V(\lambda)^{-1}$ and define $h_k(\lambda)$ as in the statement of the theorem. Now $e_i = \sum_k R_{ik}(\lambda)d^k$. Hence

$$\begin{aligned} \operatorname{tr}(f(M)P) &= \operatorname{tr}(gf(d)g^{-1}P) = \operatorname{tr}(f(d)g^{-1}Pg) \\ &= \sum_{i,k} f(\lambda_i) \operatorname{tr}(R_{ik}d^k g^{-1}Pg) = \sum_{i,k} f(\lambda_i) R_{ik} \operatorname{tr}(M^k P) = \sum_k h_k(\lambda) \operatorname{tr}(M^k P). \end{aligned}$$

The theorem now follows. The functions R_{ik} can be computed explicitly by inverting V . \square

As a remark, we would like to point out that by setting $f(M) = M^m$, we can derive the Cayley-Hamilton Theorem from Theorem 10.

We can now take $f = \operatorname{Exp}$ and apply Theorem 10 repeatedly to the trigonometric relations (7). The resulting trigonometric formulas are relations among the tangential data. But notice that the tangential relations are just restatements of the original relations. We must first determine the set of K -invariants of in (x_1, \dots, x_n) . These computations have been carried out for $SU(3)$ and $SL(3, \mathbb{C})/SU(3)$, giving an alternative to the methods used by Aslaksen ([As]) and Leuzinger ([Le]). The details will be published elsewhere ([AH]).

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