Linear differential equations in exponential extensions

Anne FREDET

Laboratoire Gage - École polytechnique, 91 128 Palaiseau cedex, France

Abstract

We present an algorithm to compute rational solutions of linear differential equations with coefficients in exponential extensions of monomial extensions of a base field. We focus on the system of generators describing the extension and show why some of the generators sets are more "suitable" than others. This results partially improves and generalizes the method presented in [Sin91] to find liouvillian solutions of linear differential equations with coefficients in liouvillian extension of $C(x)$.

1. Introduction

In the setting of linear differential equations, one is usually interested in *rational* solutions, i.e. solutions in the field that defines the coefficients of the equation.

In [Sin91] M.F. Singer presents a method to find liouvillian, and then rational solutions of linear differential equations with coefficients in almost all liouvillian extensions of $C(x)$ (see [Sin91, Theorem 4.2]) . This method although effective was not very efficient. In this article, we focus on the exponential extensions. We first outline the method presented in [Sin91] to compute the rational solutions of linear differential equations, and then present improvements provided by a suitable choice of the set of the exponential elements defining the extension.

Let L be a linear differential equation with coefficients in a differential field $K(t)$. There are mainly three steps to find rational solutions y of L: first, to compute the normal part of the denominator of y i.e. compute a polynomial $P \in K[t]$ such that if $L(y) = 0$ then yP in $K[t]$ or in $K[t, t^{-1}]$ if t is exponential. (yP is *reduced* - see [Bro97, Definition 3.5.2]). Then, after a change of variable, one is interested by polynomial solutions of L (or Laurent polynomial solutions if t is exponential). The second step deals with bounds on the degree (and the valuation) of polynomial solutions. In the last step, one computes the coefficients of the polynomial solutions.

Let (K, D) be a differential field and θ transcendent over K such that $\frac{D\theta}{\theta}$ is in

K, and the constant field is not extended. Consider a monic linear differential operator

$$L = \sum_{i=0}^{n} A_i D^i \text{ where } A_i \text{ is in } K(\theta) \text{ and } A_n = 1$$

In the method given in [Sin91], there are still three steps to compute the rational solutions of $L(y) = 0$:

1. Compute the normal part of the denominator:
   The *normal* part of a polynomial is the part which is prime with its derivative (see [Bro97, Definition 3.4.2]). Assume that $L(y) = 0$ for some y in $K(\theta)$. Given an irreducible polynomial p, consider the p-adic expansion of y i.e. $y = \frac{u}{p^\alpha} + \cdots$ where α is in $\mathbb{N}_{>0}$ and u in $K[\theta]$ satisfies $\deg_\theta(u) < \deg_\theta(p)$. Let be the following p-adic expansions of the A_i's: $A_i = \frac{a_{i,\alpha_i}}{p^{\alpha_i}} + \cdots$ where α_i is in $\mathbb{Z}$ and a_{i,α_i} in $K[\theta]$ such that $\deg_\theta(a_{i,\alpha_i}) < \deg_\theta(p)$. Then $A_i y^{(i)} = \frac{u(Dp)^i \alpha \cdots (\alpha+i-1)}{p^{i\alpha+\alpha_i}} + \cdots$. The leading term of $L(y)$ vanishes and then two exponents must be equal: $i\alpha + \alpha_i = j\alpha + \alpha_j$ for some $i \neq j$ and then $\alpha_i \neq \alpha_j$ for some $i \neq j$. Furthermore, the exponent considered is the maximal on $\{\alpha_i + i\alpha \ / \ 1 \leq i \leq n\}$ and $\alpha_n = 0$ which implies that $\alpha_i > 0$ for some i. The polynomials p that appear in the denominator of a solution are factors of the denominator of the coefficients. One also finds a bound on the order α, using indicial equation. A change of variable reduces the problem of finding Laurent polynomial solutions of linear differential equations (still denoted L for convenience).

2. Find a bound for the degree and the valuation of Laurent polynomial solutions:
   Let

   $$Y = \sum_{i=\delta}^{\gamma} y_i \theta^i \text{ with } \begin{cases} \delta, \gamma \text{ in } \mathbb{Z} \text{ where } \delta \leq \gamma, \\ y_i \text{ in } K \text{ and } y_\delta \neq 0, y_\gamma \neq 0 \end{cases}$$

   and

   $$L = \sum_{i=\nu}^{\mu} \theta^i L_i \text{ with } \begin{cases} \nu, \mu \text{ in } \mathbb{Z} \text{ where } \nu \leq \mu, \\ L_i \text{ in } K[D] \text{ and } L_\nu \neq 0, L_\mu \neq 0 \end{cases}$$

   If one writes $L(Y)$ with respect to the power of θ, one observes that

   $$L(Y) = 0 \text{ implies that } L_\nu(y_\delta t^\delta) = 0 \text{ and } L_\mu(y_\gamma t^\gamma) = 0$$

   Then one computes the exponential solutions of L_ν and L_μ i.e. u and v in K such that $L_\nu(e^{\int u}) = 0$ and $L_\mu(e^{\int v}) = 0$. Given an exponential solution $e^{\int w}$, one wants to know if $e^{\int w} = f\theta^\beta$ for some f in K and β in $\mathbb{Z}$. This is equivalent to consider solutions of the equation $y' + wy = 0$ in $K(\theta)$. Using [Sin91, Lemma 3.5] (see also Risch's work, [RC79], [SSC85]), one can find an answer to the question in almost any liouvillian extension of $C(x)$.

3. Compute the coefficients of Laurent polynomial solutions:
   Writing $L(Y)$ with respect to the powers of θ, one has a linear differential system with coefficients in K in the form $AY = B$. Using non-commutative linear algebra (see [Poo60] or [Chy98] for example), one finds matrices U and V in $K[D]$ such that U has a left inverse, V has a right inverse, and $UAV = C$ is diagonal. So Y is a solution of $AY = B$ if and only if $W = V^{-1}Y$ is a solution of $CW = UB$. This system is equivalent to $n+1$ linear differential equations with coefficients in K.

An alternative method to the p-adic expansions for computing the normal part of the denominator is proposed in [Bro92]. In [BF99], several improvements are obtained on the last two steps in the case of extensions of the form $C(x, \exp(\int f(x)dx))$, and in [Fre00] this method is adapted to extensions of $C(x)$ generated by iterated logarithms and exponentials. In this article, we propose improvements for the two last steps when considering exponential extensions with more general expression.

In the third step, we remark that the system has a special form: if $L(Y) = 0$ for $Y = y_\delta\theta^\delta + \ldots + y_\gamma\theta^\gamma$ then $M\overrightarrow{Y} = 0$ where

$$M = \begin{pmatrix} * & 0 & 0 & \cdots & \\ * & * & 0 & \cdots & \\ \vdots & & \ddots & & \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \\ \vdots & & & & \\ * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & * \end{pmatrix} \text{ and } \overrightarrow{Y} = \begin{pmatrix} y_\delta \\ y_{\delta+1} \\ \vdots \\ y_{\gamma-1} \\ y_\gamma \end{pmatrix}$$

We use this special form to avoid writing the system. This method relies in recurrence relation presented in [ABP95] to solve linear differential equations with coefficients in $C(x)$.

In the second step, we remark that if the extension is defined with a suitable system of generators, we can avoid the computation of all the exponential solutions of L_μ and L_ν, and focus on solutions with the form $f\theta^\beta$ for f in K and β in $\mathbb{Z}$. Moreover we find β giving us the bounds on the degree and the valuation of Laurent polynomial solutions of L. Let us illustrate this with an example:

EXAMPLE 1: *Consider the following linear differential equation:*

$$L(y) := (1+6x)y'' + (-60x^2 - 13 - 52x)y' + (96x^3 + 28 + 104x + 208x^2)y = 0$$

Assume we are interested by the solutions in $C(x, \exp(x^2), \exp(x^2+x))$. *We denote* $\theta_1 = \exp(x^2)$ *and* $\theta_2 = \exp(x^2+x)$. *If there exists* Y *in* $C(x)[\theta_1, \theta_2, \theta_1^{-1}, \theta_2^{-1}]$ *such that* $L(Y) = 0$ *then there are solutions with the form* $Y = f\theta_1^{\gamma_1}\theta_2^{\gamma_2}$ *with* f *in* $C(x)$ *and* γ_1, γ_2 *in* $\mathbb{Z}$ *(see [Ros75]).*
Therefore, the following equalities hold:

$$\begin{aligned} Y' &= (f' + (2x)\gamma_1 f + (2x+1)\gamma_2 f)\theta_1^{\gamma_1}\theta_2^{\gamma_2} \\ &= (2x(\gamma_1+\gamma_2)f + \gamma_2 f + f')\theta_1^{\gamma_1}\theta_2^{\gamma_2} \\ Y'' &= (4x^2(\gamma_1+\gamma_2)^2 + \cdots)\theta_1^{\gamma_1}\theta_2^{\gamma_2} \end{aligned}$$

Expanding f *as series at the infinity* ($f = cx^\alpha + \cdots$ *with* $c \neq 0$ *and* α *is in* $\mathbb{Z}$), *we conclude that the leading term of* Y' *is* $2(\gamma_1+\gamma_2)c$ *and the leading term of* Y'' *is* $4(\gamma_1+\gamma_2)^2 c$. *Therefore* $L(Y) = 0$ *implies that*

$$c(6 \times 4(\gamma_1+\gamma_2)^2 - 60 \times 2(\gamma_1+\gamma_2) + 96) = 0$$

Since $c \neq 0$, *we have a finite set of choices for* $\gamma_1+\gamma_2$, *but neither for* γ_1 *nor* γ_2. *If we consider* $C(x, \exp(x^2), \exp(x))$ *as the base field (which is differentially isomorphic to* $C(x, \exp(x^2+x), \exp(x^2))$ *), the previous computation provides us a finite set of choices for the exponents of* $\exp(x^2)$ *(here* $\{1, 4\}$*). After a change of variable, we proceed in the same way and find a finite set of choices for the exponents of* $\exp(x)$. *We have then a finite set of choices for* (γ_1, γ_2). *Changing the variables reduces the problem to find rational solutions of linear differential equations with coefficients in* $C(x)$.

The previous example shows us that if the system of generators is suitable it is possible to find bounds directly by using expansion at infinity.

This paper is organized as follows: in the next section we present the improvements of Singer's method in *flat* exponential extensions of $C(x)$. Such extensions are defined by adding simultaneously several exponential variables over $C(x)$. We introduce the notion of *well-defined* extension when the system of generators is "suitable" and present an algorithm that given a set of generators for an exponential extension, computes a suitable set of generators such that this extension is well-defined. We prove that in such extensions the computation of Laurent polynomial solutions of linear differential equations requires only expansions at infinity or p-adic expansions for some irreducible polynomial p. Finally we generalize this approach to exponential extensions of monomial extensions. This allows us to consider extensions that are not liouvillian, and extend some of the results presented in [Sin91].

Acknowledgements
I thank J.-A. Weil for his support and suggestions, M.F. Singer for his comments, and specially his "naive" idea for the proof of proposition 3.3. and M. Bronstein for helping me to clarify the definitions.

2. Exponential extensions over $C(x)$

We denote by $\mathrm{Const}(K) = \{a \in K / a' = 0\}$ the *constant field* of K.

In this section, we consider the exponential extensions over $C(x)$. An element θ is *exponential* over $C(x)$ if θ is transcendent over $C(x)$, $\frac{\theta'}{\theta}$ is in $C(x)$ and $\mathrm{Const}(C(x,\theta)) = \mathrm{Const}(C(x)) = C$. We also say that $C(x,\theta)$ is an *exponential extension* of $C(x)$.

We add simultaneously several exponential variables over $C(x)$. Let us introduce the notion of *flat exponential extension*:

Definition 1: *A* flat exponential extension *of $C(x)$ is an extension $C(x, \theta_1, \ldots, \theta_l)$ such that, for any $c_i \in \mathbb{Q}$, $\prod \theta_i^{c_i}$ is exponential over $C(x)$.*

As observed in example 1, if the system of generators is suitable, the bounds on the degree and the valuation of Laurent polynomial solution of linear differential equations can be directly computed. We now specify the notion of suitable system of generators:

Definition 2: *A flat exponential extension $C(x, \theta_1, \ldots, \theta_l)$ of $C(x)$ is a* well-defined *extension if denoting $\frac{\theta_i'}{\theta_i}$ by g_i then for any subset $\mathcal{N} \in \{1, \ldots, l\}$*

1. *g_i can be written*

$$g_i \;=\; c_i x^{\alpha_i} + \cdots \text{ where } c_i \text{ is in } C \text{ and } \alpha_i \text{ in } \mathbb{Z}$$

   and if we define $\alpha = \max_{j \in \mathcal{N}}(\alpha_j)$ then $\alpha \geq 0$ and the $(c_i)_{i \in \mathcal{N}}$'s such that $\alpha_i = \alpha$ are $\mathbb{Q}$-linearly independent,

2. *or we have the following p-adic expansions for some irreducible polynomial p:*

$$g_i \;=\; \frac{u_i}{p^{\alpha_i}} + \cdots \text{ where } u_i \in C[x], \deg_x(u_i) < \deg_x(p) \text{ and } \alpha_i \in \mathbb{Z}$$

   and if we define $\alpha = \max_{j \in \mathcal{N}}$ then $\alpha \geq 0$ and the $(u_i)_{i \in \mathcal{N}}$'s such that $\alpha_i = \alpha$ are $\mathbb{Q}$-linearly independent. Furthermore they are $\mathbb{Q}$-linearly independent with $p' \mod p$ if $\max(\alpha_j) = 1$.

If $C(x, \theta_1, \ldots, \theta_l)$ is well-defined then for any subset $\mathcal{N} = \{n_1, \ldots, n_k\} \subset \{1, \ldots, l\}$ the extension $C(x, \theta_{n_1}, \ldots, \theta_{n_k})$ is well-defined.

Remark 1: *A flat extension is well-defined if, for some valuation, the logarithmic derivatives of the exponential variables have different orders or leading terms $\mathbb{Q}$-linearly independent, eventually also $\mathbb{Q}$-linearly independent with $p' \mod p$ in the case of order* 1*.*

Example 2: *The extension $C(x, e^{\int x^3+x}, e^{\int x^3+3x})$ is a flat exponential extension of $C(x)$. It is not well-defined but is isomorphic to $C(x, e^{\int x^3+x}, e^{\int 2x})$, which is well-defined.*

2.1. Laurent polynomial solutions of linear differential solutions

2.1.1. Riccati equation

Let u be in $C(x)$ and $y = e^{\int u}$. Then $y^{(i)} = P_i(u, u', \ldots, u^{(n-1)})e^{\int u}$ where the P_i's are polynomials such that

$$\begin{aligned} P_0 &= 1 , \\ P_i &= P'_{i-1} + uP_{i-1} \text{ for } i \geq 1 \end{aligned}$$

Let $L = A_n D^n + A_{n-1}D^{n-1} + \cdots + A_0$ be a linear differential operator, with coefficients in $C(x)$. Let $y = e^{\int u}$ with u in $C(x)$. Then

$$L(y) = 0 \Leftrightarrow A_n P_n(u, \ldots, u^{(n-1)}) + A_{n-1}P_{n-1}(u, \ldots, u^{(n-2)}) + \cdots + A_0 = 0$$

This equation is the *Riccati equation associated to* L *.*
We recall the following lemmas, adapted from [Sin91, Lemma 2.2]

Lemma 1: *Let u be in $C(x)$ with the following expansion: $u = u_\beta x^\beta+$ lower order terms. If $\beta \geq 1$ then $P_i = (u_\beta)^i x^{i\beta}+$ lower order terms in x.*

Proof: By induction on i. □

Lemma 2: *Let $p(x)$ be an irreducible polynomial in $C[x]$. Let u be in $C(x)$ with the following p-adic expansion: $u = \frac{u_\beta}{p^\beta}+$higher order terms with $\beta > 0, u_\beta \neq 0$ and $\deg_x(u_\beta) < \deg_x(p)$.*

1. *If $\beta > 1$ then $P_i(u, \ldots, u^{(n-1)}) = \frac{v_{i,\beta}}{p^{i\beta}}+$ higher order terms, where $v_{i,\beta} \equiv (u_\beta)^i \mod p$.*
2. *If $\beta = 1$ and if u_β is prime with $p' \mod p$ then $P_i(u, \ldots, u^{(i-1)}) = \frac{v_i}{p^i}+$ higher order terms, where $v_i \equiv \prod_{j=0}^{i-1}(u_1 - jp') \mod p$.*

Proof: By induction on i. □

2.1.2. Bounds on the degree and the valuation

At this point of the computation, we need to bound the degree and the valuation of Laurent polynomial solutions of linear differential equations with coefficients in a flat well-defined exponential extension of $C(x)$. We first consider the solutions in the exponential extension of linear differential equations with coefficients in $C(x)$:

Proposition 2.1: *Let $K = C(x, \theta_1, \ldots, \theta_l)$ be a well-defined extension of $C(x)$ and let $L = \sum_{i=1}^{n} A_i D^i$ be a linear differential operator with coefficients in $C(x)$. If there exists $Y = f\theta_1^{\beta_1} \ldots \theta_l^{\beta_l}$ such that $L(Y) = 0$ for some f in $C(x)$ and $(\beta_1, \ldots, \beta_l)$ in $\mathbb{Z}^l$ then there exists a computable finite set of choices for $(\beta_1, \ldots, \beta_l)$.*

Proof: Denote $\frac{\theta_i'}{\theta_i}$ by g_i. We proceed by induction on l. The case $l = 0$ is trivial. The case $l = 1$ is in [BF99]. Assume that $l > 1$ and that there exists $Y = f\theta_1^{\beta_1}\dots\theta_l^{\beta_l}$ where f is in $C(x)$ and $(\beta_1,\dots,\beta_l)$ in $\mathbb{Z}^l$ such that $L(Y) = 0$. Then, from definition 2, there are two possibilities:

- In the first one, we write

$$g_i \quad = \quad c_i x^{\alpha_i} + \cdots \text{ where } c_i \text{ is in } C \text{ and } \alpha_i \text{ in } \mathbb{Z},$$

  with some i in $\{1,\dots,l\}$ such that $\alpha_i \geq 0$ and the $(c_i)_{i=1,\dots,l}$ such that $\alpha_i = \max_j(\alpha_j)$ are $\mathbb{Q}$-linearly independent. We define $\alpha = \max(\alpha_j)$ and so we have $Y = f\theta_1^{\beta_1}\dots\theta_l^{\beta_l} = \exp(\int(\sum_{i|\alpha_i=\alpha}\beta_i c_i)x^\alpha + \cdots$. From lemma 1 we have $P_i = (\sum_{i|\alpha_i=\alpha}\beta_i c_i)^i x^{i\alpha} +$ lower order terms. As $L(Y) = 0$ we conclude that $\sum A_i P_i = 0$ and then the leading term of $\sum A_i P_i$ vanishes. We denote

$$A_i = a_i x^{\psi_i} + \cdots \text{ where } a_i \text{ is in } C \text{ and } \psi_i \text{ in } \mathbb{Z}.$$

  This gives us $\sum_{i|\psi_i+i\alpha=\max(\psi_j+j\alpha)} a_i(\sum_{i|\alpha_i=\alpha}\beta_i c_i)^i = 0$. The $(c_i)_{i|\alpha_i=\max(\alpha_j)}$ are $\mathbb{Q}$-linearly independent and so we can compute a finite set of choices for $(\beta_i)_{i|\alpha_i=\max(\alpha_j)}$. By using a change a variables and the induction hypothesis, we get a finite set of choices for $(\beta_1,\dots,\beta_l)$.

- in the second possibility we consider the following expansions for some irreducible polynomial p:

$$g_i \quad = \quad \frac{u_i}{p^{\alpha_i}} + \cdots \text{ where } u_i \text{ is in } C[x],\ \deg_x(u_i) < \deg_x(p) \text{ and } \alpha_i \text{ in } \mathbb{Z}$$

  Then there exists i such that $\alpha_i \geq 0$ and the $(u_i)_{i=1,\dots,l}$ such that $\alpha_i = \max_j(\alpha_j)$ are $\mathbb{Q}$-linearly independent and $\mathbb{Q}$-linearly independent with p' mod p if $\alpha_i = 1$. Let $\alpha = \max(\alpha_j)$.
  In this case, $Y = f\theta_1^{\beta_1}\dots\theta_l^{\beta_l} = \exp(\int\frac{(\sum_{i|\alpha_i=\alpha}\beta_i u_i)}{p^\alpha} + \cdots$. Using lemma 2 , we have

  1. If $\alpha > 1$ then $P_i(u,\dots,u^{(n-1)}) = \frac{v_{i,\alpha}}{p^{i\alpha}} +$ higher order terms, where

$$v_{i,\alpha} \equiv (\sum_{i|\alpha_i=\alpha}\beta_i u_i)^i \mod p$$

  2. If $\alpha = 1$ then $P_i(u,\dots,u^{(i-1)}) = \frac{v_i}{p^i} +$ higher order terms, where

$$v_i \equiv \prod_{j=0}^{i-1}((\sum_{i|\alpha_i=1}\beta_i u_i) - jp') \mod p$$

  But $L(Y) = 0$ is equivalent to $\sum A_i P_i = 0$ and then the leading term of $\sum A_i P_i$ vanishes. We denote

$$A_i = \frac{a_i}{p^{\psi_i}} + \cdots \text{ where } a_i \text{ is in } C[x], \deg_x(a_i) < \deg_x(p) \text{ and } \psi_i \text{ in } \mathbb{Z}.$$

If $\alpha > 1$, this gives $\sum_{i|\psi_i+i\alpha=\max(\psi_j+j\alpha)} a_i(\sum_{i|\alpha_i=\alpha}\beta_i u_i)^i \equiv 0 \mod p$.

If $\alpha = 1$, this gives $\sum_{i|\psi_i+i=\max(\psi_j+j} a_i(\sum_{i|\alpha_i=\alpha}\prod_{j=0}^{i-1}((\sum_{i|\alpha_i=\max(\alpha_j)}\beta_i u_i) - jp'))^i \equiv 0 \mod p$. If we divide the equation by p and we write the remainder with respect to x by vanishing the coefficients we get a system of equations with coefficients in C. The $(u_i)_{i|\alpha_i=\max(\alpha_j)}$ are $\mathbb{Q}$-linearly independent, so we have a finite set of choices for $(\beta_i)_{i|\alpha_i=\max(\alpha_j)}$. By using a change of variables and the induction hypothesis we finally get a finite set of choices for $(\beta_1,\ldots,\beta_l)$. □

Notation: we denote $\mathcal{L}$ for $\{1,\ldots,l\}$, $\theta_\mathcal{L}$ for $\theta_1,\ldots,\theta_l$ and $\theta_\mathcal{L}^{-1}$ for $\theta_1^{-1},\ldots,\theta_l^{-1}$

THEOREM 2.2: *Let $K = C(x,\theta_\mathcal{L})$ be a well-defined extension of $C(x)$, $L = \sum_{i=1}^n A_i D^i$ be a linear differential operator with coefficients in $C(x)[\theta_\mathcal{L},\theta_\mathcal{L}^{-1}]$, $b_1,\ldots,b_k$ be elements of $C(x)[\theta_\mathcal{L},\theta_\mathcal{L}^{-1}]$.*
There exists computable $(m_1,\ldots,m_l)$ and $(M_1,\ldots,M_l)$ such that for Y in $C(x)[\theta_\mathcal{L},\theta_\mathcal{L}^{-1}]$ and some constant parameters c_j's if $L(Y)=\sum_j c_j b_j$ holds then $val_{\theta_i}(Y) \geq m_i$ and $\deg_{\theta_i}(Y) \leq M_i$

Proof: The proof is done for M_1 and m_1. Suppose that $L(Y) = \sum c_j b_j$ where Y is in $C(x)[\theta_\mathcal{L},\theta_\mathcal{L}^{-1}]$ and the c_j's are some constant parameters. We denote

$$L = \sum_{(j_1,\ldots,j_l)=\nu}^{\mu} \theta_1^{j_1}\ldots\theta_l^{j_l} L_j \text{ where } \begin{cases} \nu \text{ and } \mu \text{ in } \mathbb{Z}^l, \\ \nu \leq \mu \text{ with respect to the lexicographic order,} \\ L_i \text{ is in } C(x)[D] \text{ and } L_\nu \neq 0 \text{ and } L_\mu \neq 0. \end{cases}$$

and

$$Y = \sum_{(i_1,\ldots,i_l)=\delta}^{\gamma} y_i\theta_1^{i_1}\ldots\theta_l^{i_l} \text{ where } \begin{cases} \delta \text{ and } \gamma \text{ in } \mathbb{Z}^l, \\ \delta \leq \gamma \text{ with respect to the lexicographic order,} \\ y_i \text{ is in } C(x) \text{ and } y_\delta \neq 0, y_\gamma \neq 0. \end{cases}$$

We have

$$L(Y) \;=\; L_\nu(y_i\theta_1^{\delta_1}\ldots\theta_l^{\delta_l}) + \cdots + L_\mu(y_i\theta_1^{\gamma_1}\ldots\theta_l^{\gamma_l}).$$

and if $L(Y) = \sum c_j b_j$ then

- either the valuation of $L(Y) := \nu + \delta$ is greater than the valuation of the b_j's. In this case we get a minimal bound for δ. Or $L_\nu(y_i\theta_1^{\delta_1}\ldots\theta_l^{\delta_l}) = 0$ and by proposition 2.1, we get a finite sets $\mathcal{S}_m$ such that $(\delta_1,\ldots,\delta_l)$ is in $\mathcal{S}_m$. Then $m_1 = \max\{\delta_1/(\delta_1,\ldots,\delta_l) \in \mathcal{S}_m\}$.
- either the degree of $L(Y) := \mu + \gamma$ is lowest than the degree of the b_j's . In this case we get a maximal bound for γ. Or $L_\mu(y_i\theta_1^{\gamma_1}\ldots\theta_l^{\gamma_l}) = 0$ and by proposition 2.1, we then get a finite set $\mathcal{S}_M$ such that $(\gamma_1,\ldots,\gamma_l)$ is in $\mathcal{S}_M$. Then $M_1 = \max\{\gamma_1/(\gamma_1,\ldots,\gamma_l) \in \mathcal{S}_M\}$. □

REMARK 2: *This provides us bounds for the degree and the valuation of Y, seen as a Laurent polynomial with respect to θ_1, and bounds for the degree or the valuation of the leading term and the lowest term, seen as Laurent polynomial with respect to θ_2, and so on.*

2.1.3. Computation of the coefficients

In order to compute the coefficients, we proceed by induction on l: we assume that we can compute the Laurent polynomial solutions of linear differential equations with coefficients in $C(x)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$. Let $L = \sum_{i=1}^{n} A_i D^i$ be a linear differential operator with coefficients in $C(x)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and let $b_1, \ldots, b_k$ be elements of $C(x)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$.
Using the previous theorem, we compute bounds M_1 for the degree and m_1 for the valuation of Laurent polynomial solutions with respect to θ_1. Up to a change of variables, we can assume that $m_1 = 0$ and $\operatorname{val}_{\theta_1}(L) = 0$. Let

$$L = \sum_{k=0}^{\mu} \theta_1^k L_k \text{ with } L_k \text{ in } C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}][D]$$

Suppose that there exist Y in $C(x)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and constant parameters c_j such that $L(Y) = \sum c_j b_j$. If $M_1 = 0$ then Y is in $C(x)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$. In this case,

$$\begin{aligned}\sum_{k=0}^{\mu} \theta_1^l L_k(Y) &= \sum c_j b_j \\ &= \sum_{k=\min(\operatorname{val}_{\theta_1}(b_j))}^{\max(\deg_{\theta_1}(b_j))} \theta_1^k Q_k(c_j)\end{aligned}$$

where the polynomials Q_k have coefficients in $C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}]$. Equalizing the coefficients in θ_1 on both sides, we have conditions over the c_j's or linear differential equations with coefficients in $C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}]$ that can be solved by hypothesis.
If $M_1 > 0$, we decompose Y: $Y = y_0 + \theta_1 q$ where $y_0 = Y(0)$ is in $C(x)[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$ and q is in $C(x)[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ with $\deg_{\theta_1}(q) < M_1$. We have

$$L(Y) = L(y_0 + \theta_1 q) = L_0(y_0) + (L - L_0)(y_0) + L(\theta_1 q)$$

By the induction hypothesis, we can solves$L_0(y_0) = \sum c_j b_j(0)$ and so we have $f_1, \ldots, f_r$ in $C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}]$ and a matrix $\mathcal{M}$ with constant coefficients such that for all solution y_0, we have

$$y_0 = \sum_{i=1}^{r} d_i f_i \text{ and } \mathcal{M}(d_1, \ldots, d_r, c_1, \ldots, c_h)^T = 0.$$

If we change Y into $\sum d_i f_i + \theta_1 q$ in the previous equation we conclude

$$\begin{aligned}\sum c_j b_j &= L(Y) = L(y_0) + L(\theta_1 q) \\ &= \sum c_j b_j(0) + (L - L_0)(\sum d_i f_i) + \theta_1 \hat{L}(q)\end{aligned}$$

Then it follows that

$$\hat{L}(q) = \sum c_j \frac{b_j - b_j(0)}{\theta_1} - \sum d_i \frac{1}{\theta_1}(L - L_0) f_i$$

Finally we solve this equation with $M-1$ as bound for the degree of q in θ_1.

REMARK 3: *This method was introduced in [BF99] and generalized to difference equations in [Bro00].*

Link with the recurrence equation

This method is analogous to the one presented in [ABP95], where a recurrence is used to compute the coefficients of polynomial solutions of a linear differential equation with coefficients in $C(x)$. Let us consider

$$L = \sum_{k=0}^{\mu} \theta_1^k L_k \text{ with } L_k \text{ in } C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}][D]$$

Let f be an element of $C(x)[\theta_{\mathcal{L}\setminus\{1\}}, \theta^{-1}_{\mathcal{L}\setminus\{1\}}]$ and γ_1 be in $\mathbb{Z}$. We have

$$D^i(\theta_1^{\gamma_1} f) = \theta_1^{\gamma_1}(D + \gamma_1 g_1)^i(f) \text{ for all } i \text{ in } \mathbb{N}$$

where g_1 is the logarithmic derivative of θ_1: $g_1 = \frac{\theta_1'}{\theta_1}$. Consequently, for any operator L_k in $C(x)[\theta_2, \ldots, \theta_l, \theta_2^{-1}, \ldots, \theta_l^{-1}]$, we have

$$L_k(\theta_1^{\gamma_1} f) = \theta_1^{\gamma_1} S_{D \to D + \gamma_1 g_1}(L_k)(f)$$

where $S_{D \to D+\gamma_1 g_1}(L_k)$ is the resulting operator obtained if it is replaced D by $D + \gamma_1 g_1$ in the operator L_k. Then the following holds

$$L(f\theta_1^{\gamma_1}) = \sum_{k=0}^{\mu} \theta_1^k L_k(f\theta_1^{\gamma_1}) = \sum_{k=0}^{\mu} \theta_1^{k+\gamma_1} S_{D \to D+\gamma_1 g_1}(L_k)(f)$$

Assuming a polynomial solution of the form $Y = \sum_{i=0}^{M_1} \theta_1^i Y_i$ (eventually up to a change of variable), we have

$$\begin{aligned} L(Y) &= \sum_{k=0}^{\mu} \sum_{i=0}^{M_1} \theta^{i+k} S_{D \to D + i g_1}(L_k)(y_i) \\ &= \sum_{j=0}^{\mu+M_1} \theta_1^j \sum_{k=0}^{\mu} S_{D \to D+(j-k)g_1}(L_k)(y_{j-k}) \end{aligned}$$

The equality $L(y) = \sum \theta^h \hat{b_h}$ gives us a recurrence relation over the coefficients. We find again the results used for computing the bounds: If $j = k = 0$ then

$L_0(y_0) = b_0$ and if $j = \mu + M_1$ and $k = \mu$ then $S_{D\to D+M_1 g_1}(L_\mu)(y_{M_1}) = b_{\mu+M_1}$. The operator considered for the i-th step of the iteration in the specialization is $S_{D\to D+ig_1} L_0$. The important difference with the recurrence presented in [ABP95] is that the α_i are differential operators.

Improvements

There are several ways to improve the computation of the coefficients. Let us sketch several cases:

1. Using the asymptotic.
   Assuming that the extension is well-defined, the logarithmic derivatives of the exponential variables have different orders or the same order and with $\mathbb{Q}$-linearly independent leading terms, for some asymptotic scale in the sense of [RSSH96]. When computing bounds for the degree and the valuation of polynomial solutions, we obtain first the exponents according to the logarithmic derivatives being leading. It is interesting to consider the solutions as polynomial in this variables first.

   EXAMPLE 3: *Let us considers the Laurent polynomial solutions of linear differential equation with coefficients in* $C[x, \exp(x), \exp(x^2)]$*, it is preferable to consider the solution as a Laurent polynomial in* $\exp(x^2)$*, with coefficients in* $C[x, \exp(x), \exp(x)^{-1}]$.

2. Choosing the variable for the iteration
   Using different orders to bound the degree and the valuation of Laurent polynomial solutions in several variables θ_i, we have a choice for the main variable to compute the coefficients (for example, the variable with smallest difference between the degree and the valuation).

   EXAMPLE 4: *Let us consider the field* $C(x, \exp(x^2), \exp(\sqrt{2}x^2)) = C(x, \theta_1, \theta_2)$ *with the usual derivation. Let*

$$
\begin{aligned}
L \;=\; & ((6\sqrt{2}x - 10x)\theta_2^3\theta_1^7 + (1-12x^2)\theta_1^6 + (2x + 4\sqrt{2}x)\theta_2^4\theta_1 + (1 - 2x^2\sqrt{2})\theta_2)D^2 \\
& +((10 - 24x^2\sqrt{2} + 68x^2 - 6\sqrt{2})\theta_2^3\theta_1^7 + (12x + 144x^3)\theta_1^6 \\
& +(-68x^2 - 4\sqrt{2} - 24x^2\sqrt{2} - 2)\theta_2^4\theta_1 + (2\sqrt{2}x + 8x^3)\theta_2)D + (-576x^3\sqrt{2} + 624x^3)\theta_2^3\theta_1^7 \\
& +(-12 - 144x^2)\theta_1^6 + (80x^3 + 104x^3\sqrt{2})\theta_2^4\theta_1 + (-2\sqrt{2} - 8x^2)\theta_2
\end{aligned}
$$

   - *Considering the lexicographic order with* $\theta_1 > \theta_2$ *we have*

$$
\begin{aligned}
L \;=\; & ((6\sqrt{2} - 10)x\theta_1^7\theta_2^3 + \ldots + (1 - 2\sqrt{2}x^2)\theta_2)D^2 \\
& +(((-24\sqrt{2} + 68)x^2 + 10 - 6\sqrt{2})\theta_1^7\theta_2^3 + \ldots + (2\sqrt{2}x + 8x^3)\theta_2)D \\
& +(-576\sqrt{2} + 624)x^3\theta_1^7\theta_2^3 + \ldots + (-8x^2 - 2\sqrt{2})\theta_2
\end{aligned}
$$

     - *In order to bound the degree of a polynomial solution in* $C(x, \theta_1, \theta_2)$*, we have to solve the following equation*

$$(-576\sqrt{2}+624)+(-24\sqrt{2}+68)\times 2(\gamma_1+\sqrt{2}\gamma_2)+(6\sqrt{2}-10)\times 4(\gamma_1+\sqrt{2}\gamma_2)^2 = 0$$

Hence we get that $\{\gamma_1, \gamma_2\}$ *must belong to* $\{(6,0),(1,3)\}$. *Therefore* 6 *is a bound for the degree in* θ_1 *of a Laurent polynomial solutions and* 0 *is a bound for the degree in* θ_2 *of the leading term in* θ_1.

– *For the valuation, we have to solve the following equation*

$$8 \times 2(\gamma_1 + \sqrt{2}\gamma_2) - 2\sqrt{2} \times 4(\gamma_1 + \sqrt{2}\gamma_2)^2 = 0$$

Thus we conclude that $\{\gamma_1, \gamma_2\}$ *is in* $\{(0,1),(0,0)\}$

- *Considering the lexicographic order with* $\theta_1 < \theta_2$:
We have

$$\begin{aligned} L \;=\; & ((4\sqrt{2}+2)x\theta_1\theta_2^4 + \ldots + (1-12x^2)\theta_1^6)D^2 \\ & +(((-68-24\sqrt{2})x^2 - 4\sqrt{2}-2)\theta_1\theta_2^4 + \ldots + (12x+144x^3)\theta_1^6)D \\ & +(104\sqrt{2}+80)x^3\theta_1\theta_2^4 + \ldots + (-12-144x^2)\theta_1^6 \end{aligned}$$

– *In order to bound the degree of a polynomial solution* $C(x, \theta_1, \theta_2)$, *we have to solve the following equation*

$$(104\sqrt{2}+80)+(-68-24\sqrt{2})\times 2(\gamma_1+\sqrt{2}\gamma_2)+(4\sqrt{2}+2)\times 4(\gamma_1+\sqrt{2}\gamma_2)^2 = 0$$

Hence we conclude that $\{\gamma_1, \gamma_2\}$ *must belong to* $\{(0,1),(1,3)\}$

– *For the valuation, we have to solve the following equation*

$$-12 \times 4(\gamma_1 + \sqrt{2}\gamma_2)^2 + 144 \times 2(\gamma_1 + \sqrt{2}\gamma_2) = 0$$

Thus, we conclude that $\{\gamma_1, \gamma_2\}$ *must be in* $\{(6,0),(0,0)\}$

If there exists Y *in* $C(x)[\theta_1, \theta_2, \theta_1^{-1}, \theta_2^{-1}]$ *such that* $L(y) = 0$ *then* $\deg_{\theta_1}(Y) \leq 6$, $val_{\theta_1}(Y) \geq 0, \deg_{\theta_2}(Y) \leq 3$, $val_{\theta_2}(Y) \geq 0$. *So, by considering first the recurrence with* θ_2, *we have four linear differential equations with coefficients in* $C(x)[\theta_1, \theta_1^{-1}]$ *to compute the coefficients of* Y *(instead of seven linear differential equations in* $C(x)[\theta_2, \theta_2^{-1}]$ *if we start with* θ_1*).*

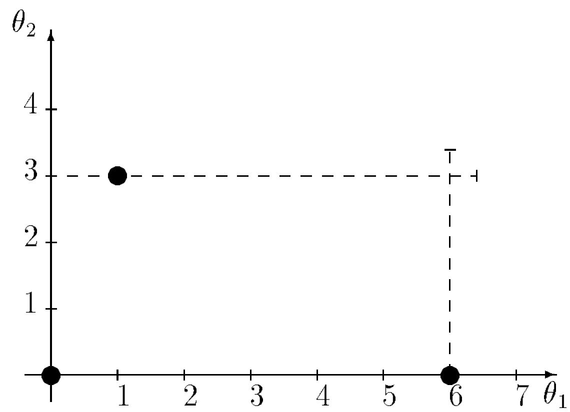
3. Using several variables
We can compute the bounds on several variables and use a recurrence on these variables simultaneously.

EXAMPLE 5: *Let us consider the field* $C(x, \exp(x^2), \exp(\sqrt{2}x^2)) = C(x, \theta_1, \theta_2)$ *with the usual derivation and the following equation:*

$$\begin{aligned} L \;=\; & ((6\sqrt{2}x - 10x)\theta_2^3\theta_1^7 + (1-12x^2)\theta_1^6 + (2x+4\sqrt{2}x)\theta_2^4\theta_1 + (1-2x^2\sqrt{2})\theta_2)D^2 \\ & +((10-24x^2\sqrt{2}+68x^2-6\sqrt{2})\theta_2^3\theta_1^7 + (12x+144x^3)\theta_1^6 \\ & +(-68x^2-4\sqrt{2}-24x^2\sqrt{2}-2)\theta_2^4\theta_1 + (2\sqrt{2}x+8x^3)\theta_2)D + (-576x^3\sqrt{2}+624x^3)\theta_2^3\theta_1^7 \\ & +(-12-144x^2)\theta_1^6 + (80x^3+104x^3\sqrt{2})\theta_2^4\theta_1 + (-2\sqrt{2}-8x^2)\theta_2 \end{aligned}$$

In the previous example, we show how to compute bounds on the degree and the valuation of Laurent polynomial solutions of $L(y) = 0$ *in*

$C(x)[\theta_1, \theta_2, \theta_1^{-1}, \theta_2^{-1}]$. Using the lexicographic order, with $\theta_1 > \theta_2$, we found that the extremal points are $(6, 0)$ for the degree and $(0, 0)$ for the valuation. Using the lexicographic order, with $\theta_2 > \theta_1$, we found that the extremal points are $(1, 3)$ for the degree and $(0, 0)$ for the valuation. Plotting this points in a graph, and considering the box they define, we have a finite set of possibilities for the exponents of θ_1 and θ_2



So, the monomials in the solution are of the form $\theta_1^{\beta_1}\theta_2^{\beta_2}$ where $0 \le \beta_1 \le 6$ and $0 \le \beta_2 \le 3$. i.e. $Y = \sum_{i=0}^{6}\sum_{j=0}^{3} y_{i,j}\theta_1^i\theta_2^j$. A change of variables leads us to solve linear differential equations in $C(x)$.

2.2. Rewriting the exponential extension

2.2.1. Algorithm

We are now interested in rewriting the system of generators defining a flat exponential extension of $C(x)$ such that this extension is well-defined.

PROPOSITION 2.3: *Let $E = C(x, \theta_{\mathcal{L}})$ be a flat exponential extension of $C(x)$. There exists a computable set of generators $(\hat{\theta_{\mathcal{L}}})$ such that $C(x, \hat{\theta_{\mathcal{L}}})$ is an algebraic extension of $C(x, \theta_{\mathcal{L}})$ and well-defined.*

Proof: We give here an algorithm computing such set of generators. In order to do this, we need two sets $\mathcal{M}$ and $\mathcal{T}$ such that

- $\mathcal{M} \cup \mathcal{T} = \{1, \ldots, l\}$, and
- if $C(x, \theta_{j_1}, \ldots, \theta_{j_m})$ is not well-defined then $\{j_1, \ldots, j_m\} \subset \mathcal{T}$.

Starting with $\mathcal{M} = \emptyset$ and $\mathcal{T} = \{1, \ldots, l\}$, we denote $g_j = \frac{\theta_j'}{\theta_j}$. For $\mathcal{N} \subset \{1, \ldots, l\}$, we define $\mathcal{P}_\infty^{\mathcal{N}} = \{j \in \mathcal{N} / \deg_x(g_j) = \max(0, \max_k(\deg_x(g_k)))\}$ and, for any irreducible polynomial p, $\mathcal{P}_p^{\mathcal{N}} = \{j \in \mathcal{N} / \operatorname{val}_p(\frac{\theta_j'}{\theta_j}) = \min(-1, \min_k(\operatorname{val}_p(g_k)))\}$. The hypothesis is that if there exists $\mathcal{N} = \{n_1, \ldots, n_k\} \subset \{1, \ldots, l\}$ such that $C(x, \theta_{n_1}, \ldots, \theta_{n_k})$ is not well-defined then $\mathcal{N} \subset \mathcal{T}$.
There are two possibilities:

1. First let us assume that $\mathcal{P}_\infty^{\mathcal{T}} \neq \emptyset$:
   We define $\eta_\infty = \max_{j\in\mathcal{T}}(\deg_x(g_j))$. Since $\mathcal{P}_\infty^{\mathcal{T}} \neq \emptyset$ we conclude that $\eta_\infty \geq 0$. Let us consider $\mathcal{J} = \{j \in \mathcal{T} / \deg_x(g_j) = \eta_\infty\}$ and $g_j = u_j x^{\eta_\infty} + \cdots$. There are two cases:
   (a) the $(u_j)_{j\in\mathcal{J}}$'s are $\mathbb{Q}$-linearly independent and then we substitute $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{J}$ for $\mathcal{M}$ and $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{J}$ for $\mathcal{T}$.
   (b) the $(u_j)_{j\in\mathcal{J}}$'s are $\mathbb{Q}$-linearly dependent:
      We can find a subset $\mathcal{B} \subset \mathcal{J}$ such that $(u_j)_{j\in\mathcal{B}}$ is a $\mathbb{Q}$-basis of $(u_j)_{j\in\mathcal{J}}$:
      i. For all c_j in $\mathbb{Q}$, $\sum_{j\in\mathcal{B}} c_j u_j = 0 \Rightarrow c_j = 0$
      ii. There exists d in $\mathbb{Z}$, $d \neq 0$ such that, for all $j \in \mathcal{J}\backslash\mathcal{B}$, there exist $c_{j,l} \in \mathbb{Z}$, not all equal to zero, such that $u_l = \frac{1}{d}\sum_{l\in\mathcal{B}} c_{j,l} u_l$
      In this case,
      i. for $j \in \mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = \theta_j^{\frac{1}{d}}$
      ii. for $j \in \mathcal{J}\backslash\mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = \frac{\theta_j}{\prod_{l\in\mathcal{B}} \hat{\theta}_l^{c_{j,l}}}$
      iii. we substitute $\mathcal{M}$ by $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\mathcal{T}$ by $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{B}$.

   In both cases, we have to prove that if there exists $\mathcal{N} = \{n_1, \ldots, n_k\} \subset \mathcal{T}$ such that $C(x, \theta_{n_1}, \ldots, \theta_{n_k})$ is not well-defined then $\mathcal{N} \subset \hat{\mathcal{T}}$. To perform this task we remark that for any $\mathcal{N} \subset \mathcal{T}$, if $\mathcal{N} \cap \mathcal{J} \neq \emptyset$ (or if $\mathcal{N} \cap \mathcal{B} \neq \emptyset$ in the second case) we can consider the expansions at infinity and the leading terms are $\mathbb{Q}$-linearly independent.
2. Secondly, let us suppose that $\mathcal{P}_\infty^{\mathcal{T}} = \emptyset$:
   In this case, there exists p such that $\mathcal{P}_p^{\mathcal{T}}$ is not empty. Let us define $\eta_p = -\min_{j\in\mathcal{T}}(\operatorname{val}_p(g_j))$. Since $\mathcal{P}_p^{\mathcal{T}} \neq \emptyset$ we conclude that $\eta_p \geq 1$. We define $\mathcal{J} = \{j \in \mathcal{T} / \operatorname{val}_p(g_j) = -\eta_p\}$ and $g_j = \frac{u_j}{p^{\eta_p}} + \cdots$ There are two cases
   (a) If $\eta_p \geq 1$: There are again two possibilities:
      i. the $(u_j)_{j\in\mathcal{J}}$'s are $\mathbb{Q}$-linearly independent:
         Then we define $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{J}$ and $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{J}$.
      ii. the $(u_j)_{j\in\mathcal{J}}$'s are $\mathbb{Q}$-linearly dependent:
         We can find a subset $\mathcal{B} \subset \mathcal{J}$ such that $(u_j)_{j\in\mathcal{B}}$ is a $\mathbb{Q}$-basis:
         A. For all $c_j \in \mathbb{Q}$ such that $\sum_{j\in\mathcal{B}} c_j u_j = 0$ we conclude that $c_j = 0$
         B. There exists $d \in \mathbb{Z}, d \neq 0$ such that for all $j \in \mathcal{J}\backslash\mathcal{B}$, there exist $c_{j,l} \in \mathbb{Z}$ non all equal to zero such that $u_l = \frac{1}{d}\sum_{l\in\mathcal{B}} c_{j,l} u_l$
         In this case,
         - for $j \in \mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = \theta_j^{\frac{1}{d}}$
         - for $j \in \mathcal{J}\backslash\mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = \frac{\theta_j}{\prod_{l\in\mathcal{B}} \hat{\theta}_l^{c_{j,l}}}$
         - we define $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{B}$.
   (b) If $\eta_p = 1$: we denote $q = p' \mod p$ i.e. q is the remainder of p' divided by p. There are two possibilities:

i. the $(u_j)_{j\in\mathcal{J}}$'s and q are $\mathbb{Q}$-linearly independent:
we then denote $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{J}$ and $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{J}$.

ii. the $(u_j)_{j\in\mathcal{J}}$'s and q are $\mathbb{Q}$-linearly dependent:
We can find a finite subset $\mathcal{B} \subset \mathcal{J}$ such that $(q, (u_j)_{j\in\mathcal{B}})$ is a $\mathbb{Q}$-basis of $(q, (u_j)_{j\in\mathcal{J}})$. Let us remark that $\mathcal{B}$ can eventually be empty, but this occur only a finite number of cases (there is only a finite number of irreducible polynomials that appear in the denominator of the g_j's whose the logarithmic derivative is linearly dependent of the leading term of the p-adic expansion, as the extension is a flat exponential extension).

A. For all $c_0, c_j \in \mathbb{Q}$, $c_0 q + \sum_{j\in\mathcal{B}} c_j u_j = 0 \Rightarrow c_0 = 0, c_j = 0$

B. There exist $d_1, d_2 \in \mathbb{Z}$ non all equal to zero such that, for any $l \in \mathcal{J}\backslash\mathcal{B}$, there exist $c_{0,l}, c_{j,l} \in \mathbb{Z}$ non all equal to zero such that $d_1 u_l = c_{0,l} q + \frac{1}{d_2}(\sum_{l\in\mathcal{B}} c_{j,l} u_l)$

In this case,

A. for $j \in \mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = \theta_j^{\frac{1}{d_2}}$

B. for $j \in \mathcal{J}\backslash\mathcal{B}$, we substitute θ_j by $\hat{\theta}_j = (\frac{1}{p^{c_{0,l}}} \frac{\theta_j}{\prod_{l\in\mathcal{B}} \hat{\theta}_l^{c_{j,l}}})^{\frac{1}{d_1}}$

C. we denote $\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ and $\hat{\mathcal{T}} = \mathcal{T}\backslash\mathcal{B}$.

In both cases, we substitute $\hat{\mathcal{M}}$ for $\mathcal{M}$ and $\hat{\mathcal{T}}$ for $\mathcal{T}$. We have to prove that if there exists $\mathcal{N} \subset \mathcal{T}$, $\mathcal{N} = \{n_1, \ldots, n_k\}$, such that $C(x, \theta_{n_1}, \ldots, \theta_{n_k})$ is not well-defined then $\mathcal{N} \subset \hat{\mathcal{T}}$. Let us note that for all $\mathcal{N} \subset \mathcal{T}$, if $\mathcal{N} \cap \mathcal{J} \neq \emptyset$ (or if $\mathcal{N} \cap \mathcal{B} \neq \emptyset$) we can consider the p-adic expansion and the leading terms are $\mathbb{Q}$-linearly independent.

This algorithm stops because the number of iterations where the cardinal of $\mathcal{M}$ does not strictly increase is finite. □

2.2.2. Improvements

There are several ways to improve the previous algorithm. The key idea is to avoid as many possible rewriting and algebraic extension.

- If the u_j's are pairwise linearly dependent over $\mathbb{Q}$, we can modify the set of generators without extending the extension. For example, if $l = 2$ and if $\frac{u_1}{u_2} = \frac{p}{q}$, then from Bezout theorem, there exist a and b such that $ap+bq = 1$. In this case, we define $\hat{g}_1 = ag_1 + bg_2$, $\hat{g}_2 = -qg_1 + pg_2$ and $\hat{\theta}_1 = \exp(\int g_1)$, $\theta_2 = \exp(\int g_2)$ (the extension is isomorphic because $g_1 = p\hat{g}_1 + q\hat{g}_2$, and $g_2 = -v\hat{g}_1 + u\hat{g}_2$).

EXAMPLE 6: *If we consider* $C(x, e^{\int 2x^2+3x+1}, e^{\int 3x^2+x})$*, the algorithm outputs* $C(x, e^{\int x^2+\frac{3}{2}x+\frac{1}{2}}, e^{\int -\frac{1}{6}x-\frac{1}{2}})$ *or* $C(x, e^{\int x^2+\frac{1}{3}}, e^{\int \frac{1}{6}x+\frac{1}{2}})$ *(depending on the choice for the* $\mathbb{Q}$*-basis* $\mathcal{B}$*). This extension is isomorphic to* $C(x, e^{\int x^2-2x-1}, e^{\int 7x+3})$*, which is well-defined, and can be found using Bezout theorem.*

- First we have to consider the sets that provide the smallest number of rewritings i.e. the sets $\mathcal{P}_p^{\mathcal{N}}$ such that the leading terms of the p-adic expansions of the logarithmic derivatives of the exponentials variables are $\mathbb{Q}$-linearly independent.

3. Exponential extensions over monomial extensions of a base field

In this section, we generalize the previous algorithm to exponential extensions of monomial extensions of a base field. We now consider a differential field (k, D). An element t is *monomial* over k if t is transcendent over k, Dt is a polynomial in t and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. The extension $k(t)$ is a *monomial extension* of k.

DEFINITION 3: *A* flat monomial extension *of k is an extension $k(t_1, \ldots, t_s)$ such that the t_i are monomial over k and algebraically independent.*

Notation : we denote $\mathcal{S}$ for $\{1, \cdots, s\}$ and $t_{\mathcal{S}}$ for $t_1, \ldots, t_s$.

Let $k(t_{\mathcal{S}})$ be a flat monomial extension of k. We turn our attention into the addition of exponential elements over $k(t_{\mathcal{S}})$. In order to mimic the method used in section 2, we limit the extensions considered. The method relies in the hypothesis that, for some valuation, the orders of the logarithmic derivatives of the exponential variables are greater than the orders of the logarithmic derivatives of the functions of $C(x)$, or the leading terms are $\mathbb{Q}$-linearly independent. This hypothesis is always true in exponential extensions of $C(x)$ but not in exponential extensions of monomial extensions of a base field. Therefore we restrict the exponential extensions considered:

DEFINITION 4: *An element θ is* effectively exponential over $k(t_{\mathcal{S}})$ *if*

1. $\frac{D\theta}{\theta}$ *is in* $k(t_{\mathcal{S}})\backslash k$
2. *for all c in $\mathbb{Q}$ and f in $k(t_{\mathcal{S}})$ there exists i such that $\frac{D\theta}{\theta} + c\frac{Df}{f}$ has a $\frac{1}{t_i}$-adic expansion with the form $u_\beta t_i^\beta + \cdots$ where β is an integer such that*
   (a) *either* $\beta < 0$,
   (b) *or β is greater than or equal to* $\max(1, \deg_{t_i}(Dt_i))$.

A differential extension F of $k(t_{\mathcal{S}})$ is an effectively exponential extension over $k(t_{\mathcal{S}})$ *if $F = k(t_{\mathcal{S}}, \theta)$ where $\theta \in F^\star$ is effectively exponential over $k(t_{\mathcal{S}})$.*

EXAMPLE 7: *Let t be transcendent over $C(x)$ and such that $Dt = t^2 + x$ (i.e. $t = tg(x)$). Then $C(x, t, e^{\int t})$ is not effectively exponential over $C(x, t)$ because the order of $\frac{De^{\int t}}{e^{\int t}} = t$ is 1, which is not greater than* $\max(1, \deg_t(Dt)) = 2$. *Therefore the condition 2 is false for $c = 0$.*

EXAMPLE 8: *Let t be transcendent over $C(x)$ and such that $Dt = t^2 + x$ (i.e. $t = tg(x)$). Then $C(x, t, e^{\int t^3})$ is effectively exponential over $C(x, t)$ because conditions 1 and 2b are true.*

In the previous definition, in the case $\beta = \text{val}_{\frac{1}{t_i}}(\frac{D\theta}{\theta}) < 0$ there exists $p \in k[t_{\mathcal{S}}]$ such that either $\text{val}_p(\frac{D\theta}{\theta}) < -1$, or $\text{val}_p(\frac{D\theta}{\theta}) = -1$ and the leading term in the p-adic expansion is $\mathbb{Q}$-linearly independent to $Dp \mod p$.

The definition implies that for all c in $\mathbb{Q}$, for all f in $k(t_{\mathcal{S}})^\star$, we have $\theta \neq f^c$. This means that $n\frac{D\theta}{\theta} \neq \frac{Dv}{v}$ for all integer $n \neq 0$, for all $v \in k(t_{\mathcal{S}})^\star$ and θ is exponential over $k(t_{\mathcal{S}})$. As consequence (see [Bro97, Theorem 5.1.2]), θ is transcendent over $k(t_{\mathcal{S}})$ and $\text{Const}(k(t_{\mathcal{S}}, \theta)) = \text{Const}(k(t_{\mathcal{S}})) = \text{Const}(k)$.

Let us consider the addition of several exponential variables:

DEFINITION 5: *E is a* flat effectively exponential extension of $k(t_{\mathcal{S}})$ *if there are $\theta_1, \dots, \theta_l$ such that $E = k(t_{\mathcal{S}}, \theta_1, \dots, \theta_l)$ and for all c_i in $\mathbb{Q}$, $\prod_{i=1}^{l} \theta_i^{c_i}$ is effectively exponential over $k(t_{\mathcal{S}})$.*

REMARK 4: *A flat effectively exponential extension is a flat monomial extension and we can proceed by induction in the extension generated by adding iteratively the exponential variables which are not flat.*

Let us focus our attention in the system of generators defining the flat effectively exponential extensions:

DEFINITION 6: *A* well-defined *extension of $k(t_{\mathcal{S}})$ is a flat exponential extension $k(t_{\mathcal{S}}, \theta_1, \dots, \theta_l)$ such that if we define $g_i = \frac{D\theta_i}{\theta_i}$ then for any subset $\mathcal{N}$ of $\mathcal{L}$:*

1. *either we denote*

$$\mathcal{P}_i^{\mathcal{N}} = \{j \in \mathcal{N} /\, \deg_{t_i}(\frac{D\theta_j}{\theta_j}) = \max\left(\deg_{t_i}(Dt_i), 1, \max_k(\deg_{t_i}\frac{D\theta_k}{\theta_k})\right)\} \text{ for } i = 1, \dots, s$$

   and there exists i such that $\mathcal{P}_i^{\mathcal{N}} \neq \emptyset$ and if for j in $\mathcal{P}_i^{\mathcal{N}}$ we denote $\frac{D\theta_j}{\theta_j} = u_j t_i^\beta + \cdots$ then the (u_j)'s are $\mathbb{Q}$-linearly independent,

2. *or for all i, $\mathcal{P}_i^{\mathcal{N}} = \emptyset$ and if we denote*

$$\mathcal{P}_p^{\mathcal{N}} = \{j \in \mathcal{N} /\ val_p(\frac{D\theta_j}{\theta_j}) = \min\left(-1, \min_k(\ val_p(\frac{D\theta_k}{\theta_k}))\right)\} \text{ for any irreducible polynomial } p,$$

   there exists p such that $\mathcal{P}_p^{\mathcal{N}} \neq \emptyset$ and if for j in $\mathcal{P}_p^{\mathcal{N}}$, we denote $\frac{D\theta_j}{\theta_j} = \frac{u_j}{p^\beta} + \cdots$ then

   (a) $\beta > 2$ and the (u_j)'s are $\mathbb{Q}$-linearly independent, or

   (b) $\beta = 1$ and the (u_j) and $Dp \mod p$ are $\mathbb{Q}$-linearly independent.

3.1. Laurent polynomial solutions of linear differential solutions

3.1.1. Riccati equation

We consider a monomial extension $K(t)$ of a differential field (K, D). For u in $K(t)$, we still define

$$\begin{aligned} P_0 &= 1 \,, \\ P_i &= DP_{i-1} + uP_{i-1} \text{ pour } i \geq 1 \end{aligned}$$

As in subsection 2.1.1, if we consider a linear differential operator $L = A_nD^n + A_{n-1}D^{n-1} + \cdots + A_0$ with coefficients in $K(t)$ then

$$L(y) = 0 \Leftrightarrow A_nP_n(u, \ldots, D^{n-1}u) + A_{n-1}P_{n-1}(u, \ldots, D^{n-2}u) + \cdots + A_0 = 0$$

We have adaptations of lemmas 1 and 2, but some properties of the polynomial used for the p-adic expansions have to be distinguish: a polynomial p in $K[t]$ is *special* if Dp divides p and p is *normal* if Dp is prime with p (see [Bro97, Definition 3.4.2]). We remark that an irreducible polynomial is either special or normal. We have:

Lemma 3: *Let u be in $K(t)$ with the following expansion: $u = u_\beta t^\beta$+terms with lower order, with β in $\mathbb{Z}$ and note $\eta = \deg_t(Dt)$. If $\beta \geq \max(\eta, 1)$ then $P_i = (u_\beta)^i t^{i\beta}$+higher order terms.*

Proof: By induction on i. □

Lemma 4: *Let $p(t)$ be a polynomial in $K[t]$ normal and irreducible. Let u be in $K(t)$ with the following p-adic expansion: $u = \frac{u_\beta}{p^\beta}$+higher order terms with β in $\mathbb{N}_{>0}$, $u_\beta \neq 0$ and $\deg_t(u_\beta) < \deg_t(p)$.*

1. *If $\beta > 1$ then $P_i(u, \ldots, D^{i-1}u) = \frac{v_{i,\beta}}{p^{i\beta}}$+higher order terms, where $v_{i,\beta} \equiv (u_\beta)^i \mod p$*
2. *If $\beta = 1$ and if u_β is prime with $Dp \mod p$ then $P_i(u, \ldots, D^{i-1}u) = \frac{v_i}{p^i}$+higher order terms, where $v_i \equiv \prod_{j=0}^{i-1}(u_1 - jDp) \mod p$.*

Proof: By induction on i. □

Lemma 5: *Let $p(t)$ be a polynomial in $K[t]$ special and irreducible. Let u be in $K(t)$ with the following p-adic expansion: $u = \frac{u_\beta}{p^\beta}$+higher order terms, with $\beta > 0, u_\beta \neq 0$. Then $P_i(u, \ldots, D^{i-1}u) = \frac{v_{i,\beta}}{p^{i\beta}} + \cdots$ where $v_{i,\beta} \equiv (u_\beta)^i \mod p$.*

Proof: We note that if p is special then p divides Dp: $Dp = qp$ for q in $K[t]$. Then we proceed by induction on i. □

3.1.2. Bounds on the degree and the valuation

We look for bounds on the degree and the valuation of Laurent polynomial solutions of linear differential equations with coefficients in a flat effectively exponential extension of a flat monomial extension. Furthermore, we assume that this extension is well-defined. In order to compute the bounds, we have to consider the order of the logarithmic derivatives of functions in a monomial extension:

LEMMA 6: *Let (K, D) be a differential field, and t be monomial over K such that $Const(K(t)) = Const(K)$. Let p be an irreducible polynomial in $K[t]$, and f be in $K(t)$. Let val_p denote the order at p and $val_{\frac{1}{t}}$ denote the order at $t = \infty$ (i.e. the degree at t).*

1. *If p is normal then the p-adic expansion of $\frac{Df}{f}$ is $\frac{\nu q}{p} + \cdots$ where $\nu = val_p(f)$ and $q \equiv Dp \mod p$.*
2. *If p is special then $val_p(\frac{Df}{f}) \geq 0$.*
3. $val_{\frac{1}{t}}(\frac{Df}{f}) \geq \deg_t(Dt) - 1$.

Proof: 1. Let p be normal. Let $f = p^\nu g$ with ν in $\mathbb{Z}$ and g prime with p. Then $\frac{Df}{f} = \frac{\nu q}{p} + \cdots$

2. If p is special then $\mathrm{val}_p(Df) \geq \mathrm{val}_p(f)$ and $\mathrm{val}_p(\frac{Df}{f}) = \mathrm{val}_p(Df) - \mathrm{val}_p(f) \geq 0$
3. for all f in $K(t)$

$$\begin{aligned} \mathrm{val}_{\frac{1}{t}}(\frac{Df}{f}) &= \mathrm{val}_{\frac{1}{t}}(Df) - \mathrm{val}_{\frac{1}{t}}(f) \\ &\geq (\mathrm{val}_{\frac{1}{t}}(f) - 1 + \deg_t(Dt)) - \mathrm{val}_{\frac{1}{t}}(f) = \deg_t(Dt) - 1 \square \end{aligned}$$

PROPOSITION 3.1: *Let $K = k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ be a well-defined extension of $k(t_{\mathcal{S}})$. Let $L = \sum_{i=1}^n A_i D^i$ be a linear differential operator with coefficients in $k(t_{\mathcal{S}})$. There exists a computable set such that if there exists $Y = f\theta_1^{\gamma_1} \ldots \theta_l^{\gamma_l}$ such that $L(Y) = 0$ for some f in $k(t_{\mathcal{S}})$ and γ_i in $\mathbb{Z}$ then $(\gamma_1, \ldots, \gamma_l)$ belongs to this set.*

Proof: The proof consists in the generalization of the algorithm given in the proof of proposition 2.1 by using lemma 6 (see appendix). □

THEOREM 3.2: *Let $K = k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ be a well-defined extension of $k(t_{\mathcal{S}})$. Let $L = \sum_{i=1}^n A_i D^i$ be a linear differential operator with coefficients in $k(t_{\mathcal{S}})[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ and $b_1, \ldots, b_k$ be elements of $k(t_{\mathcal{S}})[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$.*
We can find $(m_1, \ldots, m_l)$ and $(M_1, \ldots, M_l)$ such that if for Y in $k(t_{\mathcal{S}})[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ we have $L(Y) = \sum_j c_j b_j$ for some constant parameters c_j's then $val_{\theta_i}(Y) \geq m_i$ et $\deg_{\theta_i}(Y) \leq M_i$

Proof: To prove this theorem we proceed as in the proof of theorem 2.2, using proposition 3.1 □

3.1.3. Computation of the coefficients

We proceed as in section 2.1.3 (see appendix).

3.2. Rewriting of the exponential extension

Given a flat effectively exponential extension of a flat monomial extension, we want to find a system of generators such that the extension is well-defined:

PROPOSITION 3.3: *Let $E = k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ be a flat effectively exponential extension over $k(t_{\mathcal{S}})$. We can compute a set of generators $(\hat{\theta_{\mathcal{L}}})$ such that $k(t_{\mathcal{S}}, \hat{\theta_{\mathcal{L}}})$ is an algebraic extension of $k(t_{\mathcal{S}}, \hat{\theta_{\mathcal{L}}})$ which is flat effectively exponential over $k(t_{\mathcal{S}})$ and well-defined.*

Proof: The proof is an algorithm that compute such set of generators. This algorithm is a variation of the one presented for the proposition 2.3 and we rewrite it briefly in appendix. □

3.3. Examples

We can find the polynomial solutions of linear differential equations with coefficients in monomial extensions that are not liouvillian. Consider the extension

$$\begin{array}{cl} C(x,t,\theta) & \text{where } \theta \text{ is transcendent over } C(x,t) \text{ such that } \dfrac{D\theta}{\theta} = t^3 \\ | & \\ C(x,t) & \text{where } t \text{ is transcendent over } C(x) \text{ such that } Dt = t^2+1 \\ | & \\ C(x) & \text{where } Dx = 1 \end{array}$$

Let

$$\begin{aligned} L \;=\; & (\mathbf{2xt^6} + xt^4 - t^3 + xt^2)D^2 \\ & +(-\mathbf{8xt^9} - 16xt^7 + 2t^6 - 20xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D \\ & +\mathbf{6xt^{12}} + 13xt^{10} + 3t^9 + 24xt^8 + 11t^7 + 26xt^6 + 13t^5 + 19xt^4 + 6xt^2 - 2t + 2x \end{aligned} \tag{1}$$

We search for solutions of $L(y) = 0$ in $C(x)[t, \theta, \theta^{-1}]$. From [Ros75], we find that if such a solution exists it can be searched with the form $f\theta^\gamma$ where f is in $C(x)[t]$ and γ in $\mathbb{Z}$. we have

$$\begin{aligned} Y &= f\theta^\gamma \text{ with } f \text{ in } C(x)[t],\ \gamma \text{ in } \mathbb{Z} \\ DY &= (\gamma t^3 f + Df)\theta^\gamma \\ D^2Y &= (\gamma^2 t^6 f + \cdots)\theta^\gamma \end{aligned}$$

So $Q(\gamma) = 0$ where $Q = 2Z^2 - 8Z + 6 = 2(Z-1)(Z-3)$.

- Computing the solutions

– If $\gamma = 1$:
A change of variables gives the equation

$$\begin{aligned} L_1 &= (2xt^6 + xt^4 - t^3 + xt^2)D^2 \\ &+(-\mathbf{4xt^9} - 14xt^7 - 18xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D \\ &+\mathbf{4xt^{10}} + 4t^9 + 14xt^8 + 10t^7 + 26xt^6 + 12t^5 + 20xt^4 + 6xt^2 - 2t + 2x \end{aligned}$$

We have

$$\begin{aligned} Y &= y_\gamma t^\gamma + \cdots \text{ with } \gamma \text{ in } \mathbb{Z},\ y_\gamma \text{ in } C(x) \\ DY &= \gamma y_\gamma t^{\gamma+1} + \cdots \\ D^2Y &= \gamma(\gamma+1)t^{\gamma+2} + \cdots \end{aligned}$$

So $Q(\gamma) = 0$ where $Q = -4Z + 4 = 4(Z-1)$. A change of variable provides us that the solutions of $L_1(Y) = 0$ have the form $c_1 xt$ where c_1 is an arbitrary constant.

– If $\gamma = 3$:
A change of variables gives us the equation

$$\begin{aligned} L_2 &= (2xt^6 + xt^4 - t^3 + xt^2)D^2 \\ &+(\mathbf{4xt^9} - 10xt^7 - 4t^6 - 14xt^5 + 2t^4 - 6xt^3 + 2t^2 - 2xt)D \\ &-\mathbf{8xt^{10}} + 8t^7 + 26xt^6 + 10t^5 + 22xt^4 + 6xt^2 - 2t + 2x \end{aligned}$$

So $Q(\gamma) = 0$ where $Q = 4Z - 8 = 4(Z-2)$. A change of variable provides us that the solutions of $L_2(Y) = 0$ have the form c_2t^2 where c_2 is an arbitrary constant.

Solutions of $L(Y) = 0$ where L is defined in (1) are $c_1xt\theta + c_2t^2\theta^3$ where c_1 and c_2 are arbitrary constants.

⋆

Algorithms concerning polynomial solutions of linear differential equations with coefficients in extensions generated by iterated logarithms and exponential was presented in [Fre00]. Let us consider the example 3.9.1 of [Sin91] : we search the polynomial solutions of

$$L(y) := (x^2 \ln^2 x)y'' + (x \ln^2 x - 3x \ln x)y' + 3y = 0 \tag{2}$$

We consider $C[l_0, l_1] = \mathbb{Q}[x, \ln x]$ with the derivation D such that $Dl_1 = l_1,\ Dl_0 = l_0l_1$ i.e. $D = x \ln x \frac{d}{dx}$. So, we consider

$$L(y) = D^2 - 4D + 3$$

and we search for Y in $C[l_0, l_1, l_0^{-1}, l_1^{-1}]$ such that $L(Y) = 0$.

- Bounds on the degree and the valuation with respect to l_0 :
  A solution of $L(Y) = 0$ is monomial in l_0 (because l_0 is not present in the equation - see [Ros75]). Let $Y = f l_0^\gamma$, where γ is in $\mathbb{Z}$ and f in $C[l_1, l_1^{-1}]$. So

  $$DY = (Df + \gamma l_1) l_0^\gamma \text{ and } D^2 Y = (D^2 f + \gamma l_1 + \gamma l_1 Df + \gamma^2 l_1^2) l_0^\gamma.$$

  Considering the leading term of $L(Y)$ with respect to l_1, we have $\gamma^2 = 0$ i.e. $\gamma = 0$. Then the polynomial solutions of $L(y) = 0$ are in $C[l_1, l_1^{-1}]$.
- Coefficients with respect to l_0 :
  A solution of $L(Y) = 0$ is monomial in l_1 (because l_1 is not present in the equation). Let $Y = f l_1^\gamma$, where γ is in $\mathbb{Z}$ and f in C. Then$DY = \gamma Y$, $D^2 Y = \gamma^2 Y$. So, we search γ such that

  $$\gamma^2 - 4\gamma + 3 = 0 = (\gamma - 1)(\gamma - 3)$$

Solutions of (2) are $c_1 l_1 + c_2 l_1^3$ where c_1, c_2 are constant parameters.

In this example we see that a change of the derivation can simplify the computation of polynomial solutions.

4. Conclusion

We have presented algorithms to compute efficiently Laurent polynomial solutions of linear differential equations with coefficients in exponential extensions of monomial extensions of a base field. Our claim is that a suitable system of generators of the extension improves the computation of such solutions.

References

[ABP95] Serguei Abramov, Manuel Bronstein, and Marko Petkovšek. On polynomial solutions of linear operator equations. In *Proceedings of ISSAC'95*, pages 290–296. ACM Press, 1995.

[BF99] Manuel Bronstein and Anne Fredet. Solving linear ordinary differential equations over $C(x, e^{\int f(x)dx})$. In *Proceedings of ISSAC'99*, pages 173–179. ACM Press, 1999.

[Bro92] Manuel Bronstein. On solutions of linear ordinary differential equations in their coefficient field. *Journal of Symbolic Computation*, 13(4):413–440, April 1992.

[Bro97] Manuel Bronstein. *Symbolic Integration I – Transcendental Functions.* Springer, Heidelberg, 1997.

[Bro00] Manuel Bronstein. Computer algebra algorithms for linear ordinary differential and difference equations. *E3M*, 2000.

[Chy98] Frédéric Chyzak. *Fonctions holonomes en calcul formel.* PhD thesis, INRIA, 1998.

[Fre00] Anne Fredet. Linear differential equations, iterative logarithms and orderings on monomial differential extensions. In *Proceedings of ISSAC'00*, pages 121–128. ACM Press, 2000.

[Poo60] E.G.C Poole. *Introduction to the Theory of Linear Differential Equations.* Dover Publications Inc., 1960.

[RC79] Michael Rothstein and B. F. Caviness. A structure theorem for exponential and primitive functions. *SIAM J. Comput*, 8:357–367, 1979.

[Ros75] Maxwell Rosenlicht. Differential extension fields of exponential type. *Pacific Journal of Mathematics*, 57:289–300, 1975.

[RSSH96] Daniel Richardson, Bruno Salvy, John Shackell, and Joris Van Der Hoeven. Asymptotic expansions of exp-log functions. In *Proceedings of ISSAC'96*, pages 309–313. ACM Press, 1996.

[Sin91] Michael F. Singer. Liouvillian solutions of linear differential equations with liouvillian coefficients. *Journal of Symbolic Computation*, 11:251–273, 1991.

[SSC85] Michael F. Singer, Bob D. Saunders, and Bob F. Caviness. An extension of liouvillian theorem on integration in finite terms. *SIAM Journal on Computing*, 14:966–990, 1985.

Appendix: Algorithms

Exponents of solutions

Input: A differential field (k, D)
$t_1, \ldots, t_s$ such that $k(t_{\mathcal{S}})$ is a flat monomial extension of k
$\theta_1, \ldots, \theta_l$ such that $k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ is a flat effectively exponential extension of $k(t_{\mathcal{S}})$, well-defined
$L(y) = D^n y + \cdots + A_1 Dy + A_0 y$ a linear differential operator with coefficients in $k(t_{\mathcal{S}})$, $A_n = 0$

Output: A finite set $\mathcal{S}$ such that if for f in $k(t_{\mathcal{S}})$ and $(\beta_1, \ldots, \beta_l)$ in $\mathbb{Z}^l$
We have $L(f\theta_1^{\beta_1} \ldots \theta_l^{\beta_l}) = 0$ then $(\beta_1, \ldots, \beta_l)$ is in $\mathcal{S}$.

Algorithm:
If $l = 0$ it is trivial
While $l > 0$ do
Let $g_i = \frac{D\theta_i}{\theta_i}$
For $j = 1, \ldots, s$ let $\mathcal{P}_i^{\mathcal{T}} = \{j \in \mathcal{T} / \deg_{t_i}(g_j) > \max(\deg_{t_i}(Dt_i), 1)\}$.
If there exists i such that $\mathcal{P}_i^{\mathcal{T}} \neq \emptyset$
Let $\eta_i = \max_j(\deg_{t_i}(g_j))$ and $\mathcal{J} = \{j \in \mathcal{T} / \deg_{t_i}(g_j) = \eta_i\}$.
Denote $g_j = u_j t_i^{\eta_i} + \cdots$ for $j \in \mathcal{J}$ and $A_j = a_{j,\alpha_j} t_i^{\alpha_j} + \cdots$
Find the integer solutions of $\sum_{k | k\eta_i + \alpha_k = \max(\alpha_j + j\eta_i)} a_{k,\alpha_k} (\sum_{j \in \mathcal{J}} \beta_j u_j)^k$
If for all i, $\mathcal{P}_i^{\mathcal{T}} = \emptyset$
For the polynomials p such that $\mathrm{val}_p(g_j) < 0$ for some j do
Let $\mathcal{P}_p^{\mathcal{T}} = \{j \in \mathcal{T} / \mathrm{val}_p(g_j) < 0\}$, $\eta_p = -\min_j(\mathrm{val}_p(g_j))$ and $\mathcal{J} = \{j \in \mathcal{T} / \mathrm{val}_p(g_j) = \eta_p\}$.
Denote $g_j = \frac{u_j}{p^{\eta_p}} + \cdots$ for $j \in \mathcal{J}$ and $A_j = a_{j,\alpha_j} p^{\alpha_j} + \cdots$
If p is normal and $\eta_p \geq 2$, compute the integer solutions of
$\sum_{k | k\eta_i - \alpha_k = \max(\alpha_j + j\eta_i)} a_{k,\alpha_k} (\sum_{j \in \mathcal{J}} \beta_j u_j)^k \mod p$
If p is normal and $\eta_p = 1$, compute the integer solutions of
$\sum_{k | k\eta_i - \alpha_k = \max(\alpha_j + j\eta_i)} a_{k,\alpha_k} (\sum_{j \in \mathcal{J}} \beta_j \prod_{h=0}^{k-1} (\sum_{j \in \mathcal{N}_p} \beta_j u_j - hDp)) \mod p$
If p is special, compute the integer solutions of
$\sum_{k | k\eta_i - \alpha_k = \max(\alpha_j + j\eta_i)} a_{k,\alpha_k} (\sum_{j \in \mathcal{J}} \beta_j u_j)^k \mod p$
Loop
Change the variable and repeat

Polynomial solutions

Input: A differential field (k, D)

$t_1, \ldots, t_s$ such that $k(t_{\mathcal{S}})$ is a flat monomial extension of k

we assume that one can compute the rational solutions of linear differential equations with coefficients in $k(t_{\mathcal{S}})$.

$\theta_1, \ldots, \theta_l$ such that $k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ is a flat effectively exponential extension of $k(t_{\mathcal{S}})$ well-defined

$L(y) = D^n y + \cdots + A_1 D y + A_0 y$ a linear differential operator, where A_i are in $(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ and $A_n = 1$

$b_1, \ldots, b_m$ in $k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$

Output: $f_1, \ldots, f_h$ in $k(t_{\mathcal{S}})[\theta_{\mathcal{L}}, \theta_{\mathcal{L}}^{-1}]$ an a matrix $\mathcal{M}$ with constant coefficients such that any polynomial solution of $L(y) = c_1 b_1 + \cdots + c_m b_m$ with c_i constant has the form $d_1 f_1 + \cdots + d_h f_h$ where the d_i's are constant parameters such that $\mathcal{M}(d_1, \ldots, d_h, c_1, \ldots, c_m)^T = 0$

Algorithm: By induction over l. Let $L = \sum_{k=\mu}^{\nu} \theta_1^k L_k$

Let $\mathcal{S}_m$ and $\mathcal{S}_M$ be the sets of the exponents of solutions of L_μ and L_ν.

Let $M_1 = \{\max(\gamma_1)/(\gamma_1, \ldots, \gamma_l) \in \mathcal{S}_M\}$ and $m_1 = \{\min(\gamma_1)/(\gamma_1, \ldots, \gamma_l) \in \mathcal{S}_m\}$

Change the variable $\hat{y} = y\theta^{-m_1}$, multiply L by a power of θ_1 such that $\mathrm{val}_{\theta_1}(L) = 0$.

For i from 0 to $M_1 - m_1$ do

$L = L_0 + \theta_1 \ldots$

Solve $L(y_0) = c_1 b_1(0) + \cdots + c_m b_m(0)$.

Compute f_i in $k(t_{\mathcal{S}})[\theta_{\mathcal{L}\setminus\{1\}}, \theta_{\mathcal{L}\setminus\{1\}}^{-1}]$ and $\mathcal{M}$ with constant coefficients such that $y = \sum d_i f_i$ and $\mathcal{M}(d_i, c_j)^T = 0$.

Repeat with $\hat{L} = \sum c_j \frac{b_j - b_j(0)}{\theta_1} - \sum d_i \frac{1}{\theta_1}(L - L_0) f_i$.

Loop

Rewriting of the extension

Input: A differential field (k, D)
$t_1, \ldots, t_s$ such that $k(t_{\mathcal{S}})$ is a flat monomial extension of k
$\theta_1, \ldots, \theta_l$ such that $k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$ is a flat effectively exponential extension of $k(t_{\mathcal{S}})$
Output: $\hat{\theta}_1, \ldots, \hat{\theta}_l$ such that $k(t_{\mathcal{S}}, \hat{\theta}_{\mathcal{L}})$ is an algebraic extension of $k(t_{\mathcal{S}}, \theta_{\mathcal{L}})$,
flat over $k(t_{\mathcal{S}})$ and well-defined
Algorithm: Let $\mathcal{T} = \{1, \ldots, l\}$, $\mathcal{M} = \emptyset$ and $g_i = \frac{D\theta_i}{\theta_i}$
While $\mathcal{T} \neq \emptyset$ do
For $j = 1, \ldots, s$ let $\mathcal{P}_i^{\mathcal{T}} = \{j \in \mathcal{T} / \deg_{t_i}(g_j) > \max(\deg_{t_i}(Dt_i), 1)\}$.
While there exists i such that $\mathcal{P}_i^{\mathcal{T}} \neq \emptyset$ do
Let $\eta_i = \max_j(\deg_{t_i}(g_j))$, $\mathcal{J} = \{j \in \mathcal{T} / \deg_{t_i}(g_j) = \eta_i\}$.
For $j \in \mathcal{J}$, denote $g_j = u_j t_i^{\eta_i} + \cdots$
Let $\mathcal{B}$ be a basis of the $(u_j)_{j \in \mathcal{J}}$
Compute $d \in \mathbb{Z}$ and $c_{j,l} \in \mathbb{Z}$ such that for any $j \in \mathcal{J} \backslash \mathcal{B}$ one has $u_j = \sum_{l \in \mathcal{B}} c_{j,l} u_l$.
$\hat{\mathcal{M}} \leftarrow \mathcal{M} \cup \mathcal{B}$, $\hat{\mathcal{T}} \leftarrow \mathcal{T} \backslash \mathcal{B}$
$\hat{\theta}_j \leftarrow \theta_j^{1/d}$ for $j \in \mathcal{B}$ and $\hat{\theta}_j \leftarrow \theta_j / (\prod_{l \in \mathcal{B}} \hat{\theta}_l^{c_{j,l}})$ for $j \in \mathcal{J} \backslash \mathcal{B}$
Loop
If for all i, $\mathcal{P}_i^{\mathcal{T}} = \emptyset$
For any polynomial p such that $\mathrm{val}_p(g_j) < 0$ for some j in $\mathcal{T}$ do
Let $\mathcal{P}_p^{\mathcal{T}} = \{j \in \mathcal{T} / \mathrm{val}_p(g_j) < 0\}$, $\eta_p = -\min_j(\mathrm{val}_p(g_j))$
and $\mathcal{J} = \{j \in \mathcal{T} / \mathrm{val}_p(g_j) = \eta_p\}$.
For $j \in \mathcal{J}$, one denotes $g_j = \frac{u_j}{p^{\eta_p}} + \cdots$
If $\eta_p > 1$: let $\mathcal{B}$ be a basis of the $(u_j)_{j \in \mathcal{J}}$'s
Compute $d \in \mathbb{Z}$ and $c_{j,l} \in \mathbb{Z}$ such that
for all $j \in \mathcal{J} \backslash \mathcal{B}$ one has $u_j = \sum_{l \in \mathcal{B}} c_{j,l} u_l$.
Let $\hat{\mathcal{M}} \leftarrow \mathcal{M} \cup \mathcal{B}$, $\hat{\mathcal{T}} \leftarrow \mathcal{T} \backslash \mathcal{B}$
$\hat{\theta}_j \leftarrow \theta_j^{1/d}$ for $j \in \mathcal{B}$ and $\hat{\theta}_j \leftarrow \theta_j / (\prod_{l \in \mathcal{B}} \hat{\theta}_l^{c_{j,l}})$ for $j \in \mathcal{J} \backslash \mathcal{B}$
If $\eta_p = 1$, let $q = Dp \mod p$ and $\mathcal{B}$ be a basis of the $(q, (u_j)_{j \in \mathcal{J}})$
Find d_1, $d_2 \in \mathbb{Z}$ and $c_{j,l} \in \mathbb{Z}$ such that
for all $j \in \mathcal{J} \backslash \mathcal{B}$, $d_1 u_j = c_{0,l} q + \frac{1}{d_2} \sum_{l \in \mathcal{B}} c_{j,l} u_l$.
Let $\hat{\mathcal{M}} \leftarrow \mathcal{M} \cup \mathcal{B}$, $\hat{\mathcal{T}} \leftarrow \mathcal{T} \backslash \mathcal{B}$
$\hat{\theta}_j \leftarrow \theta_j^{1/d_2}$ for $j \in \mathcal{B}$ and $\hat{\theta}_j \leftarrow (\frac{1}{p^{c_{0,l}}} \frac{\theta_j}{\prod_{l \in \mathcal{B}} \hat{\theta}_l^{c_{j,l}}})^{1/d_1}$ for $j \in \mathcal{J} \backslash \mathcal{B}$
Loop
Loop