

# Institution Morphisms<sup>1</sup>

Joseph Goguen

Department of Computer Science & Engineering  
University of California at San Diego

Grigore Roşu<sup>2</sup>

NASA Ames Research Center, RIACS

## Abstract

Institutions formalize the intuitive notion of logical system, including both syntax and semantics. A surprising number of different notions of morphism have been suggested for forming categories with institutions as objects, and a surprising variety of names have been proposed for them. One goal of this paper is to suggest a terminology that is both uniform and informative to replace the current rather chaotic nomenclature. Another goal is to investigate the properties and interrelations of these notions. Following brief expositions of indexed categories, twisted relations, and Kan extensions, we demonstrate and then exploit the duality between institution morphisms in the original sense of Goguen and Burstall, and the “plain maps” of Meseguer, obtaining simple uniform proofs of completeness and cocompleteness for both resulting categories; because of this duality, we prefer the name “comorphism” over “plain map.” We next consider “theoroidal” morphisms and comorphisms, which generalize signatures to theories, finding that the “maps” of Meseguer are theoroidal comorphisms, while theoroidal morphisms are a new concept. We then introduce “forward” and “semi-natural” morphisms, and appendices discuss institutions for hidden algebra, universal algebra, partial equational logic, and a variant of order sorted algebra supporting partiality.

## 1 Introduction

Many different logics are used in computer science, including (many variants of) first order, higher order, Horn clause, type theoretic, equational, temporal, modal, and infinitary logics. To capture the fact that many general results about logics do not depend on the particular logic chosen, Goguen and Burstall [29] developed *institutions*, formalizing the notion of a logical system with varying non-logical symbols (sets of such symbols are traditionally called “signatures” in this field). The main ingredient of an institution is a satisfaction relation between its models and its sentences, an abstract form of Tarski’s classic semantic definition of truth [70], and the main requirement is that this relation should be consistent with respect to signature morphisms, which intuitively means that satisfaction is invariant under change of notation. The formalization only assumes abstract categories (or classes) of signatures, sentences and models, without assuming any particular structure for them; the covariance of sentences and contravariance of models under signature morphisms is captured by appropriate functors.

Many papers have been written on institutions, both theoretical and applied, in the twenty years since the earliest formulation [5, 6]; for example, institutions have been used to study lambda calculus, second order logic, and many variants of equational logic, modal logic, higher order logic, and first order logic. The main original paper on institutions [29] already contains several significant results, including a number of equivalent definitions for institutions, cocompleteness for categories of theories, colimit preservation for the functor on theories induced by a signature morphism, a theory

---

<sup>1</sup>The research reported here was supported in part by National Science Foundation grant CCR-9901002.

<sup>2</sup>Also Fundamentals of Computing, Faculty of Mathematics, University of Bucharest, Romania.

of constraints (including freeness and generation constraints, among others), several ways of building new institutions from old, and deduction as sentence morphisms (see the discussion after Definition 11), despite an apparently common belief that institutions do not handle deduction. Among other examples, Mosses showed that his unified algebra is an institution [47], Goguen showed that (his original version of) hidden algebra is an institution [25], Roşu gave an institution for order sorted logic [52], and Mossakowski gave a hierarchy of institutions for total, partial and order sorted logics [45]. One important application of institutions is a uniform approach to modularization for specifications; in fact, this was a major motivation [5, 27, 29]; among many papers on this topic, we mention [18, 33] and [56], which all add inclusion systems to institutions. Much other interesting work with institutions has been done by Tarlecki [62, 63, 64, 65, 67], Sannella and Tarlecki [58, 59, 60], Cerioli [7], Mossakowski [45], and Diaconescu [19, 14, 15], among others; [67] in particular is an important paper with goals and results similar to those of this paper. Burstall and Diaconescu [4] generalize “hiding” from algebra to an arbitrary institution, and apply this to both many sorted and order sorted algebra.

Many variations on the institution concept have appeared. For example, Mayoh introduced “galleries” [42], which Goguen and Burstall extended to “generalized institutions” [28], allowing non-Boolean values for satisfaction. Poigné’s “foundations” and “rich institutions” [50] further abstracted institutions by requiring that sentences form a fibration, although this gets very complex; Fiadeiro and Sernadas [23] introduced “ $\pi$ -institutions” and Meseguer [43] studied “general logics,” each combining institutions with classical entailment relations; [43] is a gem that contains many interesting ideas. Salibra and Scollo introduced “pre-institutions” [57], where the “iff” in the satisfaction condition is split into two implications, which are then studied separately, combined, or both dropped; Ehrig, Orejas *et al.* introduced “specification logics” [22], which are indexed categories of models, with no sentences; Căzănescu introduced “truth systems” [11], a sort of compromise between institutions and charters, allowing inference in a designated model; and Pawłowski introduced “context institutions” [49] to deal with variable contexts and substitutions. Diaconescu introduced “many sorted institutions” [19], which assign a sort set to each signature, and Grothendieck (or fibred) institutions [20], which combine multiple institutions in a single structure; the latter was developed for the semantics of the CafeOBJ language [21]. Section 3 of this paper introduces the “close variants” of the institution concept, which share its mathematical properties.

Although the variants of institution all have interesting properties, and are no doubt worth studying, some can be seen as special kinds of institution<sup>3</sup>, and the others have close natural relationships to institutions. It seems to us that the original institution concept captures the essence of logical system, which is the intimate dance between syntax and semantics, including deduction. We feel that most structures that weaken the institution definition are somehow pathological<sup>4</sup>. There are tendencies both to focus on syntax at the expense of semantics, and on semantics at the expense of syntax; the first occurs especially in intuitionistic logic and type theory, while the second is more common in computer science. This paper treats institutions in the original sense, believing that most concepts and results are easily adapted to the variant notions. (However, the notions of charter and parchment [28] formalize genuinely different notions, though still closely related to institutions.)

Over the last fifteen years, there have been even more variations on institution morphisms than on institutions, even discounting those that are adaptations of morphism concepts to other institution-like formalisms; moreover, these notions have been given many different names, includ-

---

<sup>3</sup>E.g., [28] shows that the “ $\pi$ -institutions” of [23] are really institutions.

<sup>4</sup>For example, the main example used to motivate the “pre-institutions” of [57] is an unnatural version of hidden algebra where the morphisms fail to preserve all the relevant structure.

ing morphism, map, mapping, coding, encoding, representation, representation map, embedding, simulation, transformation, and more, most of which do little or nothing to suggest their nature. This paper tries to bring some order to this chaos by exploring their properties and relationships, and by introducing names that suggest their meaning. Goguen and Burstall introduced “morphisms” [29], which are perhaps the most natural, since they include structure forgetting (and hence embedding or representation); but because institution morphisms in this sense do not capture all the important relationships, researchers have introduced many variants. Perhaps the most important of these is dual to institution morphisms, introduced by Meseguer [43] under the name “plain map;” this was later renamed “representation” by Tarklecki [67] and “plain representation” by Mossakowski [45], but because of the duality, we prefer the name *comorphism*. Cerioli introduced the special case of “simulations” [7], Tarlecki introduced “codings” [66], a further weakening, and Meseguer introduced “simple institution maps” [43], which generalize comorphisms by mapping signatures to theories; some variations, including “conjunctive maps” which take a sentence to a set of sentences, were studied by Mossakowski [45], who with Kreowski also introduced “embeddings of institutions” [38], to formalize equivalence of logical frameworks; Sannella and Tarlecki introduced “semi-morphisms” [61, 67], which only have models, for relating specification and implementation languages, and Salibra and Scollo introduced “transformations” [57], which map models to sets of models. Diaconescu introduced “extra theory morphisms” [15] for the semantics of multiparadigm languages like CafeOBJ [16]. It is very helpful to look at examples to gain an understanding of this rocky terrain, and we shall often do so.

We had originally hoped to survey and systematize all the distinct notions of morphism for institutions in the original sense, but we found even this limited goal impractical at less than monograph length; however, we do hope to have covered the most important notions. Section 2 gives brief expositions of indexed categories, twisted relations, and Kan extensions, followed in Section 3 by several equivalent definitions for institutions and their close variants, especially as functors from signatures to twisted relations; a subsection considers “inclusive institutions,” which are institutions with inclusions. The functor formulations allow easy proofs in Section 4 for completeness and cocompleteness results; we also advance the hypothesis that morphisms are in general more natural than comorphisms. Section 5 considers “theoroidal” morphisms and comorphisms, which generalize from signature morphisms to theory morphisms; what we call theoroidal comorphisms were introduced by Meseguer, while theoroidal morphisms appear to be a new concept. Section 6 introduces the new notion of forward morphism, while Section 7 considers semi-natural morphisms and comorphisms, which weaken morphisms by removing one naturality condition. A summary of the paper appears in Section 8, along with a list of some open problems. Appendices A and B discuss partial equational logic, a variant of order sorted algebra that supports partiality, their corresponding institutions, and an appropriate morphism between them; Appendix C gives two institutions for hidden algebra, and Appendix D introduces a new abstract institution for universal algebra. The institutions in these appendices, which draw on the authors’ prior work on more concrete applications, are used in examples in the body of this paper.

**Dedication** This paper is dedicated, most warmly and respectfully, to Prof. Rod Burstall on the occasion of his retirement from the University of Edinburgh. Rod was the cofounder of the institution of institutions and has always been an enthusiastic supporter of its further development. He is also a very close and very dear friend, and one of the most insightful, kind and intelligent people we have ever known. We salute his very distinguished past, and we wish him every success and happiness in his future.

## 2 Preliminaries

We assume the reader familiar with basic categorical concepts, including limits, colimits, functor categories, and adjoints. We use semicolon for morphism composition, written in diagrammatic order, that is, if  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are morphisms, then  $f;g: A \rightarrow C$  is their composition. We let  $\mathbf{C}(a,b)$  denote the morphisms  $a \rightarrow b$  in a category  $\mathbf{C}$ , and we let  $|\mathbf{C}|$  denote the objects of  $\mathbf{C}$ ; also we use “;” for vertical composition of natural transformations and “ $\circ$ ” for their horizontal composition. The reader is assumed familiar with the fact that  $\mathbf{Cat}$  (and thus also  $\mathbf{Cat}^{op}$ ) and  $\mathbf{Set}$  are both complete and cocomplete [40].

### 2.1 Indexed Categories

Institutions, with their variation of syntax and semantics over signatures of non-logical symbols, are an instance of a general categorical notion capturing structures that vary over other structures. Let  $\mathbf{Ind}$  be any category, with objects called indices.

**Definition 1** An **indexed category** is a functor  $C: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$ ; when  $i \in |\mathbf{Ind}|$ , we may write  $C_i$  for  $C(i)$ . Given an indexed category  $C$ , then  $\mathbf{Flat}(C)$  is the category having pairs  $(i, a)$  as objects, where  $i$  is an object in  $\mathbf{Ind}$  and  $a$  is an object in  $C_i$ , and having pairs  $(\alpha, f): (i, a) \rightarrow (i', a')$  as morphisms, where  $\alpha \in \mathbf{Ind}(i, i')$  and  $f \in C_i(a, C_\alpha(a'))$ . ■

The following gives sufficient conditions for the flattening of an indexed category to be complete or cocomplete [69]:

**Theorem 2** *If  $C: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$  is an indexed category, then:*

1. *If  $\mathbf{Ind}$  is complete, if  $C_i$  is complete for each  $i \in |\mathbf{Ind}|$ , and if  $C_\alpha: C_j \rightarrow C_i$  is continuous for each  $\alpha: i \rightarrow j$ , then  $\mathbf{Flat}(C)$  is complete.*
2. *If  $\mathbf{Ind}$  is cocomplete, if  $C_i$  is cocomplete for each  $i \in |\mathbf{Ind}|$ , and if  $C_\alpha: C_j \rightarrow C_i$  has a left adjoint for each  $\alpha: i \rightarrow j$ , then  $\mathbf{Flat}(C)$  is cocomplete.*

Given an indexed category  $C: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$ , define the indexed category  $C^{op}: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$  by  $C_i^{op}$  is  $(C_i)^{op}$  and  $C_\alpha^{op}: C_j^{op} \rightarrow C_i^{op}$  is  $(C_\alpha)^{op}$  for  $\alpha \in \mathbf{Ind}(i, j)$ . The following is direct from Theorem 2, but is worth stating explicitly because it is so easy to become confused by the dualities involved:

**Corollary 3** *If  $C: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$  is an indexed category, then:*

1. *If  $\mathbf{Ind}$  is complete, if  $C_i$  is cocomplete for each  $i \in |\mathbf{Ind}|$ , and if  $C_\alpha: C_j \rightarrow C_i$  is cocontinuous for each  $\alpha: i \rightarrow j$ , then  $\mathbf{Flat}(C^{op})$  is complete.*
2. *If  $\mathbf{Ind}$  is cocomplete, if  $C_i$  is complete for each  $i \in |\mathbf{Ind}|$ , and if  $C_\alpha: C_j \rightarrow C_i$  has a right adjoint for each  $\alpha: i \rightarrow j$ , then  $\mathbf{Flat}(C^{op})$  is cocomplete.*

### 2.2 Functor Categories and Kan Extensions

Given categories  $\mathbf{T}$  and  $\mathbf{S}$ , let  $\mathbf{T}^{\mathbf{S}}$  denote the category of functors from  $\mathbf{S}$  to  $\mathbf{T}$  having natural transformations as morphisms, and for any functor  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$ , let  $\mathbf{T}^\Phi: \mathbf{T}^{\mathbf{S}'} \rightarrow \mathbf{T}^{\mathbf{S}}$  denote the functor defined by  $\mathbf{T}^\Phi(I') = \Phi;I'$  for a functor  $I': \mathbf{S}' \rightarrow \mathbf{T}$ , and by  $\mathbf{T}^\Phi(\sigma) = 1_\Phi \circ \sigma$  for a natural transformation  $\sigma: I' \Rightarrow J'$ , where  $I', J': \mathbf{S}' \rightarrow \mathbf{T}$  are functors. Also let  $\mathbf{T}^-: \mathbf{Cat}^{op} \rightarrow \mathbf{Cat}$  denote the functor that takes a category  $\mathbf{S}$  to  $\mathbf{T}^{\mathbf{S}}$  and a functor  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$  to  $\mathbf{T}^\Phi$ . Note that  $\mathbf{T}^-: \mathbf{Cat}^{op} \rightarrow \mathbf{Cat}$  is an indexed category for any category  $\mathbf{T}$ .

**Proposition 4** *If  $\mathbf{T}$  is complete (cocomplete) then  $\mathbf{T}^{\mathbf{S}}$  is complete (cocomplete) for any category  $\mathbf{S}$ , and  $\mathbf{T}^{\Phi}$  is continuous (cocontinuous) for any functor  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$ .*

**Proof:** Hint: Limits (colimits) in  $\mathbf{T}^{\mathbf{S}}$  are built “pointwise” [40], p. 112.  $\square$

**Definition 5** Given functors  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$  and  $I: \mathbf{S} \rightarrow \mathbf{T}$ , a **right Kan extension of  $I$  along  $\Phi$**  is a pair containing a functor  $I': \mathbf{S}' \rightarrow \mathbf{T}$  and a natural transformation  $\mu: \Phi; I' \Rightarrow I$  which is universal from  $\mathbf{T}^{\Phi}$  to  $I$ , that is, for every  $J': \mathbf{S}' \rightarrow \mathbf{T}$  and  $\mu': \Phi; J' \Rightarrow I$  there is a unique natural transformation  $\sigma: J' \Rightarrow I'$  such that  $\mu' = (1_{\Phi} \circledast \sigma); \mu$ . Dually, a **left Kan extension of  $I$  along  $\Phi$**  is a functor  $I': \mathbf{S}' \rightarrow \mathbf{T}$  and a natural transformation  $\mu: I \Rightarrow \Phi; I'$  which is universal from  $I$  to  $\mathbf{T}^{\Phi}$ , that is, for every  $J': \mathbf{S}' \rightarrow \mathbf{T}$  and  $\mu': I \Rightarrow \Phi; J'$  there is a unique natural transformation  $\sigma: I' \Rightarrow J'$  such that  $\mu' = \mu; (1_{\Phi} \circledast \sigma)$ .  $\blacksquare$

The rest of this section contains general categorical results that are used later in the paper; the first may be found in [40].

**Proposition 6** *Given a small category  $\mathbf{S}$ , then:*

1. *If  $\mathbf{T}$  is complete then any functor  $I: \mathbf{S} \rightarrow \mathbf{T}$  has a right Kan extension along any  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$  and  $\mathbf{T}^{\Phi}$  has a right adjoint.*
2. *If  $\mathbf{T}$  is cocomplete then any functor  $I: \mathbf{S} \rightarrow \mathbf{T}$  has a left Kan extension along any  $\Phi: \mathbf{S} \rightarrow \mathbf{S}'$  and  $\mathbf{T}^{\Phi}$  has a left adjoint.*

**Theorem 7**  $\mathbf{T}^-$  *contravariantly lifts adjoints to functor category adjoints.*

**Proof:** Hint: If  $\langle \Phi, \Phi', \eta, \epsilon \rangle: \mathbf{S} \rightarrow \mathbf{S}'$  is an adjoint (with  $\Phi'$  a left adjoint to  $\Phi$ ), then so is  $\langle \mathbf{T}^{\Phi'}, \mathbf{T}^{\Phi}, \mathbf{T}^{\eta}, \mathbf{T}^{\epsilon} \rangle: \mathbf{T}^{\mathbf{S}} \rightarrow \mathbf{T}^{\mathbf{S}'}$ , where  $(\mathbf{T}^{\eta})_{I'} = \eta \circledast 1_{I'}$  and  $(\mathbf{T}^{\epsilon})_I = \epsilon \circledast 1_I$  for all functors  $I: \mathbf{S} \rightarrow \mathbf{T}$  and  $I': \mathbf{S}' \rightarrow \mathbf{T}$ .  $\square$

Then using the same notation, we have:

**Corollary 8**  *$\text{Nat}(\Phi; I', I) \simeq \text{Nat}(I', \Phi'; I)$ , naturally in both  $I$  and  $I'$ . More precisely, a natural transformation  $\mu: \Phi; I' \Rightarrow I$  goes to  $(\eta \circledast 1_{I'}); (1_{\Phi'} \circledast \mu)$  and conversely, a natural transformation  $\mu': I' \Rightarrow \Phi'; I$  goes to  $(1_{\Phi} \circledast \mu'); (\epsilon \circledast 1_I)$ .*

## 2.3 Twisted Relations

Twisted relations were introduced in [29] and further explored in [54]:

**Definition 9** Let  $\mathbf{Trel}$  be the category of **twisted relations**, with triples  $\langle A, \mathcal{R}, B \rangle$  as its objects, where  $A$  is a category,  $B$  is a set and  $\mathcal{R} \subseteq |A| \times B$ , and with pairs  $\langle F, g \rangle: \langle A, \mathcal{R}, B \rangle \rightarrow \langle A', \mathcal{R}', B' \rangle$  as its morphisms, where  $F: A' \rightarrow A$  is a functor and  $g: B \rightarrow B'$  is a function such that the diagram

$$\begin{array}{ccc} |A| & \xrightarrow{\mathcal{R}} & B \\ F \uparrow & & \downarrow g \\ |A'| & \xrightarrow{\mathcal{R}'} & B' \end{array}$$

commutes, in the sense that for any  $a' \in |A'|$  and  $b \in B$ , we have  $a' \mathcal{R}' g(b)$  iff  $F(a') \mathcal{R} b$ .  $\blacksquare$

There are four natural variants of this definition, arising from the four choices of one of sets or categories for the left and right components of the triples; let us call these the **original variants**, since they already appear in [29]. Those variants where the right component is category-valued give rise to institutions that allow deduction, whereas those where the left component is category-valued give rise to institutions that allow morphisms of models (see the discussion after Definition 11). It is not hard to see that the following holds for all four of the original variants, generalizing the proof given in [54]:

**Proposition 10** *Trel is both complete and cocomplete.*

### 3 Institutions

Here finally is the main basic concept of this paper:

**Definition 11** An **institution**  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  consists of a category **Sign** whose objects are called **signatures**, a functor  $\mathbf{Mod}: \mathbf{Sign} \rightarrow \mathbf{Cat}^{op}$  giving for each signature  $\Sigma$  a category of  $\Sigma$ -**models**, a functor  $\mathbf{Sen}: \mathbf{Sign} \rightarrow \mathbf{Set}$  giving for each signature a set of  $\Sigma$ -**sentences**, and a  $\Sigma$ -indexed relation  $\models = \{\models_{\Sigma} \mid \Sigma \in \mathbf{Sign}\}$  with  $\models_{\Sigma} \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$ , such that for any signature morphism  $\varphi: \Sigma \rightarrow \Sigma'$ , the following diagram commutes,

$$\begin{array}{ccc} \Sigma & |\mathbf{Mod}(\Sigma)| & \xrightarrow{\models_{\Sigma}} \mathbf{Sen}(\Sigma) \\ \varphi \downarrow & \mathbf{Mod}(\varphi) \uparrow & \downarrow \mathbf{Sen}(\varphi) \\ \Sigma' & |\mathbf{Mod}(\Sigma')| & \xrightarrow{\models_{\Sigma'}} \mathbf{Sen}(\Sigma') \end{array}$$

that is, the following **satisfaction condition**

$$m' \models_{\Sigma'} \mathbf{Sen}(\varphi)(f) \quad \text{iff} \quad \mathbf{Mod}(\varphi)(m') \models_{\Sigma} f$$

holds for all  $m' \in |\mathbf{Mod}(\Sigma')|$  and  $f \in \mathbf{Sen}(\Sigma)$ . ■

We often write only  $\varphi$  instead of  $\mathbf{Sen}(\varphi)$  and  $\lrcorner_{\varphi}$  instead of  $\mathbf{Mod}(\varphi)$ ; the functor  $\lrcorner_{\varphi}$  is called the **reduct** functor associated to  $\varphi$ . With this notation, the satisfaction condition becomes

$$m' \models_{\Sigma'} \varphi(f) \quad \text{iff} \quad m' \lrcorner_{\varphi} \models_{\Sigma} f .$$

We also use the satisfaction notation with a set of sentences  $F$  on its right side, letting  $m \models_{\Sigma} F$  mean that  $m$  satisfies each sentence in  $F$ , and further extend this notation by letting  $F \models_{\Sigma} F'$  mean that  $m \models_{\Sigma} F'$  for any  $\Sigma$ -model  $m$  with  $m \models_{\Sigma} F$ . We may omit the subscript  $\Sigma$  in  $\models_{\Sigma}$  when it can be inferred from context. The **closure** of a set of  $\Sigma$ -sentences  $F$ , denoted  $F^{\bullet}$ , is the set of all  $f$  in  $\mathbf{Sen}(\Sigma)$  such that  $F \models_{\Sigma} f$ . The sentences in  $F^{\bullet}$  are often called the **theorems** of  $F$ . Closure is obviously a closure operator, i.e., it is extensive, monotonic and idempotent.

There are four natural variants of the definition of institution, arising from choosing one of **Cat** or **Set** for the targets of the functors **Sen** and **Mod**, being sure that the target of **Mod** is dualized, to remain contravariant; since these already appear in [29], we again call them the **original variants**. The two variants where **Sen** is **Cat**-valued allow deduction via morphisms among sentences (as advocated for example by Lambek and (Phil) Scott [39]) with conjunction appearing as categorical product. In case the category is a partially ordered set, its morphisms can represent an entailment relation; let us call this an **entailment variant**. Let us write  $f \vdash_{\Sigma} f'$

when there is a morphism in  $\mathbf{Sen}(\Sigma)$  from  $f$  to  $f'$ . But there is no reason to restrict to such variants; we could instead use multicategories as advocated by Meseguer [43] with their forgetful functor to sets, or any other appropriate such structure, allowing proofs to be represented. Notice that the notation  $f \vdash f'$  still makes sense for non-entailment variants, although it elides the specific proof. Twisted relations are easily adapted to such variants, as are Proposition 10 and the later completeness results that build upon it. We will informally call these, and any other variants that arise just by substituting other appropriate functors into the twisted category definition, the **close variants** of the institution concept, because technically they proceed in the same way. A yet more categorical definition of institution is given in [29], taking the target categories to be comma categories constructed to be twisted relation categories; general properties of comma categories then replace arguments about twisted relations.

**Example 12** We briefly discuss some institutions that are especially relevant to this paper.

1. Classical unsorted equational logic, the institution of which we denote  $\mathbf{EL}$ , goes back to Birkhoff [2]; it is the one sorted special case of the many sorted equational logic discussed in the next item.
  2. Many sorted equational logic, the institution of which we denote  $\mathbf{MSEL}$ , was first shown to be an institution in early drafts of [29]. Here signatures and algebras are the usual overloaded many sorted signatures and algebras (but we do allow empty carriers), which go back to Goguen [24]; sentences are explicitly universally quantified pairs of terms, and satisfaction is defined in the obvious way. Proving the satisfaction condition does take a bit of work (see [29]), but as with many other examples, this can be alleviated by using charters [28].
  3. Order sorted equational logic, the institution for which we denote  $\mathbf{OSEL}$ , has overloaded order sorted signatures and algebras, with explicitly universally quantified pairs of terms as sentences, and with the obvious satisfaction; see e.g. [30] for details. The first proof that this is an institution was probably given by Han Yan [71] for a case that also included so called sort constraints; see also the proofs in [52] and [45], noting that there are many variants of order sorted algebra [30].
  4. Among the many variants of first order logic, we first mention the one with many sorted function and predicate symbols in its signature, plus of course the usual logical symbols and the models (though we allow empty carriers); let  $\mathbf{MSFOL}$  denote this institution, and let  $\mathbf{FOL}$  denote its unsorted variant; proofs for their satisfaction conditions are sketched in [29].
  5. Many sorted first order logic with equality, denoted  $\mathbf{MSFOLE}$ , enriches  $\mathbf{MSFOL}$  by allowing equations as atoms, rather than just predicates; a proof that this forms an institution is sketched in [29]. The unsorted special case is denoted  $\mathbf{FOLE}$ .
  6. Many sorted Horn clause logic is the same as  $\mathbf{MSFOL}$  except that only Horn clauses are allowed as sentences; let us denote this institution  $\mathbf{MSHCL}$ , its unsorted variant by  $\mathbf{HCL}$ , its variant with equations as additional atoms  $\mathbf{MSHCLE}$ , and its unsorted variant with equations as atoms  $\mathbf{HCLE}$ ; proof sketches again may be found in [29].
  7. Partial equational logic, denoted  $\mathbf{PEL}$ , is discussed in Appendix A.
  8. Supersorted order sorted equational logic, denoted  $\mathbf{OSEL}^?$ , is discussed in Appendix B.
  9. Two hidden equational logics, denoted  $\mathbf{HEL}_1$  and  $\mathbf{HEL}_2$ , are discussed in Appendix C.
- Of course there are many many other examples, some of which have a very different character. ■

### 3.1 Some Basics of Institutions

We review some basics from [29]:

**Proposition 13** *For any morphism  $\varphi: \Sigma \rightarrow \Sigma'$  and sets  $F, F'$  of  $\Sigma$ -sentences:*

1. Closure Lemma:  $\varphi(F^\bullet) \subseteq \varphi(F)^\bullet$ ;
2.  $\varphi(F^\bullet)^\bullet = \varphi(F)^\bullet$ ;
3.  $(F^\bullet \cup F')^\bullet = (F \cup F')^\bullet$ .

**Definition 14** A **specification** or **presentation** is a pair  $(\Sigma, F)$  where  $\Sigma$  is a signature and  $F$  is a set of  $\Sigma$ -sentences. A **specification morphism** from  $(\Sigma, F)$  to  $(\Sigma', F')$  is a signature morphism  $\varphi: \Sigma \rightarrow \Sigma'$  such that  $\varphi(F) \subseteq F'^\bullet$ . Specifications and specification morphisms give a category denoted **Spec**. A **theory**  $(\Sigma, F)$  is a specification with  $F = F^\bullet$ ; the full subcategory of theories in **Spec** is denoted **Th**. ■

The inclusion functor  $\mathcal{U}: \mathbf{Th} \rightarrow \mathbf{Spec}$  is an equivalence of categories, having a left-adjoint-left-inverse  $\mathcal{F}: \mathbf{Spec} \rightarrow \mathbf{Th}$ , given by  $\mathcal{F}(\Sigma, F) = (\Sigma, F^\bullet)$  on objects and identity on morphisms; note that  $\mathcal{F}$  is also a right adjoint of  $\mathcal{U}$ , so that **Th** is a reflective and coreflective subcategory of **Spec**. It is also known [29] that **Th** is cocomplete whenever **Sign** is cocomplete, and that **Th** has pushouts whenever **Sign** does. The following construction for pushouts in **Th** is a special case of the general colimit creation result proved in [29]:

**Proposition 15** *For theory morphisms  $\varphi_1: (\Sigma, F) \rightarrow (\Sigma_1, F_1)$  and  $\varphi_2: (\Sigma, F) \rightarrow (\Sigma_2, F_2)$ , if*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ \varphi_2 \downarrow & & \downarrow \varphi'_1 \\ \Sigma_2 & \xrightarrow{\varphi'_2} & \Sigma' \end{array}$$

*is a pushout in **Sign**, then*

$$\begin{array}{ccc} (\Sigma, F) & \xrightarrow{\varphi_1} & (\Sigma_1, F_1) \\ \varphi_2 \downarrow & & \downarrow \varphi'_1 \\ (\Sigma_2, F_2) & \xrightarrow{\varphi'_2} & (\Sigma', F') \end{array}$$

*is a pushout in **Th**, where  $F' = (\varphi'_1(F_1) \cup \varphi'_2(F_2))^\bullet$ .*

**Definition 16** A theory morphism  $\varphi: (\Sigma, F) \rightarrow (\Sigma', F')$  is **conservative** iff for any  $(\Sigma, F)$ -model  $m$  there is some  $(\Sigma', F')$ -model  $m'$  such that  $m' \upharpoonright_\varphi = m$ . A signature morphism  $\varphi: \Sigma \rightarrow \Sigma'$  is **conservative** iff it is conservative as a morphism of void theories, i.e.  $\varphi: (\Sigma, \emptyset^\bullet) \rightarrow (\Sigma', \emptyset'^\bullet)$ . ■

The following is not difficult to prove (see [56]):

**Proposition 17** *Given  $\varphi: \Sigma \rightarrow \Sigma'$ ,  $f \in \mathbf{Sen}(\Sigma)$  and  $F \subseteq \mathbf{Sen}(\Sigma)$ , then:*

1.  $F \models_\Sigma f$  implies  $\varphi(F) \models_{\Sigma'} \varphi(f)$ .
2. If  $\varphi$  is conservative, then  $F \models_\Sigma f$  iff  $\varphi(F) \models_{\Sigma'} \varphi(f)$ .



The next result (explicit in [54] for the notion of institution in Definition 11, and implicit in [29]) says that an institution over a category of signatures **Sign** can be regarded as a functor with target **Trel**, and *vice versa*; this also holds for the close variants of the institution and twisted relation concepts (when they are appropriately correlated). Theorem 26 extends this result from objects to morphisms and comorphisms.

**Proposition 18** *There is a bijection (i.e., a one-to-one correspondence between classes) between institutions over **Sign** and functors **Sign**  $\rightarrow$  **Trel**.*

Every institution  $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  has an associated functor  $\mathbf{Sign} \rightarrow \mathbf{Trel}$  taking a signature  $\Sigma \in |\mathbf{Sign}|$  to the triple  $\langle \mathbf{Mod}(\Sigma), \models_{\Sigma}, \mathbf{Sen}(\Sigma) \rangle$ , and taking a signature morphism  $\varphi: \Sigma \rightarrow \Sigma'$  to the “twisted” morphism  $\langle \mathbf{Mod}(\varphi), \mathbf{Sen}(\varphi) \rangle$ ; and moreover, every functor  $I: \mathbf{Sign} \rightarrow \mathbf{Trel}$  has an associated institution  $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  such that if  $I(\Sigma) = \langle A_{\Sigma}, \mathcal{R}_{\Sigma}, B_{\Sigma} \rangle$ , then  $\mathbf{Mod}(\Sigma) = A_{\Sigma}$ ,  $\mathbf{Sen}(\Sigma) = B_{\Sigma}$  and  $\models_{\Sigma} = \mathcal{R}_{\Sigma}$ , and such that for a signature morphism  $\varphi: \Sigma \rightarrow \Sigma'$ , if  $I(\varphi) = \langle F_{\varphi}, g_{\varphi} \rangle$ , then  $\mathbf{Mod}(\varphi) = F_{\varphi}$  and  $\mathbf{Sen}(\varphi) = g_{\varphi}$ . Therefore we can use the tuple and functor notations interchangeably for institutions.

An institution where the **Sen** functor is category-valued is said to be **complete** iff for any two  $\Sigma$ -sentences  $f, f'$ , we have

$$f \vdash_{\Sigma} f' \quad \text{iff} \quad f \models_{\Sigma} f' .$$

We can define compactness in the same style, provided  $\mathbf{Sen}(\Sigma)$  has suitable extra structure, such as that of an infinitary multicategory<sup>5</sup>: an institution is **compact** iff whenever  $f \vdash_{\Sigma} f'$  then  $f_0 \vdash_{\Sigma} f'$  for some finite  $f_0 \subseteq f$ .

### 3.2 Inclusive Institutions

In many categories, among the monics are some especially simple and natural maps which may be called *inclusions*. Although many professional category theorists are loathe to consider them, because of their desire to identify things that are isomorphic, inclusions are in fact a natural concept, the use of which can greatly simplify some applications, especially where syntax is the object of study. For example, we really do prefer a *subsignature* to be given by an inclusion, so that the exact same symbols are involved; and the same holds for modules in both programming and specification. At the end of [29], axiomatizing and then exploiting inclusions for modularization was listed among the open problems. A first solution was given in [18] with the formal notion of *inclusion system*, which was then used to significantly simplify the semantics of module systems over an institution. The abstract notion of inclusion system was further studied and simplified in a series of papers [37, 12, 13, 53]. Here we briefly summarize the current state, and sketch some applications.

There is a well-known correspondence between certain small categories and partially ordered sets, or **posets** for short; these categories have exactly one object  $A$  for each element  $a$  in the set, a morphism from  $A$  to  $B$  iff  $a \leq b$ , and they satisfy anti-symmetry, in that if there is a morphism from  $A$  to  $B$  and another from  $B$  to  $A$  then  $A = B$ ; hereafter, we will identify posets with their corresponding categories. Sums and products correspond to unions and intersections, respectively, and a poset with finite sums and products is a lattice, with all the usual properties thereof. Of course, things generalizes from sets to classes, which we will call **poclasses**; we let  $\hookrightarrow$  denote the poclass morphisms.

---

<sup>5</sup>While an ordinary multicategory has finite lists as objects, our notion of **infinitary multicategory** is a monoidal category with arbitrary subsets of a given infinite set as its objects, and with union as its multiplication; we hope to develop this notion, which in this form only works for entailment variants, in more detail at some later time.

**Definition 19** An **inclusive category**  $\mathcal{C}$  is a category with a broad subcategory<sup>6</sup>  $\mathcal{I}$  which is a poclass, called its **subcategory of inclusions**, having finite intersections and unions, such that for every pair of objects  $A, B$ , their union in  $\mathcal{I}$  is a pushout in  $\mathcal{C}$  of their intersection in  $\mathcal{I}$ .  $\mathcal{C}$  is **distributive** iff  $\mathcal{I}$  is distributive. A functor between two inclusive categories is an **inclusive functor** (or **preserves inclusions**) iff it takes inclusions in the source category to inclusions in the target category. ■

This notion of inclusion is similar to that of (weak) inclusion systems [18, 37, 12, 13, 53], except that no factorization properties are assumed; however, the weaker notion is adequate for many purposes. Also, sums and products are not needed for many applications. Inclusive categories can play a similar role to factorization systems [36, 48], but tend to have smoother proofs.

The following enriches an institution with inclusions [56]:

**Definition 20** An **inclusive institution** is an institution with its category of signatures and its **Sen** functor both inclusive. It is **distributive** iff its category of signatures is distributive, and is **semiexact** iff the functor  $\mathbf{Mod} : \mathbf{Sign} \rightarrow \mathbf{Cat}^{op}$  preserves the pushouts<sup>7</sup>, i.e. it takes pushouts in  $\mathbf{Sign}$  to pullbacks in  $\mathbf{Cat}$ . ■

The term semiexactness was introduced in [18] as a weakening of *exactness*, which says that  $\mathbf{Mod}$  preserves general colimits; exactness seems to have first appeared in [60], and was used by Tarlecki [63] on abstract algebraic institutions and by Meseguer [43] on general logics. Although many sorted logics tend to be exact, their unsorted variants tend to be only semiexact.

The category of theories,  $\mathbf{Th}$ , inherits many properties from  $\mathbf{Sign}$ . One of the most important of these is that  $\mathbf{Th}$  is cocomplete if  $\mathbf{Sign}$  is. Moreover,

**Proposition 21** *For an inclusive institution:*

1.  $\mathbf{Th}$  is inclusive; and
2.  $\mathbf{Th}$  has pushouts that preserve inclusions if  $\mathbf{Sign}$  has pushouts that preserve inclusions.

It is often more convenient to speak of a **theory extension** instead of a theory inclusion.

Inspired by Goguen and Tracz’s “implementation oriented” (i.e., more concrete) semantics for modularization [33], Roşu [56] introduced the notion of *module specification* as a generalization of a standard specification, having both public (or visible) and private symbols via inclusions of signatures, and then explored their properties and gave semantics for module composition over an arbitrary inclusive institution. More precisely, a module specification in an inclusive institution is a triple  $(\Sigma, F, \Sigma')$ , where  $\Sigma' \hookrightarrow \Sigma$  and  $F$  is a set of  $\Sigma$ -sentences. The *visible theorems* (or the visible consequences) of a module  $(\Sigma, F, \Sigma')$  are the  $\Sigma'$ -sentences satisfied by  $F$  over  $\Sigma$ , and a model of  $(\Sigma, F, \Sigma')$  is a  $\Sigma'$ -model of its visible consequences.

For another application, inclusive institutions are an attractive alternative to Mossakowski’s “institutions with symbols” [46], which assign a set of symbols to each signature, as part of a semantics for the CASL language [10], since inclusions will automatically keep track of shared symbols in subsignatures, while allowing all the usual operations on modules, including renaming, to be (more) easily and naturally expressed. It is our view that inclusive institutions provide the most natural and easy way to formulate the semantics of specification languages like CASL [10], CafeOBJ [16], and BOBJ [31].

---

<sup>6</sup>In the sense that it has the same objects as  $\mathcal{C}$ .

<sup>7</sup>Actually, we are interested only in pushouts of inclusions, but we wish to avoid introducing a new concept.

## 4 Institution Morphisms and Comorphisms

Perhaps the two best known kinds of morphism between institutions are the original “morphisms” of Goguen and Burstall [29], and the “plain maps” of Meseguer [43], later given the better name “representations” by Tarlecki [67, 66]. We show a natural duality between these, by viewing their categories with institutions as objects as flattened indexed categories; this motivates our preference for the institution comorphism terminology, and also yields easy proofs of completeness and cocompleteness, using the fact that given a functor between signature categories, any institution over the source signature category extends to an institution over the target signature category along that functor in two canonical ways, given by the left and right Kan extensions. Arrais and Fiadeiro [41] showed that given an adjunction between signature categories, an institution morphism gives rise to an institution comorphism and *vice versa*. We show that this result is a natural consequence of the fact that an adjoint between signature categories lifts contravariantly to functor categories.

The original morphisms for institutions introduced with the institution concept in [29] seem to be the most natural notion. In particular, they include structure forgetting, and hence structure embedding or representation relationships. Our examples will show that morphic formulations are usually simpler and more natural in other contexts as well.

**Definition 22** Given institutions  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  and  $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ , an **institution morphism** from  $\mathbb{I}$  to  $\mathbb{I}'$  consists of a functor  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ , a natural transformation  $\beta: \mathbf{Mod} \Rightarrow \Phi; \mathbf{Mod}'$ , and a natural transformation  $\alpha: \Phi; \mathbf{Sen}' \Rightarrow \mathbf{Sen}$ , such that the following **satisfaction condition** holds for each  $\Sigma \in |\mathbf{Sign}|$ ,  $m \in |\mathbf{Mod}(\Sigma)|$  and  $f' \in \mathbf{Sen}'(\Phi(\Sigma))$ ,

$$m \models_{\Sigma} \alpha_{\Sigma}(f') \quad \text{iff} \quad \beta_{\Sigma}(m) \models'_{\Phi(\Sigma)} f'.$$

We let  $\mathcal{INS}$  denote the category of institutions with institution morphisms. ■

Note that the functor  $\Phi$  on signatures and the natural transformation  $\beta$  on models go in the same direction in this definition, while the natural transformation  $\alpha$  goes in the opposite direction.

Meseguer [43] introduced a dual of the institution morphisms of Goguen and Burstall under the name “plain map,” later renamed “representation” by Tarlecki [67, 68]; however, we prefer the name “comorphism” in order to emphasize the important duality between these concepts.

**Definition 23** Given institutions  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  and  $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ , an **institution comorphism** from  $\mathbb{I}$  to  $\mathbb{I}'$  consists of  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ , a natural transformation  $\beta: \Phi; \mathbf{Mod}' \Rightarrow \mathbf{Mod}$ , and a natural transformation  $\alpha: \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$ , such that the following **(co-)satisfaction condition** holds for each  $\Sigma \in |\mathbf{Sign}|$ ,  $m' \in |\mathbf{Mod}'(\Phi(\Sigma))|$ , and  $f \in \mathbf{Sen}(\Sigma)$ ,

$$\beta_{\Sigma}(m') \models_{\Sigma} f \quad \text{iff} \quad m' \models'_{\Phi(\Sigma)} \alpha_{\Sigma}(f).$$

We let  $co\mathcal{INS}$  denote the category of institutions and institution comorphisms. ■

Cerioli introduced the special case of *simulation* [7], which in addition requires that  $\beta$  be a surjective partial natural transformation.

It is characteristic of our subject that the same example can often be presented in more than one way. For example, consider the relationship between the institutions of equational logic and first order logic with equality, for simplicity restricted to the unsorted versions. Since signatures for first order logic with equality are pairs  $(\Pi, \Sigma)$  where  $\Pi$  gives the predicate symbols and  $\Sigma$  gives the function symbols, we can capture the relationship between the two kinds of signature with a forgetful functor sending  $(\Pi, \Sigma)$  to  $\Sigma$ , or with an embedding functor sending  $\Sigma$  to  $(\emptyset, \Sigma)$ . A perhaps insufficiently emphasized small insight from category theory is that it is often better to deal with

forgetful functors than with functors going in the other direction. For example, the forgetful functor from groups to sets better expresses the relationship between these two than the free group functor; and we can see a similar phenomenon in our little example that the forgetful functor avoids the (admittedly rather small) arbitrariness of introducing the empty set. Although intuitively we have an embedding of equational signatures into first order with equality signatures, it is more natural to use the forgetful functor than the embedding functor. The examples below extend this insight from signatures to institutions.

**Example 24** We give some examples of morphisms and comorphisms for embeddings.

1. First some more details of the embedding of equational logic into first order logic with equality. Let  $\Phi$  denote the forgetful functor which on objects sends  $(\Pi, \Sigma)$  to  $\Sigma$ , let  $\beta_\Sigma$  be the forgetful functor sending a  $(\Pi, \Sigma)$ -model to the corresponding  $\Sigma$ -algebra, and let  $\alpha_{(\Pi, \Sigma)}$  send a  $\Sigma$ -equation to the same equation viewed as a  $(\Pi, \Sigma)$ -sentence (which may require adding quantifiers). It is now easy to check the naturality and satisfaction conditions.
2. A contrasting case is the embedding of unsorted equational logic into many sorted equational logic, because here there is no natural forgetful functor for the signatures; therefore this is better seen as a comorphism, with  $\Phi$  mapping an unsorted signature to the corresponding one sorted signature, and with the obvious  $\alpha$  and  $\beta$ .
3. On the other hand, if we modify the many sorted equational logic institution to provide distinguished elements in its sort sets<sup>8</sup>, then there is a natural forgetful functor from many sorted signatures to unsorted signatures, and we get an institution morphism. We encourage the reader to work out the details of this as an exercise.
4. An example similar to the first above (but simpler) is the embedding of Horn clause logic into first order logic. Here the signature categories are the same in the two institutions, consisting of just indexed sets of predicate symbols, and  $\Phi$  is the identity functor. The two model categories are also the same, and  $\beta_\Pi$  consists of all identity functors (where  $\Pi$  is a signature of predicate symbols). Finally, each  $\alpha_\Pi$  is the inclusion of the  $\Pi$ -Horn clauses into the first order  $\Pi$ -sentences. Since so many of the structures in this example are the same, there is no significant difference between using this morphism and using the corresponding comorphism to represent the relationship of the two institutions; moreover, these two are dual in the sense of Section 4.1.
5. There is also a comorphism from equational logic to first order logic with equality. Let  $\Phi'$  send an equational signature  $\Sigma$  to the first order signature  $(\Sigma, \emptyset)$ , let  $\alpha'$  send a  $\Sigma$ -equation to the corresponding  $(\Sigma, \emptyset)$ -sentence, and let  $\beta'$  send a  $(\Sigma, \emptyset)$ -model to the corresponding  $\Sigma$ -algebra. We will see in Section 4.1 that this comorphism is dual to the morphism of item 1 above in a very natural way.

There are many more examples of a similar character. In general, it appears that the forgetful morphism versions are somewhat simpler and more natural than the comorphism versions. ■

**Example 25** There is an institution comorphism from  $\text{OSEL}$ <sup>?</sup> to  $\text{PEL}$  (these institutions of partial equational logic, and of supersorted order sorted equational logic, are defined in Appendices A and B, which also review the notation from [26] that we use here). Given a supersorted signature  $(S, \Sigma)$  and a partial  $(S', \Sigma')$ -algebra  $A'$ , it is natural to extend  $A'$  to an order sorted  $(S, \Sigma)$ -algebra

---

<sup>8</sup>This is by no means an unnatural concept. For example, in the OBJ3 system [34], every module has a “principal sort,” which is needed for computing default views [34]. We can therefore argue that these “pointed sort sets” are more natural, at least for many computer science applications.

by adding a special symbol  $\star$  called the **error element**, to the carrier of each supersort  $s?$ , and extending all partial operations to total operations having the value  $\star$  where they were undefined, and propagate error elements. A disadvantage of this construction is that it does not provide information about the origin of errors.

For any supersorted signature  $(S, \Sigma)$  and partial  $(S', \Sigma')$ -algebra  $A'$ , let  $\beta'_\Sigma(A')$  be the  $S$ -sorted family given by

1.  $(\beta'_\Sigma(A'))_{s'} = A'_{s'}$  for all  $s' \in S'$ , and
2.  $(\beta'_\Sigma(A'))_{s'??} = A'_{s'} \cup \{\star\}$  for all  $s' \in S'$ .

Then  $\beta'_\Sigma(A')$  can be given an  $(S, \Sigma)$ -algebra structure as follows, where  $\sigma$  is an operation in  $\Sigma$ :

1.  $(\beta'_\Sigma(A'))_\sigma(a_1, \dots, a_n) = A'_\sigma(a_1, \dots, a_n)$  if  $a_1, \dots, a_n$  are all different from the error element  $\star$  and  $A'_\sigma(a_1, \dots, a_n)$  is defined; and
2.  $(\beta'_\Sigma(A'))_\sigma(a_1, \dots, a_n) = \star$  if any of  $a_1, \dots, a_n$  is equal to  $\star$  or if  $A'_\sigma(a_1, \dots, a_n)$  is not defined.

We call  $\beta'_\Sigma(A')$  the **single error superextension** of  $A'$ , and it is easily seen that  $\beta'_\Sigma(A')$  is a strict  $\Sigma$ -algebra. As shown in [26],  $\beta'_\Sigma$  can be organized as a functor  $\beta'_\Sigma: \mathbf{PAlg}(\Sigma^b) \rightarrow \mathbf{OSAlg}(\Sigma)$  which is left inverse to  $\mathcal{U}_\Sigma$ , and right adjoint to  $\mathcal{U}_\Sigma$  restricted to strict algebras; moreover,  $\beta'$  is a natural transformation.

Now we can check that  $(\Phi', \beta', \alpha)$ , with  $\alpha$  as defined in Appendix B, and with  $\Phi'$  the forgetful functor  $\_{}^b$  of Appendix B, is a comorphism  $\mathbf{OSEL}^? \rightarrow \mathbf{PEL}$ . When the signature is clear from context, we prefer to write  $\_{}^\star$  for  $\beta'_\Sigma$  and to omit  $\alpha_\Sigma$ . Then the satisfaction condition for this comorphism is as follows, for  $A' \in \mathbf{PAlg}(\Sigma^b)$  and  $(\gamma, e) \in \mathbf{Sen}^?( \Sigma)$ ,

$$A'^\star \models_\Sigma (\gamma, e) \quad \text{iff} \quad A' \models_{\mathfrak{F}_{\Sigma^b}} (\gamma, e) .$$

This not entirely trivial result is proved in [26].

However, a simpler relationship between these institutions is given by an institution morphism  $\mathbf{PEL} \rightarrow \mathbf{OSEL}^?$  that we will now define. Given a many sorted signature  $(D, \Delta)$  and a partial  $\Delta$ -algebra  $A$ , it is natural to extend  $\Delta$  to a supersorted order sorted signature  $\Delta^? = (D \cup D^?, \Delta)$  by adding an error supersort  $d^?$  for each sort  $d \in D$ , extending  $A$  to an order sorted  $(D \cup D^?, \Delta)$ -algebra by adding the error element  $\star$  to the carrier of each supersort  $d^?$ , and extending all partial operations to total operations taking the value  $\star$  where they were undefined. As above, errors are propagated by these operations, and information about the origin of errors is lost.

Given a partial  $\Delta$ -algebra  $A$ , let  $\beta_\Delta(A)$  be the  $(D \cup D^?)$ -sorted family given by

1.  $(\beta_\Delta(A))_d = A_d$  for all  $d \in D$ , and
2.  $(\beta_\Delta(A))_{d^?} = A_d \cup \{\star\}$  for all  $d \in D$ .

Then  $\beta_\Delta(A)$  can be made a  $\Delta^?$ -algebra by defining  $(\beta_\Delta(A))_\sigma(a_1, \dots, a_n)$  to be  $A_\sigma(a_1, \dots, a_n)$  when  $A_\sigma(a_1, \dots, a_n)$  is defined, and  $\star$  when  $A_\sigma(a_1, \dots, a_n)$  is not defined, for  $\sigma \in \Delta$ . We call  $\beta_\Delta(A)$  the **single error superextension** of  $A$ , and it is easy to check that it is a strict  $\Delta^?$ -algebra, and that  $\beta_\Delta$  can be organized as a functor  $\beta_\Delta: \mathbf{PAlg}(\Delta) \rightarrow \mathbf{OSAlg}(\Delta^?)$  which is left inverse to  $\mathcal{U}_{\Delta^?}$ , and right adjoint to  $\mathcal{U}_{\Delta^?}$  restricted to strict algebras; moreover,  $\beta$  is a natural transformation.

Now we can check that  $(\Phi, \beta, \alpha)$ , with  $\alpha$  as in Appendix B, and with  $\Phi$  the functor defined above, is a morphism  $\mathbf{PEL} \rightarrow \mathbf{OSEL}^?$ . As above, when the signature is clear, we may write  $\_{}^\star$  for  $\beta_\Delta$  and omit  $\alpha_\Delta$ , so the satisfaction condition for this institution morphism, for  $A \in \mathbf{PAlg}(\Delta)$  and  $(\gamma, e) \in \mathbf{Sen}^?( \Delta^?)$ , is

$$A \models_{\mathfrak{F}_\Delta} (\gamma, e) \quad \text{iff} \quad A^\star \models_{\Delta^?} (\gamma, e) ,$$

which is not difficult to check.

Let us now compare the morphism and the comorphism. It is clear from the constructions that there are many similarities. But it is also clear that  $\beta$  is significantly simpler to construct than  $\beta'$ , and that  $\Phi$  is simpler than  $\Phi'$ . It also turns out that the morphism satisfaction condition is significantly easier to check than the comorphism condition. All this seems to confirm our hypothesis about the greater naturality of morphisms over comorphisms<sup>9</sup>. ■

The following extends Proposition 18 to morphisms and to comorphisms; of course, it holds for all close variants, and proofs for the case of Definition 11 can be found in [54].

**Theorem 26**  $\mathcal{INS}$  is isomorphic to  $\mathbf{Flat}((\mathbf{Trel-})^{op})$ , and  $co\mathcal{INS}$  is isomorphic to  $\mathbf{Flat}(\mathbf{Trel-})$ .

Therefore we can use morphisms in  $\mathbf{Flat}((\mathbf{Trel-})^{op})$  instead of institution morphisms whenever this simplifies the exposition. The intuition behind this isomorphism is that any institution morphism  $\langle \Phi, \beta, \alpha \rangle$  as in Definition 22 corresponds to a morphism  $\langle \Phi, \mu \rangle$  in  $\mathbf{Flat}((\mathbf{Trel-})^{op})$ ,

$$\begin{array}{ccc}
 \mathbf{Sign} & & \\
 \downarrow \Phi & \searrow \mathbb{I} & \\
 & \mu \updownarrow & \mathbf{Trel} \\
 & \swarrow \mathbb{I}' & \\
 \mathbf{Sign}' & & 
 \end{array}$$

where  $\mu : \Phi; \mathbb{I}' \Rightarrow \mathbb{I}$  is the natural transformation defined as  $\mu_\Sigma = \langle \beta_\Sigma, \alpha_\Sigma \rangle$  for each  $\Sigma$  in  $\mathbf{Sign}$ .

Similarly, we can use morphisms in  $\mathbf{Flat}(\mathbf{Trel-})$  instead of institution comorphisms whenever this simplifies the exposition. The intuition is that any institution comorphism  $\langle \Phi, \beta, \alpha \rangle$  as in Definition 23 corresponds to a morphism  $\langle \Phi, \mu \rangle$  in  $\mathbf{Flat}(\mathbf{Trel-})$ ,

$$\begin{array}{ccc}
 \mathbf{Sign} & & \\
 \downarrow \Phi & \searrow \mathbb{I} & \\
 & \mu \downarrow \updownarrow & \mathbf{Trel} \\
 & \swarrow \mathbb{I}' & \\
 \mathbf{Sign}' & & 
 \end{array}$$

where  $\mu : \mathbb{I} \Rightarrow \Phi; \mathbb{I}'$  is the natural transformation defined by  $\mu_\Sigma = \langle \beta_\Sigma, \alpha_\Sigma \rangle$ .

The following is now an immediate corollary of Theorem 26, using Theorem 2, Propositions 10, 4 and 4, and Corollary 3; of course it holds for all close variants of institutions, and proofs for the case of Definition 11 can be found in [54].

**Corollary 27**  $\mathcal{INS}$  and  $co\mathcal{INS}$  are both complete.

The completeness of  $\mathcal{INS}$  was first shown by Tarlecki in [63] for the notion of institution in Definition 11, and the completeness of  $co\mathcal{INS}$  was shown by Tarlecki in [68], again for the notion of institution in Definition 11.

---

<sup>9</sup>On the other hand, it is interesting to note that it is the comorphism that involves the forgetful functor here, and that the authors only uncovered the morphism recently. Perhaps such phenomena help to explain why much of the literature seems to prefer comorphisms over morphisms.

## 4.1 Duality of Institution Morphisms and Comorphisms

Arrais and Fiadeiro [41] observed that an adjoint pair of functors between two signature categories induces a bijection between associated institution morphisms and comorphisms. This nice result follows easily from the fact that the functor  $\mathbf{Trel}$  contravariantly lifts adjoint pairs to functor categories (Theorem 7), via its Corollary 8; details are in [54] for the case of Definition 11, but of course this and everything else in this subsection holds for all close variants of institutions.

**Theorem 28** *If  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$  has a left adjoint  $\Phi': \mathbf{Sign}' \rightarrow \mathbf{Sign}$  then for any institutions  $\mathbb{I}: \mathbf{Sign} \rightarrow \mathbf{Trel}$  and  $\mathbb{I}': \mathbf{Sign}' \rightarrow \mathbf{Trel}$  there is a bijection between institution morphisms  $\langle \Phi, \mu \rangle: \mathbb{I} \rightarrow \mathbb{I}'$  and institution comorphisms  $\langle \Phi', \mu' \rangle: \mathbb{I}' \rightarrow \mathbb{I}$ . Moreover, this bijection is natural in  $\mathbb{I}$  and  $\mathbb{I}'$ .*

The bijection of Corollary 8 takes a natural transformation  $\mu: \Phi; \mathbb{I}' \Rightarrow \mathbb{I}$  to  $(\eta; 1_{\mathbb{I}'}); (1_{\Phi'}; \mu)$ , and its inverse takes a natural transformation  $\mu': \mathbb{I}' \Rightarrow \Phi'; \mathbb{I}$  to  $(1_{\Phi}; \mu'); (\epsilon; 1_{\mathbb{I}})$ , where  $\eta$  and  $\epsilon$  are the unit and the counit of the adjunction, respectively. Translating that into a more institutional language, by the construction of isomorphisms in Theorem 26, one gets exactly the construction of [41]:

1. Any morphism  $\langle \Phi, \beta, \alpha \rangle: \mathbb{I} \Rightarrow \mathbb{I}'$  yields a comorphism  $\langle \Phi', \beta', \alpha' \rangle: \mathbb{I}' \rightarrow \mathbb{I}$ , where  $\beta'_{\Sigma'} = \beta_{\Phi'(\Sigma')}; \mathbf{Mod}'(\eta_{\Sigma'})$  and  $\alpha'_{\Sigma'} = \mathbf{Sen}'(\eta_{\Sigma'}); \alpha_{\Phi'(\Sigma')}$  for all  $\Sigma' \in |\mathbf{Sign}'|$ .
2. Any comorphism  $\langle \Phi', \beta', \alpha' \rangle: \mathbb{I}' \Rightarrow \mathbb{I}$  yields a morphism  $\langle \Phi, \alpha, \beta \rangle: \mathbb{I} \rightarrow \mathbb{I}'$ , where for all  $\Sigma \in |\mathbf{Sign}|$ ,  $\beta_{\Sigma} = \mathbf{Mod}(\epsilon_{\Sigma}); \beta'_{\Phi(\Sigma)}$ ,  $\alpha_{\Sigma} = \alpha'_{\Phi(\Sigma)}; \mathbf{Sen}(\epsilon_{\Sigma})$ .

**Example 29** The morphisms and comorphisms of Example 24 provide some good examples of the duality discussed above:

1. The functor  $\Phi'$  in item 5 of Example 24, from equational to first order signatures, is left adjoint to the functor  $\Phi$  in item 1 of Example 24, and the morphism (item 1) and comorphism (item 5) between these institutions are dual in exactly the sense of the construction above.
2. The same holds for the morphism of item 3 of Example 24, from many sorted equational logic to unsorted equational logic, and the corresponding modification of the comorphism of item 2 of Example 24, from unsorted equational logic to many sorted equational logic.
3. The same also holds for the morphism and comorphism of item 4 of Example 24, between Horn clause logic and first order logic.

And there are of course many other examples of a similar kind. On the other hand, the morphism and comorphism of Example 25 are not dual in this sense, despite the fact that their functors  $\Phi$  and  $\Phi'$  are adjoint. ■

## 4.2 Kan Extensions of Institutions

Given a morphism from its signature category, any institution can be translated in two distinct canonical ways, given by the two Kan extensions associated to the signature category morphism. The result below follows from Proposition 6, plus Proposition 10, that  $\mathbf{Trel}$  is both complete and cocomplete; as usual, everything in this subsection holds for all close variants.

**Proposition 30** *Given a small category  $\mathbf{Sign}$  and a functor  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ , any institution  $\mathbb{I}: \mathbf{Sign} \rightarrow \mathbf{Trel}$  has both a right and a left Kan extension along  $\Phi$ , and the functor  $\mathbf{Trel}^{\Phi}$  has both a right and a left adjoint.*

The limitation to small categories is insignificant for practical purposes, even though it is inconsistent with the usual formulations of signature categories; for example, in forming the category of equational signatures, we can restrict symbols to those that could be expressed in ASCII, or in an idealized L<sup>A</sup>T<sub>E</sub>X, which are countable sets.

Let **SCat** denote the category of small categories, *SINS* the category of institutions over small signature categories and institution morphisms, and *coSINS* the category of institutions over small signature categories and institution comorphisms. Since **SCat** is both complete and cocomplete, and since Theorem 26 can be adapted to categories of small signatures, we have that *SINS* and *coSINS* are both complete. Although we do not know whether *INS* and *coINS* are cocomplete, the following, in [54] for the case of Definition 11, is sufficient for practical purposes, and holds for any close variant of the institution concept:

**Theorem 31** *SINS and coSINS are both cocomplete.*

## 5 Theoroidal Morphisms

This section considers generalizations of morphisms that involve mapping theories instead of just signatures. As already mentioned, the “maps” of Meseguer [43] are comorphisms generalized in this way, which we call “theoroidal.” We will consider completeness and cocompleteness of categories with theoroidal (co)morphisms. We first define the theoroidal institution of an institution, and then theoroidal morphisms; both these concepts seem to be new, and like all else in this section, they generalize to all close variants of institutions.

**Definition 32** The **theoroidal institution**  $\mathbb{I}^{th}$  of an institution  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  is  $(\mathbf{Th}, \mathbf{Mod}^{th}, \mathbf{Sen}^{th}, \models^{th})$ , where **Th** is the category of theories of  $\mathbb{I}$ ,  $\mathbf{Mod}^{th}$  is the extension of **Mod** to theories,  $\mathbf{Sen}^{th}$  is *sign*; **Sen**, and  $\models^{th}$  is *sign*;  $\models$ , where *sign*: **Th**  $\rightarrow$  **Sign** is the functor which forgets the sentences of a theory. We may omit superscripts *th*, so that  $\mathbb{I}^{th}$  appears as  $(\mathbf{Th}, \mathbf{Mod}, \mathbf{Sen}, \models)$ . ■

It follows that theories of  $\mathbb{I}^{th}$  are pairs  $((\Sigma, F_1), F_2)$  where  $F_1, F_2$  are sets of  $\Sigma$ -sentences, and that the models of  $((\Sigma, F_1), F_2)$  in  $\mathbb{I}^{th}$  are  $(\Sigma, (F_1 \cup F_2))$ -models in  $\mathbb{I}$ . The following natural notions are important for this section:

**Definition 33** Given institutions  $\mathbb{I}$  and  $\mathbb{I}'$ , a functor  $\Phi: \mathbf{Th} \rightarrow \mathbf{Th}'$  is **signature preserving** iff there is a functor  $\Phi^\diamond: \mathbf{Sign} \rightarrow \mathbf{Sign}'$  such that  $\Phi; \text{sign}' = \text{sign}; \Phi^\diamond$ . Similarly, a functor  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$  is **signature preserving** iff there is a functor  $\Phi^\diamond: \mathbf{Th} \rightarrow \mathbf{Th}'$  such that  $\Phi; \text{sign}' = \Phi^\diamond$ . ■

The reader can check that  $\Phi^\diamond$  is unique if it exists. Now we can introduce the main concepts:

**Definition 34** A **theoroidal morphism (comorphism)** from  $\mathbb{I}$  to  $\mathbb{I}'$  is a morphism (comorphism)  $(\Phi, \beta, \alpha)$  from  $\mathbb{I}^{th}$  to  $\mathbb{I}'^{th}$  such that  $\Phi$  is signature preserving. We let *thINS* and *thcoINS* denote the categories of institutions with theoroidal morphisms and comorphisms, respectively, and we let  $_{-}^{th}: \text{thINS} \rightarrow \text{INS}$  and  $_{-}^{th}: \text{thcoINS} \rightarrow \text{coINS}$  denote the associated functors to *INS* and to *coINS*, respectively. ■

To be explicit, the theoroidal morphism satisfaction condition says that for any  $\mathbb{I}$ -theory  $(\Sigma, F)$ , any model  $m \in \mathbf{Mod}(\Sigma, F)$  and any formula  $f' \in \mathbf{Sen}'(\Sigma')$ , where  $\Sigma' = \Phi^\diamond(\Sigma)$ ,

$$m \models_{\Sigma} \alpha_{\Sigma}(f') \quad \text{iff} \quad \beta_{(\Sigma, F)}(m) \models'_{\Sigma'} f',$$

while the theoroidal comorphism satisfaction condition states that for any  $\mathbb{I}$ -theory  $(\Sigma, F)$ , any model  $m' \in \mathbf{Mod}(\Sigma', F')$  and any formula  $f \in \mathbf{Sen}(\Sigma)$ , where  $\Sigma' = \Phi^\diamond(\Sigma)$ ,



$$\beta_{(\Sigma, F)}(m') \models_{\Sigma} f \quad \text{iff} \quad m' \models'_{\Sigma'} \alpha_{\Sigma}(f) .$$

It is immediate that institutions with theoroidal morphisms (or comorphisms) form a category. But despite the simplicity of Definition 34, it can be difficult to check the satisfaction condition directly; however, it fortunately reduces to checking the condition for just the empty theories, as shown in the next two results:

**Proposition 35** *Given institutions  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  and  $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ , a signature preserving functor  $\Phi: \mathbf{Th} \rightarrow \mathbf{Th}'$ , a natural transformation  $\beta: \mathbf{Mod} \Rightarrow \Phi; \mathbf{Mod}'$  and a natural transformation  $\alpha: \Phi; \mathbf{Sen}' \Rightarrow \mathbf{Sen}$ , then  $(\Phi, \beta, \alpha)$  is a theoroidal morphism if and only if*

$$m \models_{\Sigma} \alpha_{\Sigma}(f') \quad \text{iff} \quad \beta_{(\Sigma, \emptyset^{\bullet})}(m) \models'_{\Sigma'} f' ,$$

for any empty theory  $(\Sigma, \emptyset^{\bullet}) \in \mathbf{Th}$ , any model  $m \in \mathbf{Mod}(\Sigma, \emptyset^{\bullet})$  and any formula  $f' \in \mathbf{Sen}'(\Sigma')$ , where  $\Sigma' = \Phi^{\diamond}(\Sigma)$ .

**Proof:** The “only if” part follows from the definition of theoroidal morphism. Conversely, let  $(\Sigma, F)$  be any theory in  $\mathbf{Th}$ , let  $m \in \mathbf{Mod}(\Sigma, F)$  and let  $f' \in \mathbf{Sen}'(\Sigma')$ . Then

$$\begin{aligned} m \models_{\Sigma} \alpha_{\Sigma}(f') & \text{ iff } \beta_{(\Sigma, \emptyset^{\bullet})}(m) \models'_{\Sigma'} f' && \text{(by hypothesis)} \\ & \text{ iff } \mathbf{Mod}'(\Phi(\iota))(\beta_{(\Sigma, F)}(m)) \models'_{\Sigma'} f' && \text{(by the naturality of } \beta) \\ & \text{ iff } \beta_{(\Sigma, F)}(m) \models'_{\Sigma'} f' && \text{(by the satisfaction condition in } \mathbb{I}') \end{aligned}$$

where  $\iota$  is the theory inclusion  $(\Sigma, \emptyset^{\bullet}) \hookrightarrow (\Sigma, F)$ . □

**Proposition 36** *Given institutions  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  and  $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ , a signature preserving functor  $\Phi: \mathbf{Th} \rightarrow \mathbf{Th}'$ , a natural transformation  $\beta: \Phi; \mathbf{Mod}' \Rightarrow \mathbf{Mod}$  and a natural transformation  $\alpha: \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$ , then  $(\Phi, \beta, \alpha)$  is a theoroidal comorphism if and only if*

$$\beta_{(\Sigma, \emptyset^{\bullet})}(m') \models_{\Sigma} f \quad \text{iff} \quad m' \models'_{\Sigma'} \alpha_{\Sigma}(f) ,$$

for any empty theory  $(\Sigma, \emptyset^{\bullet}) \in \mathbf{Th}$ , any model  $m' \in \mathbf{Mod}(\Phi(\Sigma, \emptyset^{\bullet}))$  and any formula  $f \in \mathbf{Sen}(\Sigma)$ , where  $\Sigma' = \Phi^{\diamond}(\Sigma)$ .

**Proof:** The “only if” part follows from the definition of theoroidal comorphism. Conversely, let  $(\Sigma, F) \in \mathbf{Th}$ , let  $m' \in \mathbf{Mod}(\Sigma', F')$ , and let  $f \in \mathbf{Sen}(\Sigma)$ , where  $\Phi(\Sigma, F) = (\Sigma', F')$ . Then

$$\begin{aligned} \beta_{(\Sigma, F)}(m') \models_{\Sigma} f & \text{ iff } \beta_{(\Sigma, F)}(m') \models_{\Sigma} \mathbf{Sen}(\iota)(f) && \text{(\mathbf{Sen}(\iota) is an identity)} \\ & \text{ iff } \mathbf{Mod}(\iota)(\beta_{(\Sigma, F)}(m')) \models_{\Sigma} f && \text{(by the satisfaction condition in } \mathbb{I}^{th}) \\ & \text{ iff } \beta_{(\Sigma, \emptyset^{\bullet})}(\mathbf{Mod}'(\Phi(\iota))(m')) \models_{\Sigma} f && \text{(by the naturality of } \beta) \\ & \text{ iff } \mathbf{Mod}'(\Phi(\iota))(m') \models'_{\Sigma'} \alpha_{\Sigma}(f) && \text{(by hypothesis)} \\ & \text{ iff } m' \models'_{\Sigma'} \mathbf{Sen}'(\Phi(\iota))(\alpha_{\Sigma}(f)) && \text{(by the satisfaction condition in } \mathbb{I}'^{th}) \\ & \text{ iff } m' \models'_{\Sigma'} \alpha_{\Sigma}(f) && \text{(\mathbf{Sen}(\Phi(\iota)) is an identity)} \end{aligned}$$

where  $\iota$  is the theory inclusion  $(\Sigma, \emptyset^{\bullet}) \hookrightarrow (\Sigma, F)$ . □

Meseguer [43] defined<sup>10</sup> his maps as in Proposition 36, but with the additional requirement that  $\Phi$  be  $\alpha$ -sensible<sup>11</sup>, which seems not only natural, but also technically desirable for proving properties beyond the above, as in the following:

**Conjecture 37** *With appropriate restrictions on morphisms, such as sensibility, thSINS and thcoSINS are complete and cocomplete.*

<sup>10</sup>However, Meseguer used presentations instead of theories.

<sup>11</sup>This essentially means that  $\Phi$  is completely determined by its restriction to empty theories and  $\alpha$ .

## 5.1 Simple Theoroidal Morphisms

There is an important special case of theoroidal comorphism that often occurs in practice, called “simple”<sup>12</sup> by Meseguer [43], that maps signatures to theories instead of theories to theories:

**Definition 38** A **simple theoroidal morphism (comorphism)** from  $\mathbb{I}$  to  $\mathbb{I}'$  is a morphism (comorphism)  $(\Phi, \beta, \alpha)$  from  $\mathbb{I}$  to  $\mathbb{I}'^{th}$  such that  $\Phi$  is signature preserving.  $\blacksquare$

Notice that simple theoroidal (co)morphisms reduce to ordinary (co)morphisms where signatures map to theories with no axioms. Also notice that the simple theoroidal morphism satisfaction condition says that for any signature  $\Sigma \in \mathbf{Sign}$ , any model  $m \in \mathbf{Mod}(\Sigma)$  and any formula  $f' \in \mathbf{Sen}'(\Sigma')$ , where  $\Sigma' = \Phi^\diamond(\Sigma)$ ,

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_\Sigma(m) \models_{\Sigma'} f' ,$$

while the satisfaction condition for a simple theoroidal comorphism states that for any  $\Sigma \in \mathbf{Sign}$ , any  $m' \in \mathbf{Mod}(\Sigma')$  and any formula  $f \in \mathbf{Sen}(\Sigma)$ , where  $\Sigma' = \Phi^\diamond(\Sigma)$ ,

$$\beta_\Sigma(m') \models_\Sigma f \quad \text{iff} \quad m' \models_{\Sigma'} \alpha_\Sigma(f) .$$

If  $(\Phi, \beta, \alpha) : \mathbb{I} \rightarrow \mathbb{I}'$  is a simple theoroidal morphism of institutions, then let  $(\Phi, \beta, \alpha)^{th}$  be the theoroidal morphism  $(\Phi^{th}, \beta^{th}, \alpha^{th})$  from  $\mathbb{I}$  to  $\mathbb{I}'$  defined as  $\Phi^{th}(\Sigma, F) = (\Sigma', (\alpha_\Sigma^{-1}(F) \cup F_\emptyset^\Sigma)^\bullet)$  for each theory  $(\Sigma, F) \in \mathbf{Th}$ , where  $\Phi(\Sigma) = (\Sigma', F_\emptyset^\Sigma)$ , i.e.,  $F_\emptyset^\Sigma$  is the set of  $\mathbb{I}'$ -sentences associated by  $\Phi$  to the  $\mathbb{I}$ -signature  $\Sigma$ , and where also  $\beta_{(\Sigma, F)}^{th}(m) = \beta_\Sigma(m)$  for each  $(\Sigma, F)$ -model  $m$ , and  $\alpha^{th}$  is exactly  $\alpha$ . We let the reader check that indeed  $\Phi^{th}$  is a signature preserving functor and that  $\beta^{th}$  and  $\alpha^{th}$  are natural transformations. The satisfaction condition follows by Proposition 35 using that  $\beta_{(\Sigma, \emptyset^\bullet)}^{th}$  is exactly  $\beta_\Sigma$ .

The most natural way to compose simple morphisms is as in Kleisli categories, that is, to compose the first simple theoroidal morphism with the extension of the second to a theoroidal morphism. More precisely, given two simple morphisms of institutions  $(\Phi_1, \beta_1, \alpha_1)$  from  $\mathbb{I}_1$  to  $\mathbb{I}_2$  and  $(\Phi_2, \beta_2, \alpha_2)$  from  $\mathbb{I}_2$  to  $\mathbb{I}_3$ , their composition  $(\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)$  is defined as the institution morphism  $(\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)^{th}$  from  $\mathbb{I}_1$  to  $\mathbb{I}_3^{th}$ . Unfortunately, in order to prove the associativity of morphism composition, one has to show that  $((\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)^{th})^{th}$  equals  $(\Phi_1, \beta_1, \alpha_1)^{th}; (\Phi_2, \beta_2, \alpha_2)^{th}$ , which doesn't seem to follow without further assumptions; at this time, we don't know what the weakest requirements should be.

The situation is better for simple theoroidal comorphisms, because here  $\Phi$  and  $\alpha$  go in the same direction; indeed, simple theoroidal comorphisms form a category without any additional assumptions. If  $(\Phi, \beta, \alpha) : \mathbb{I} \rightarrow \mathbb{I}'$  is a simple theoroidal comorphism of institutions, then let  $(\Phi, \beta, \alpha)^{th}$  be the theoroidal comorphism  $(\Phi^{th}, \beta^{th}, \alpha^{th})$  from  $\mathbb{I}$  to  $\mathbb{I}'$  defined as  $\Phi^{th}(\Sigma, F) = (\Sigma', (\alpha_\Sigma(F) \cup F_\emptyset^\Sigma)^\bullet)$  for each theory  $(\Sigma, F) \in \mathbf{Th}$ , where  $\Phi(\Sigma) = (\Sigma', F_\emptyset^\Sigma)$ ,  $\beta_{(\Sigma, F)}^{th}(m) = \beta_\Sigma(m)$  for each  $(\Sigma, F)$ -model  $m$ , and  $\alpha^{th}$  is exactly  $\alpha$ . We let the reader check that indeed  $\Phi^{th}$  is a signature preserving functor, that  $\beta^{th}$  is well defined (the satisfaction condition of  $(\Phi, \beta, \alpha)$  is needed), is a natural transformation, and that  $\alpha^{th}$  is also a natural transformation. The satisfaction condition follows by Proposition 36 using that  $\beta_{(\Sigma, \emptyset^\bullet)}^{th}$  is exactly  $\beta_\Sigma$ .

Simple comorphisms can be composed as expected from Kleisli, composing the first with the extension of the second, i.e., given simple comorphisms  $(\Phi_1, \beta_1, \alpha_1)$  from  $\mathbb{I}_1$  to  $\mathbb{I}_2$  and  $(\Phi_2, \beta_2, \alpha_2)$  from  $\mathbb{I}_2$  to  $\mathbb{I}_3$ , their composition  $(\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)$  is defined to be  $(\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)^{th}$  from  $\mathbb{I}_1$  to  $\mathbb{I}_3^{th}$ . To show associativity, one must show that  $((\Phi_1, \beta_1, \alpha_1); (\Phi_2, \beta_2, \alpha_2)^{th})^{th}$  equals  $(\Phi_1, \beta_1, \alpha_1)^{th}; (\Phi_2, \beta_2, \alpha_2)^{th}$ , which after some calculation reduces to showing that

<sup>12</sup>We have not yet thought of a better name for this, but we do feel that one is needed.

$$(\alpha_{2,\Sigma_2}(\alpha_{1,\Sigma_1}(F_1)) \cup (\alpha_{2,\Sigma_2}(F_0^{\Sigma_1}) \cup F_0^{\Sigma_2})^\bullet)^\bullet = (\alpha_{2,\Sigma_2}((\alpha_{1,\Sigma_1}(F_1) \cup F_0^{\Sigma_1})^\bullet) \cup F_0^{\Sigma_2})^\bullet,$$

where  $\Phi_1(\Sigma_1) = (\Sigma_2, F_0^{\Sigma_1})$  and  $\Phi_2(\Sigma_2) = (\Sigma_3, F_0^{\Sigma_2})$ , and where the last assertion follows from 3. of Proposition 13 and the following:

**Proposition 39** *If  $(\Phi, \beta, \alpha)$  is a comorphism (or a simple theoroidal co morphism or a theoroidal comorphism) of institutions from  $\mathbb{I}$  to  $\mathbb{I}'$  and  $F$  is a set of  $\Sigma$ -sentences of  $\mathbb{I}$ , then  $\alpha_\Sigma(F^\bullet) \subseteq \alpha_\Sigma(F)^\bullet$  and  $\alpha_\Sigma(F^\bullet)^\bullet = \alpha_\Sigma(F)^\bullet$ .*

**Proof:** Let  $m' \models_{\Sigma'} \alpha_\Sigma(F)$ , where  $\Sigma' = \Phi(\Sigma)$  in the case of comorphisms and  $\Sigma' = \Phi^\diamond(\Sigma)$  in the case of (simple) theoroidal comorphisms. Then  $m' \models_{\Sigma'} \alpha_\Sigma(F)$  iff (by the satisfaction condition)  $\beta_\Sigma(m') \models_\Sigma F$  iff  $\beta_\Sigma(m') \models_\Sigma F^\bullet$  iff (by the satisfaction condition)  $m' \models_{\Sigma'} \alpha_\Sigma(F^\bullet)$ . Therefore  $\alpha_\Sigma(F) \models_{\Sigma'} \alpha_\Sigma(F^\bullet)$ , which proves the inclusion. Then the equality is immediate.  $\square$

**Example 40** We consider the relationship between FOLE and FOL, unsorted first order equational logic with and without equality, respectively. First observe that there is a very simple and natural morphism FOLE  $\rightarrow$  FOL, where the functor  $\Phi$  forms the disjoint union of an FOLE signature  $\Sigma$  with the symbol “=”; for notational convenience, we may denote this signature by  $\Sigma^=$  and we assume that “=” does not occur in any FOLE signature, but is reserved for equality in FOLE sentences. Given an FOLE signature  $\Sigma$  and a FOLE  $\Sigma$ -model  $M$ , we define  $\beta_\Sigma(M)$  to be the  $\Phi(\Sigma)$ -model  $M^=$  with the equality symbol interpreted as actual identity in  $M$ ; it is easy to see that  $\beta$  is natural. Given any FOL  $\Sigma^=$ -sentence  $f'$ , let  $\alpha_\Sigma(f')$  be just  $f'$ , but with “=” now viewed as the symbol used to form equational atoms. The satisfaction condition follows easily.

Although it is certainly very simple and natural, this morphism fails to capture the familiar trick of axiomatizing equality when moving from FOLE to FOL, as is needed to use a first order theorem prover on the translations of FOLE sentences. However, it is easy to extend it to a simple theoroidal morphism, the theories of which contain axioms for equality, such as reflexivity and symmetry: let the signature map send  $\Sigma$  to  $\Psi(\Sigma) = (\Phi(\Sigma), T(\Sigma))$  where  $\Phi$  is as above and where  $T(\Sigma)$  is a  $\Phi(\Sigma)$ -theory of equality. But there is something strange about this, because the satisfaction condition holds no matter what axioms we give, including none at all — unless some of them are wrong.

On the other hand, to view this situation as a comorphism, it must be simple theoroidal with equality axioms, for the satisfaction condition to hold. We use  $\Psi$  as above for the signature to theory map, and given a FOL  $\Psi(\Sigma)$ -model  $M'$ , we define  $\beta_\Sigma(M')$  to be the reduct  $M' \upharpoonright_\Sigma$ ; it is easy to see that  $\beta$  is natural. Also, given an FOLE  $\Sigma$ -sentence  $f$ , let  $\alpha_\Sigma(f)$  be  $f$  with “=” viewed as the new predicate symbol in  $\Sigma^=$ . For the satisfaction condition to hold, the axioms in  $T(\Sigma)$  must be strong enough to force the equality symbol to be interpreted as identity in models; this will rely on a completeness theorem for equational logic.  $\blacksquare$

This example helps confirm our hypothesis that morphisms are usually simpler and more natural than comorphisms, but it also shows that morphisms may not encapsulate all the information we want to have available; the theoroidal morphism is simple and natural, and it can include all the information we want, but it is curious that this information is not necessary. The comorphism is more complex because it needs a complete set of equality axioms over the given signature and relies on a non-trivial completeness theorem. However, we have also seen that the notion of simple theoroidal morphism is rather complex, perhaps even problematic. Clearly there is more work to be done in this area.

## 6 Forward Morphisms

Both institution morphisms and comorphisms have their syntactic and semantic components going in opposite directions. But there are examples where it seems natural for these go in the same direction. We will speak of “forward morphisms” when both go in the forward direction. The following is the theoroidal version of this concept, though there is of course also a version at the ordinary level; as usual, everything works for all close variants, at both these levels:

**Definition 41** Given institutions  $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$  and  $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ , then a **theoroidal forward institution morphism, from  $\mathbb{I}$  to  $\mathbb{I}'$** , consists of

- $\Phi: \mathbf{Sign} \rightarrow \mathbf{Th}(\mathbb{I}')$  is signature preserving,
- $\beta: \mathbf{Mod} \Rightarrow \Phi; \mathbf{Mod}'$  is a natural transformation, and
- $\alpha: \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$  is a natural transformation,

such that for any signature  $\Sigma \in \mathbf{Sign}$ , any sentence  $f \in \mathbf{Sen}(\Sigma)$  and any model  $m \in \mathbf{Mod}(\Sigma)$ , the **satisfaction condition** holds

$$m \models_{\Sigma} f \quad \text{iff} \quad \beta_{\Sigma}(m) \models'_{\Phi(\Sigma)} \alpha_{\Sigma}(f) . \quad \blacksquare$$

**Example 42** There is a natural relationship between the two institutions,  $\mathbb{HEL}_1$  and  $\mathbb{HEL}_2$ , for hidden equational logic that are described in Appendix C:

- since congruent operations are declared as sentences, any signature in the first institution translates to a specification in the second;
- any model  $A$  of  $(\Sigma, , )$  in the first institution gives a model of the second, namely  $(A, \equiv_{\Sigma}^{\Gamma})$ ;
- any  $(\Sigma, , )$ -sentence is a  $\Sigma$ -sentence;

and we can see that for any  $(\Sigma, , )$ -sentence  $f$  and any hidden  $\Sigma$ -algebra  $A$ , we get  $A \models_{\Sigma}^{\Gamma} f$  iff  $(A, \equiv_{\Sigma}^{\Gamma}) \models_{\Sigma} f$ . All these say that there is an theoroidal forward morphism from  $\mathbb{HEL}_1$  to  $\mathbb{HEL}_2$ .  $\blacksquare$

Of course, we can also define forward theoroidal comorphisms in much the same way, as well as simple theoroidal versions, and these will work for all close variants. Moreover, we can “untwist” the definitions and results about twisted relations, institutions, morphisms and comorphisms to obtain forward versions of all the main results, including completeness and cocompleteness of the categories with institutions as objects, and with morphisms or comorphisms.

It is easy to give corresponding definitions for backward notions, but this is unnecessary, because a backward morphism is just a forward comorphism, and a backward comorphism is just a forward morphism; because of these relationships, it is not even necessary to introduce the terminology.

## 7 Semi-Natural Institution Morphisms and Comorphisms

The following weakens comorphisms by eliminating one of the naturality conditions; as usual, everything in this section holds for all close variants.

**Definition 43** Given institutions  $\mathbb{I} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \models \rangle$  and  $\mathbb{I}' = \langle \mathbf{Sign}', \mathbf{Sen}', \mathbf{Mod}', \models' \rangle$ , a **semi-natural institution comorphism**  $\langle \Phi, \alpha, \beta \rangle: \mathbb{I} \rightarrow \mathbb{I}'$  consists of

- a functor  $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ ,
- a family of functors  $\beta = \{\beta_{\Sigma}: \mathbf{Mod}'(\Phi(\Sigma)) \rightarrow \mathbf{Mod}(\Sigma)\}_{\Sigma \in |\mathbf{Sign}|}$ , and
- a natural transformation  $\alpha: \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$ ,

such that for every  $\Sigma \in |\mathbf{Sign}|$ ,  $m' \in \mathbf{Mod}'(\Phi(\Sigma))$  and  $f \in \mathbf{Sen}(\Sigma)$ , the following **(co-)satisfaction condition** holds:

$$\beta_{\Sigma}(m') \models_{\Sigma} f \quad \text{iff} \quad m' \models'_{\Phi(\Sigma)} \alpha_{\Sigma}(f) . \quad \blacksquare$$

The point to note here is that  $\beta$  need not be natural; this condition is not satisfied in some examples, and is not needed to ensure some significant properties. The following shows that the free superextension of a partial algebra to an order sorted algebra [26] gives rise to a semi-natural institution comorphism.

**Example 44** Another natural expansion of a partial algebra to a supersorted algebra is the free extension, which freely adds supersorted terms for operations when they are undefined. We formalize this construction in the following.

For any supersorted signature  $(S, \Sigma)$  and partial  $(S', \Sigma')$ -algebra  $A'$ , let  $\beta_{\Sigma}(A')$  be the smallest  $S$ -sorted family such that:

1.  $(\beta_{\Sigma}(A'))_{s'} = A'_{s'}$  for all  $s' \in S'$  – let us call the elements of  $(\beta_{\Sigma}(A'))_{s'}$  the **pure** elements;
2.  $\beta_{\Sigma}(A')_{s'} \subseteq \beta_{\Sigma}(A')_s$  whenever  $s' \leq s$ ; and
3.  $\sigma(a_1, \dots, a_n)$  is in  $\beta_{\Sigma}(A')_s$  and is called **impure** whenever any of  $a_1, \dots, a_n$  are impure or  $A'_{\sigma}(a_1, \dots, a_n)$  is not defined, where  $\sigma: w \rightarrow s$  is an operation with  $|w| = n$  and where  $A'_{\sigma}$  is the partial map which interprets  $\sigma: w' \rightarrow s'$  in  $A'$ .

Then  $\beta_{\Sigma}(A')$  can be given an  $(S, \Sigma)$ -algebra structure as follows:

1.  $(\beta_{\Sigma}(A'))_{\sigma}(a_1, \dots, a_n) = A'_{\sigma}(a_1, \dots, a_n)$  if  $a_1, \dots, a_n$  are all pure and  $A'_{\sigma}(a_1, \dots, a_n)$  is defined, and
2.  $(\beta_{\Sigma}(A'))_{\sigma}(a_1, \dots, a_n) = \sigma(a_1, \dots, a_n)$  if any of  $a_1, \dots, a_n$  are impure or if  $A'_{\sigma}(a_1, \dots, a_n)$  is not defined,

where  $\sigma$  is as above. We call the  $(S, \Sigma)$ -algebra  $\beta_{\Sigma}(A')$  the **free superextension** of  $A'$ . As shown in [26],  $\beta_{\Sigma}$  can be organized as a functor  $\beta_{\Sigma}: \mathbf{PAlg}(\Sigma^b) \rightarrow \mathbf{OAlg}(\Sigma)$  which is left inverse left adjoint to  $\mathcal{U}_{\Sigma}$ . When the signature is clear from the context, we prefer to use the notation  $\underline{\beta}$  instead of  $\beta_{\Sigma}$ .

Although all these constructions are very natural,  $\beta$  is still not a natural transformation. To see this, let  $\varphi = (f, g): (S_1, \Sigma_1) \rightarrow (S_2, \Sigma_2)$  be a morphism of supersorted signatures, and let  $A'$  be a partial  $\Sigma_2'$ -algebra. Then the free superextension of the  $\varphi$ -reduct of  $A'$  involves operation symbols in  $\Sigma_1$  but the  $\varphi$ -reduct of the free superextension of  $A'$  involves operation symbols in  $\Sigma_2$ , so that these two  $\Sigma_1$ -algebras cannot be equal. (However, they are isomorphic if  $\varphi$  is injective.)

Now the satisfaction condition for the semi-natural institution comorphism from  $\mathbf{OSEL}^?$  to  $\mathbf{PEL}$  can be formulated as follows: for every  $A' \in \mathbf{PAlg}(\Sigma^b)$  and  $(\gamma, e) \in \mathbf{Sen}^?( \Sigma)$ ,

$$A'^{\natural} \models_{\Sigma} (\gamma, e) \quad \text{iff} \quad A' \models_{\Sigma^b} (\gamma, e) .$$

This result is proved in [26].

Although the relationship between institutions is not quite so neat for the free superextension construction as for the single error superextension of the previous subsection, the former is more useful for many purposes, because it preserves information about why functions are undefined that is very useful for doing proofs, as well as for other purposes.  $\blacksquare$

The notion that we call semi-naturality was introduced in the context of membership equational logic by Meseguer with his “general maps of institutions” [44], where  $\alpha$  is not required to be natural, but only a signature indexed family of functions, just as with  $\beta$  in our Definition 43. At present, it is unclear how important semi-natural morphisms or comorphisms may be, or what are the general properties of their institutions. For this reason, the fact that we do not know any examples of semi-natural morphisms may be another point in favor of the morphism concept.

## 8 Summary and Further Research

Mathematicians, and even logicians, have not shown much interest in the theory of institutions, perhaps because their tendency towards Platonism inclines them to believe that there is just one true logic and model theory; it also doesn't much help that institutions use category theory extensively.

On the other hand, computer scientists, having been forcibly impressed with the need to work with a number of different logics, often for very practical reasons, have written hundreds of papers that apply or further develop the theory of institutions. Institution morphisms become especially relevant when multiple logical systems need to be used for the same application, and somehow coordinated, as often occurs in complex systems, where different logics are used for different aspects, including functional requirements, safety and liveness properties, concurrency control, real time response, data type design, and architectural structure.

We would like to emphasize certain points made in the body of this paper which, though not really new, do seem insufficiently appreciated in the current literature.

1. The notion of institution easily accomodates inference for logical systems; this was already noted in the basic early paper on institutions [29], and this theme is further developed here with our notion of "close variant." This fact makes it unnecessary to combine institutions with other, more familiar, machinery to handle inference.
2. It is easy to add a notion of inclusion to a category, and hence to an institution, and this can greatly simplify many typical applications of institutions, such as giving semantics to a specification language. In every single practical example we know, the category of signatures has a natural and obvious notion of inclusion, so it is quite harmless to assume an inclusive institution when doing specification semantics over an arbitrary institution.
3. In many cases, institution morphisms in the original sense [29] provide more natural formulations of important relationships between institutions than more recent notions.
4. Results about institutions can often be pulled out of a general categorical hat, after a little translation, generalization and/or massaging. Indeed, we are now often left feeling unsatisfied unless we have managed to do this for our major results. The use of indexed categories in Section 2.1 is one good example, the duality of morphisms and comorphisms is another, and the construction of theoroidal morphisms and comorphisms using the theoroidal institution is a third.

In this paper we have tried to bring some additional order to the menagerie of morphisms between institutions, starting with but not limited to, an improved taxonomy for the various genres and species, bringing out some unexpected relationships, and some new properties. Our new nomenclature includes the forms co-, semi-, theoroidal, and forward, among which all combinations are meaningful, and some special cases, such as simple. All of these could be adapted to various institution-like formalisms, but we argue that there is no good reason to do so.

As is often the case, it seems to us that our research has opened far more questions than it has closed, including the following:

- One general class of questions concerns properties of the various categories of institutions, the most immediate of which is how complete and cocomplete they are. Another question is, which ones can be seen as flattened indexed categories?
- One can also ask for each category of institutions, which of its morphisms admit Kan extensions? However, one should also ask for interesting applications for translating a whole logical system along a mapping of its syntax in this way.

- To what extent do the various morphisms support the reuse of logics and theorem provers in the style suggested in [29] and later in [8]?
- To what extent do the various morphisms support the “extra theory morphisms” and “Grothendieck construction” of Diaconescu in [15] and [20], respectively?
- Finally, one might wonder about applying the machinery of this paper to the rapidly evolving field of coalgebra. For example, would there be any value to coinstitutions, or to dualizing the material in Appendix D?

## References

- [1] Narjes Berregeb, Adel Bouhoula, and Michaël Rusinowitch. Observational proofs with critical contexts. In *Fundamental Approaches to Software Engineering*, volume 1382 of *Lecture Notes in Computer Science*, pages 38–53. Springer-Verlag, 1998.
- [2] Garrett Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 31:433–454, 1935.
- [3] Peter Burmeister. *A Model Theoretic Oriented Approach to Partial Algebras*. Akademie-Verlag Berlin, 1986.
- [4] Rod Burstall and Răzvan Diaconescu. Hiding and behaviour: an institutional approach. In A. William Roscoe, editor, *A Classical Mind: Essays in Honour of C.A.R. Hoare*, pages 75–92. Prentice-Hall, 1994. Also in Technical Report ECS-LFCS-8892-253, Laboratory for Foundations of Computer Science, University of Edinburgh, 1992.
- [5] Rod Burstall and Joseph Goguen. Semantics of Clear. Unpublished notes handed out at the 1978 Symposium on Algebra and Applications, Stefan Banach Center, Warsaw, Poland, 1978.
- [6] Rod Burstall and Joseph Goguen. The semantics of Clear, a specification language. In Dines Bjorner, editor, *Proceedings, 1979 Copenhagen Winter School on Abstract Software Specification*, pages 292–332. Springer, 1980. *Lecture Notes in Computer Science*, Volume 86; based on unpublished notes handed out at the Symposium on Algebra and Applications, Stefan Banach Center, Warsaw, Poland, 1978.
- [7] Maura Cerioli. *Relationships between Logical Frameworks*. PhD thesis, Universities of Genova, Pisa and Udine, 1993.
- [8] Maura Cerioli and José Meseguer. May I borrow your logic? (Transporting logical structures along maps). *Theoretical Computer Science*, 173(2):311–347, 1997.
- [9] Maura Cerioli, Till Mossakowski, and Horst Reichel. From total equational to partial first order. In E. Astesiano, H.-J. Kreowski, and B. Krieg-Bruckner, editors, *Algebraic Foundation of Information Systems Specification*, pages 31–104. Springer Verlag, 1999. Chapter 3. A preliminary version is in DISI-TR-96, 1996.
- [10] CoFI. CASL summary, 2000. [www.brics.dk/Projects/CoFi](http://www.brics.dk/Projects/CoFi).
- [11] Virgil Emil Căzănescu. Local equational logic 2. In Grzegorz Rosenberg and Arto Salomaa, editors, *Developments in Language Theory: At The Crossroads of Mathematics, Computer Science and Biology*, pages 210–221. World Scientific, 1994.
- [12] Virgil Emil Căzănescu and Grigore Roşu. Weak inclusion systems. *Mathematical Structures in Computer Science*, 7(2):195–206, 1997.
- [13] Virgil Emil Căzănescu and Grigore Roşu. Weak inclusion systems, part 2. *Journal of Universal Computer Science*, 6(1):5–21, 2000.
- [14] Răzvan Diaconescu. Category-based modularization for equational logic programming. *Acta Informatica*, 33(5):477–510, 1996.

- [15] Răzvan Diaconescu. Extra theory morphisms in institutions: logical semantics for multi-paradigm languages. *Applied Categorical Structures*, 6(4):427–453, 1998.
- [16] Răzvan Diaconescu and Kokichi Futatsugi. *CafeOBJ Report: The Language, Proof Techniques, and Methodologies for Object-Oriented Algebraic Specification*. World Scientific, 1998. AMAST Series in Computing, volume 6.
- [17] Răzvan Diaconescu and Kokichi Futatsugi. Behavioral coherence in object-oriented algebraic specification. *Journal of Universal Computer Science*, 6(1):74–96, 2000. Also Japan Advanced Institute for Science and Technology Technical Report number IS–RR–98–0017F, 1998.
- [18] Răzvan Diaconescu, Joseph Goguen, and Petros Stefanias. Logical support for modularization. In Gerard Huet and Gordon Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993.
- [19] Răzvan Diaconescu. *Category-based Semantics for Equational and Constraint Logic Programming*. PhD thesis, University of Oxford, 1994.
- [20] Răzvan Diaconescu. Grothendieck institutions. IMAR Preprint 2–2000, Institute of Mathematics of the Romanian Academy, February 2000. ISSN 250–3638.
- [21] Răzvan Diaconescu and Kokichi Futatsugi. Logical foundations of CafeOBJ. *Theoretical Computer Science*, To appear.
- [22] Hartmut Ehrig, Michael Baldamus, Felix Cornelius, and Fernando Orejas. Theory of algebraic module specification including behavioural semantics and constraints. In Maurice Nivat, Charles Rattray, Teodor Rus, and Giuseppe Scollo, editors, *Proceedings of the Second International Conference on Algebraic Methodology and Software Technology*, Workshops in Computing, pages 145–172, London, May 22–25 1992. Springer Verlag.
- [23] José Fiadeiro and Amílcar Sernadas. Structuring theories on consequence. In Donald Sannella and Andrzej Tarlecki, editors, *Recent Trends in Data Type Specification*, pages 44–72. Springer, 1988. Lecture Notes in Computer Science, Volume 332.
- [24] Joseph Goguen. Semantics of computation. In Ernest Manes, editor, *Proceedings, First International Symposium on Category Theory Applied to Computation and Control*, pages 151–163. Springer, 1975. (San Francisco, February 1974.) Lecture Notes in Computer Science, Volume 25.
- [25] Joseph Goguen. Types as theories. In George Michael Reed, Andrew William Roscoe, and Ralph F. Wachter, editors, *Topology and Category Theory in Computer Science*, pages 357–390. Oxford, 1991. Proceedings of a Conference held at Oxford, June 1989.
- [26] Joseph Goguen. Stretching first order equational logic: Proofs with partiality, subtypes and retracts. In Maria Paola Bonacina and Ulrich Furbach, editors, *Proceedings, Workshop on First Order Theorem Proving*, pages 78–85. University of Vienna, 1997. Schloss Hagenberg, Austria, October 1997; RISC-Linz Report No. 95–50; revised version with appendixes co-authored with Grigore Roşu to appear in *Journal of Symbolic Computation*.
- [27] Joseph Goguen and Rod Burstall. Introducing institutions. In Edmund Clarke and Dexter Kozen, editors, *Proceedings, Logics of Programming Workshop*, pages 221–256. Springer, 1984. Lecture Notes in Computer Science, Volume 164.
- [28] Joseph Goguen and Rod Burstall. A study in the foundations of programming methodology: Specifications, institutions, charters and parchments. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Conference on Category Theory and Computer Programming*, pages 313–333. Springer, 1986. Lecture Notes in Computer Science, Volume 240.
- [29] Joseph Goguen and Rod Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.
- [30] Joseph Goguen and Răzvan Diaconescu. An Oxford survey of order sorted algebra. *Mathematical Structures in Computer Science*, 4:363–392, 1994.



- [31] Joseph Goguen, Kai Lin, and Grigore Roşu. Circular coinductive rewriting. In *Proceedings, Automated Software Engineering (ASE'00)*. IEEE, 2000.
- [32] Joseph Goguen and Grigore Roşu. Hiding more of hidden algebra. In *FM'99 – Formal Methods*, pages 1704–1719. Springer, 1999. Lecture Notes in Computer Sciences, Volume 1709, Proceedings of World Congress on Formal Methods, Toulouse, France.
- [33] Joseph Goguen and William Tracz. An implementation-oriented semantics for module composition. In Gary Leavens and Murali Sitaraman, editors, *Foundations of Component-based Systems*, pages 231–263. Cambridge, 2000.
- [34] Joseph Goguen, Timothy Winkler, José Meseguer, Kokichi Futatsugi, and Jean-Pierre Jouannaud. Introducing OBJ. In Joseph Goguen and Grant Malcolm, editors, *Algebraic Specification with OBJ: An Introduction with Case Studies*. Academic, to appear. Also Technical Report SRI-CSL-88-9, August 1988, SRI International.
- [35] Rolf Hennicker and Michel Bidoit. Observational logic. In *Algebraic Methodology and Software Technology (AMAST'98)*, volume 1548 of *Lecture Notes in Computer Science*, pages 263–277. Springer, 1999.
- [36] Horst Herrlich and George Strecker. *Category Theory*. Allyn and Bacon, 1973.
- [37] Hendrik Hilberdink. Inclusion systems, 1996. Unpublished paper.
- [38] Hans-Jörg Kreowski and Till Mossakowski. Equivalence and difference between institutions: simulating Horn Clause Logic with based algebras. *Mathematical Structures in Computer Science*, 5(2):189–215, June 1995.
- [39] Joachim Lambek and Phil Scott. *Introduction to Higher Order Categorical Logic*. Cambridge, 1986. Cambridge Studies in Advanced Mathematics, Volume 7.
- [40] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, 1971.
- [41] M. Arrais and J.L. Fiadeiro. Unifying theories in different institutions. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 81–101. Springer, 1996. Proceedings of 11th Workshop on Specification of Abstract Data Types. Oslo, Norway, September 1995.
- [42] Brian Mayoh. Galleries and institutions. Technical Report DAIMI PB-191, Aarhus University, 1985.
- [43] José Meseguer. General logics. In H.-D. Ebbinghaus et al., editors, *Proceedings, Logic Colloquium 1987*, pages 275–329. North-Holland, 1989.
- [44] José Meseguer. Membership algebra as a logical framework for equational specification. In *Proceedings, WADT'97*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer, 1998.
- [45] Till Mossakowski. *Representations, Hierarchies, and Graphs of Institutions*. PhD thesis, University of Bremen, 1996.
- [46] Till Mossakowski. Specifications in an arbitrary institution with symbols. In *Proceedings, WADT'99*, volume 1827 of *Lecture Notes in Computer Science*, pages 252–270. Springer, 2000.
- [47] Peter Mosses. Unified algebras and institutions. In *Proceedings, Fourth Annual Conference on Logic in Computer Science*, pages 304–312. IEEE, 1989.
- [48] István Németi. On notions of factorization systems and their applications to cone-injective subcategories. *Periodica Mathematica Hungarica*, 13(3):229–335, 1982.
- [49] Wieslaw Pawlowski. Context institutions. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 436–457. Springer, 1996. Proceedings of 11th Workshop on Specification of Abstract Data Types. Oslo, Norway, September 1995.

- [50] Axel Poigné. Foundations are rich institutions, but institutions are poor foundations. In Hartmut Ehrig, Horst Herrlich, Hans-Jörg Kerowski, and Gerhard Preuss, editors, *Categorical Methods in Computer Science*, pages 82–101. Springer, 1989. Lecture Notes in Computer Science, Volume 393.
- [51] Horst Reichel. *Structural Induction on Partial Algebras*. Akademie-Verlag Berlin, 1984.
- [52] Grigore Roşu. The institution of order-sorted equational logic. *Bulletin of EATCS*, 53:250–255, 1994.
- [53] Grigore Roşu. Axiomatizability in inclusive equational logic, 1996. Submitted for publication.
- [54] Grigore Roşu. Kan extensions of institutions. *Journal of Universal Computer Science*, 5(8):482–493, 1999.
- [55] Grigore Roşu. *Hidden Logic*. PhD thesis, University of California at San Diego, 2000.
- [56] Grigore Roşu. Abstract semantics for module composition. Technical Report CSE2000–0653, University of California at San Diego, May 2000. Written August 1997.
- [57] Antonio Salibra and Giuseppe Scollo. A soft stairway to institutions. In Michel Bidoit and Christine Choppy, editors, *Recent Trends in Data Type Specification*, volume 655 of *Lecture Notes in Computer Science*, pages 310–329. Springer-Verlag, 1992.
- [58] Donald Sannella and Andrzej Tarlecki. Extended ML: an institution independent framework for formal program development. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming*, pages 364–389. Springer, 1986. Lecture Notes in Computer Science, Volume 240.
- [59] Donald Sannella and Andrzej Tarlecki. On observational equivalence and algebraic specification. *Journal of Computer and System Science*, 34:150–178, 1987. Also in Proceedings of the International Joint Conference on Theory and Practice of Software Development (TAPSOFT), Berlin. Volume 1: Colloquium on Trees in Algebra and Programming (CAAP 1985), *Lecture Notes in Computer Science 185*, pages 308–322.
- [60] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Information and Control*, 76:165–210, 1988.
- [61] Donald Sannella and Andrzej Tarlecki. Toward formal development of programs from algebraic specifications: implementations revisited. *Acta Informatica*, 25(3):233–281, April 1988.
- [62] Andrzej Tarlecki. Free constructions in algebraic institutions. In M.P. Chytil and V. Koubek, editors, *Proceedings, International Symposium on Mathematical Foundations of Computer Science*, pages 526–534. Springer, 1984.
- [63] Andrzej Tarlecki. Bits and pieces of the theory of institutions. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming*, pages 334–360. Springer, 1986. Lecture Notes in Computer Science, Volume 240.
- [64] Andrzej Tarlecki. On the existence of free models in abstract algebraic institutions. *Theoretical Computer Science*, 37:269–304, 1986.
- [65] Andrzej Tarlecki. Quasi-varieties in abstract algebraic institutions. *Journal of Computer and System Sciences*, 33(3):333–360, 1986.
- [66] Andrzej Tarlecki. Institution representation, 1987. Unpublished note, Dept. of Computer Science, Univ. of Edinburgh.
- [67] Andrzej Tarlecki. Moving between logical systems. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 478–502. Springer, 1996. Proceedings of 11th Workshop on Specification of Abstract Data Types. Oslo, Norway, September 1995.

- [68] Andrzej Tarlecki. Structural properties of some categories of institutions. Technical report, Institute of Informatics, Warsaw University, 1996.
- [69] Andrzej Tarlecki, Rod Burstall, and Joseph Goguen. Some fundamental algebraic tools for the semantics of computation, part 3: Indexed categories. *Theoretical Computer Science*, 91:239–264, 1991.
- [70] Alfred Tarski. The semantic conception of truth. *Philos. Phenomenological Research*, 4:13–47, 1944.
- [71] Han Yan. *Theory and Implementation of Sort Constraints for Order Sorted Algebra*. PhD thesis, Programming Research Group, Oxford University, draft from 1993.

## A Partial Equational Logic

In this appendix we present two different algebraic approaches to partiality, one based on partial algebra and the other based on order sorted algebra; we also give the corresponding institutions, following [26].

### A.1 Partial Algebra

Given a many sorted signature  $\Delta$ , a **partial  $\Delta$ -algebra**  $A$  is just the same as an ordinary  $\Delta$ -algebra, except that the interpretations of the symbols from  $\Delta$  in  $A$  may be *partial* functions instead of total functions. Note that even constants can be partial – which means they are undefined. Given a many sorted signature  $\Delta$ , let **PAIlg $\Delta$**  denote the category of all partial  $\Delta$ -algebras with total  $\Delta$ -homomorphisms. Unfortunately, there are multiple choices for morphisms, with no clear way to decide among them; for example, homomorphisms might be indexed sets of partial functions. However, we choose to require them to be total.

Two classic references on partial algebra, by Horst Reichel [51] and Peter Burmeister [3], are excellent sources for partial algebra and satisfaction. More recently, Cerioli, Mossakowski and Reichel in their survey [9] argue in favor of partial satisfaction and against aspects of order sorted equational logic, particularly retracts.

### A.2 Partial Satisfaction

One of the frustrations of partial algebra is the confusing plethora of definitions of satisfaction. We only consider satisfaction of unconditional equations by partial algebras, over a many sorted signature  $\Delta$ . Perhaps the most common notion, called **existential satisfaction**<sup>13</sup>, says that a partial  $\Delta$ -algebra  $A$  satisfies a  $\Delta$ -equation  $(\forall X) t = t'$  iff for every assignment  $a: X \rightarrow A$ , both  $a(t)$  and  $a(t')$  are defined, and they are equal. This notion has the disadvantage that equations like this inverse law

$$(\forall N: Nat) N * (1/N) = 1$$

are not satisfied by the rational numbers, because the left side is undefined for some values where the right side is not (namely  $N = 0$ ). Existentially satisfied equations act as if they were totally satisfied, since they require everything that they talk about to be defined. Therefore existential is not in general reflexive. These considerations suggest that existential satisfaction is too strong.

Another notion, called **strong satisfaction**, says that  $A$  satisfies  $(\forall X) t = t'$  iff for every assignment  $a: X \rightarrow A$ , if *either*  $a(t)$  or  $a(t')$  is defined, then so is the other, and they are equal. For example, the equation

---

<sup>13</sup>This name is a bit ironic, because many existentialist philosophers had serious doubts about even the possibility of genuine satisfaction.

$$(\forall N, M: \text{Nat}) \ 1/(N * M) = (1/N) * (1/M)$$

is not existentially satisfied by the rationals, but it is strongly satisfied, because the two sides are defined for exactly the same assignments (namely when  $N \neq 0$  and  $M \neq 0$ ) and they are equal for all these assignments. However, the inverse law above fails to be strongly satisfied by the rationals, because the two sides are defined for different values. Similarly, the equation

$$(\forall N, M: \text{Nat}) \ N * M = 1/((1/N) * (1/M))$$

is neither strongly nor existentially satisfied by the rationals, because the left side is defined for some assignments where the right is not (namely whenever  $N = 0$  or  $M = 0$ ). These examples suggest that strong satisfaction is also too strong.

A third notion called **weak satisfaction**, is that  $A$  satisfies  $(\forall X) t = t'$  iff for every assignment  $a: X \rightarrow A$ , if *both*  $a(t)$  and  $a(t')$  are defined, then they are equal. The difference between weak and strong satisfaction is illustrated by the equation

$$(\forall M, N: \text{Nat}) \ M - N = N - M ,$$

which is weakly satisfied on the natural numbers, because both sides are defined iff  $N = M$ ; however, it is neither strongly nor existentially satisfied by the naturals. Our intuition is that equations like the above should *not* be true, which implies that weak satisfaction is too weak. It is well known and easy to check that given a partial  $\Delta$ -algebra  $A$  and a  $\Delta$ -equation  $e$ , if  $A$  existentially satisfies  $e$  then  $A$  strongly satisfies  $e$ , and if  $A$  strongly satisfies  $e$  then  $A$  weakly satisfies  $e$ .

### A.3 A Partial Equational Logic Institution

Let **Sign** be the category of many sorted signatures, and let **Sen**: **Sign**  $\rightarrow$  **Set** be the functor that gives for each signature  $\Delta$  the set of all pairs  $(\gamma, e)$  where  $\gamma$  is a type of satisfaction, i.e., an element in the set  $\{\text{weak}, \text{strong}, \text{existential}\}$ , and  $e$  is a  $\Delta$ -equation. Let **PAIg**: **Sign**  $\rightarrow$  **Cat**<sup>op</sup> be the functor that gives for any signature  $\Delta$  the category of partial  $\Delta$ -algebras. If  $A$  is a partial  $\Delta$ -algebra and  $e$  is a  $\Delta$ -equation, let us write  $A \models_{\Delta}(\gamma, e)$  whenever  $A$  partially  $\gamma$ -satisfies  $e$ . Then

**Proposition 45**  $\text{PEL} = \langle \text{Sign}, \text{Sen}, \text{PAIg}, \{\models_{\Delta}\}_{\Delta \in |\text{Sign}|} \rangle$  is an institution.

## B Supersorted Order Sorted Equational Logic

Goguen [26] shows how order sorted equational logic with retracts can effectively handle both calculations and proofs for partial functions. There are two order sorted approaches to partiality, one using subsorts of definition and the other using error supersorts [26]. Here we concentrate on the second, and show how the partial algebra concepts can be naturally adapted to (total) order sorted algebra. As a consequence, a new institution appears [26], which we call supersorted order sorted equational logic, or simply **OSEL**<sup>?</sup>.

### B.1 Supersorted Signatures

Given an order sorted signature  $\Sigma$ , let **OAlg** <sub>$\Sigma$  denote the category of all  $\Sigma$ -algebras with  $\Sigma$ -homomorphisms. Call an order sorted signature  $\Sigma$  with sort set  $S$  **supersorted** iff  $S$  is the disjoint union of subsets  $S'$  and  $S''$  such that  $S'$  and  $S''$  are isomorphic (as ordered sets), with  $<$  the least ordering on  $S$  including  $S'$  and  $S''$  (as ordered sets) such that  $s' < s''$  whenever  $s' \in S'$  and  $s'' \in S''$  are corresponding sort symbols. Call the sorts in  $S'$  **pure**, and given a  $\Sigma$ -algebra  $A$ , call its elements</sub>

having sorts in  $S'$  its **pure** elements. Also, let us call a  $\Sigma$ -algebra  $A$  **strict** iff each of its operations returns an impure value whenever (one or more) of its arguments is impure.

Let a **morphism of supersorted signatures**, from  $(S_1, \Sigma_1)$  to  $(S_2, \Sigma_2)$ , be a pair  $(f, g)$  where  $f: S_1 \rightarrow S_2$  is such that  $f(s_1) \in S_2'$  and  $f(s_1^?) = (f(s_1))^?$  for each  $s_1 \in S_1'$ , and where  $g = \{g_{w,s}: (\Sigma_1)_{w,s} \rightarrow (\Sigma_2)_{f(w),f(s)}\}$  is such that  $g_{w,s}(\sigma) = g_{u,t}(\sigma)$  whenever  $w' = u'$ ,  $s' = t'$  and  $\sigma \in (\Sigma_1)_{w,s} \cap (\Sigma_1)_{u,t}$ . Notice that  $f$  can be restricted to source  $S_1'$  and target  $S_2'$ , and let  $f': S_1' \rightarrow S_2'$  denote such a restriction of  $f$ ; note that  $f(w)' = f'(w')$  for every  $w \in S_1^*$ . If  $\mathbf{Sign}^?$  denotes the supersorted signatures and their morphisms, then

**Fact 46**  $\mathbf{Sign}^?$  is a category.

## B.2 Super Satisfaction

We present order sorted versions for the various kinds of partial satisfaction presented in Section A.1. Given a  $\Sigma$ -equation  $e = (\forall X) t = t'$ , we can make the following definitions:  $A$  **existentially supersatisfies**  $e$  iff for every pure assignment  $a: X \rightarrow A$ , both  $a(t)$  and  $a(t')$  are pure and they are equal. Similarly,  $A$  **strongly supersatisfies**  $e$  iff for every pure assignment  $a: X \rightarrow A$ , if either  $a(t)$  or  $a(t')$  are pure, then both are pure and they are equal. And finally,  $A$  **weakly supersatisfies**  $e$  iff for every pure assignment  $a: X \rightarrow A$ , if  $a(t)$  and  $a(t')$  are both pure, then they are equal.

## B.3 The Supersorted Order Sorted Equational Logic Institution

Let  $\mathbf{Sen}^?: \mathbf{Sign}^? \rightarrow \mathbf{Set}$  denote the functor that maps a supersorted signature to the set of all pairs  $(\gamma, e)$  where  $\gamma$  is a type of supersatisfaction (i.e., an element in the set  $\{weak, strong, existential\}$ ) and  $e$  is a standard equation over that signature quantified with variables of non-error sorts<sup>14</sup>. Let  $\mathbf{OSAlg}: \mathbf{Sign}^? \rightarrow \mathbf{Cat}^{op}$  be the usual functor that gives for any supersorted signature  $\Sigma$  the category of order sorted  $\Sigma$ -algebras. If  $A$  is an order sorted  $\Sigma$ -algebra and  $e$  is a  $\Sigma$ -equation, let us write  $A \models_{\Sigma} (\gamma, e)$  when  $A$   $\gamma$ -satisfies  $e$ . Then we have

**Fact 47**  $\mathbf{OSEL}^? = \langle \mathbf{Sign}^?, \mathbf{Sen}^?, \mathbf{OSAlg}, \{ \models_{\Sigma} \}_{\Sigma \in |\mathbf{Sign}^?}|} \rangle$  is an institution.

## B.4 Forgetting the Errors

Let  $(S, \Sigma)^b = (S', \Sigma')$  for any supersorted signature  $(S, \Sigma)$ , and note that  $(S, \Sigma)^b$  is indeed a signature whenever  $(S, \Sigma)$  is a supersorted signature, because the operations in  $\Sigma'$  only involve sorts in  $S'$ . Now if  $(f, g): (S_1, \Sigma_1) \rightarrow (S_2, \Sigma_2)$  is a morphism of supersorted signatures, define  $(f, g)^b$  to be the pair  $(f', g')$  where  $g'$  is the family  $\{g'_{w',s'}: (\Sigma_1')_{w',s'} \rightarrow (\Sigma_2')_{f(w'),f(s')}\}$  with  $g'_{w',s'}(\sigma) = g_{w,s}(\sigma)$ . Then we have

**Fact 48**  $\_b: \mathbf{Sign}^? \rightarrow \mathbf{Sign}$  is a functor.

We now define a natural transformation  $\alpha: \mathbf{Sen}^? \Rightarrow \_b; \mathbf{Sen}$  as follows: for any supersorted signature  $(S, \Sigma)$  and any  $(S, \Sigma)$ -equation  $(\gamma, e)$ , let  $\alpha_{\Sigma}(\gamma, e)$  be the  $(S', \Sigma')$ -equation obtained from  $(\gamma, e)$  replacing each operation  $\sigma: w \rightarrow s$  by  $\sigma: w' \rightarrow s'$ . Then indeed

**Fact 49**  $\alpha: \mathbf{Sen}^? \Rightarrow \_b; \mathbf{Sen}$  is a natural transformation.

**Fact 50** Given a supersorted signature  $(S, \Sigma)$ , then  $\mathcal{U}_{\Sigma}: \mathbf{OSAlg}(\Sigma) \rightarrow \mathbf{PAlg}(\Sigma^b)$  is a functor. Moreover,  $\mathcal{U}: \mathbf{OSAlg} \Rightarrow \_b; \mathbf{PAlg}$  is a natural transformation.

<sup>14</sup>For  $\mathbf{Sen}^?$  to be a functor, we need the rather technical result that the equations quantified by non-error variables are mapped to equations quantified by non-error sorts. However, this is a consequence of the fact that non-error sorts are mapped to non-error sorts.

## C Two Hidden Equational Logic Institutions

There are two rather different ways to present hidden logic as an institution in two interesting ways, depending on whether the declaration of an operation to be behavioral is considered part of the signature, or as a separate sentence; we first approached this issue in [32]. A thorough exposition of hidden algebra may be found [55].

The first institution, denoted  $\mathbb{H}\mathbb{E}\mathbb{L}_1$ , follows the institution of hidden algebra initially presented in [25], the institution of observational logic in [35], and the coherent hidden algebra approach in [16, 17], while the second, which we simply call  $\mathbb{H}\mathbb{E}\mathbb{L}_2$ , seems more promising for future research. Our approach also avoids the infinitary logic used in observational logic. Only the fixed-data case is investigated here, but we hope to extend it to the loose-data case soon (see [55] for more on the terminology of hidden logic). We fix a data  $\Psi$ -algebra  $D$ .

### C.1 The First Institution

The institution  $\mathbb{H}\mathbb{E}\mathbb{L}_1$  is built as follows:

**Signatures:** The category **Sign** has hidden signatures over a fixed data algebra  $D$  as objects. A morphism of hidden signatures  $\phi: (\langle \cdot, \Sigma_1 \rangle) \rightarrow (\langle \cdot, \Sigma_2 \rangle)$  is the identity on the visible signature  $\Psi$ , takes hidden sorts to hidden sorts, and if a behavioral operation  $\delta_2$  in  $\langle \cdot, \Sigma_2 \rangle$  has an argument sort in  $\phi(H_1)$  then there is some behavioral operation  $\delta_1$  in  $\langle \cdot, \Sigma_1 \rangle$  such that  $\delta_2 = \phi(\delta_1)$ . **Sign** is indeed a category, and the composition of two hidden signature morphisms is another. Indeed, let  $\psi: (\langle \cdot, \Sigma_2 \rangle) \rightarrow (\langle \cdot, \Sigma_3 \rangle)$  and let  $\delta_3$  be an operation in  $\langle \cdot, \Sigma_3 \rangle$  having an argument sort in  $(\phi; \psi)(H_1)$ . Then  $\delta_3$  has an argument sort in  $\psi(H_2)$ , so there is an operation  $\delta_2$  in  $\langle \cdot, \Sigma_2 \rangle$  with  $\delta_3 = \psi(\delta_2)$ . Also  $\delta_2$  has an argument sort in  $\phi(H_1)$ , so there is some  $\delta_1$  in  $\langle \cdot, \Sigma_1 \rangle$  with  $\delta_2 = \phi(\delta_1)$ . Therefore  $\delta_3 = (\phi; \psi)(\delta_1)$ , i.e.,  $\phi; \psi$  is also a morphism of hidden signatures.

**Sentences:** Given a hidden signature  $(\langle \cdot, \Sigma \rangle)$ , let  $\mathbf{Sen}(\langle \cdot, \Sigma \rangle)$  be the set of all  $\Sigma$ -equations. If  $\phi: (\langle \cdot, \Sigma_1 \rangle) \rightarrow (\langle \cdot, \Sigma_2 \rangle)$  is a hidden signature morphism, then  $\mathbf{Sen}(\phi)$  is the function taking a  $\Sigma_1$ -equation  $e = (\forall X) t = t' \text{ if } t_1 = t'_1, \dots, t_n = t'_n$  to the  $\Sigma_2$ -equation

$$\phi(e) = (\forall X') \phi(t) = \phi(t') \text{ if } \phi(t_1) = \phi(t'_1), \dots, \phi(t_n) = \phi(t'_n),$$

where  $X'$  is  $\{x : \phi(s) \mid x : s \in X\}$ . Then  $\mathbf{Sen}: \mathbf{Sign} \rightarrow \mathbf{Set}$  is indeed a functor.

**Models:** Given a hidden signature  $(\langle \cdot, \Sigma \rangle)$ , let  $\mathbf{Mod}(\langle \cdot, \Sigma \rangle)$  be the category of hidden  $\Sigma$ -algebras and their morphisms. If  $\phi: (\langle \cdot, \Sigma_1 \rangle) \rightarrow (\langle \cdot, \Sigma_2 \rangle)$  is a hidden signature morphism, then  $\mathbf{Mod}(\phi)$  is the usual reduct functor,  $\_|\phi$ . Unlike [1, 35], etc., this allows models where not all operations are congruent.

**Satisfaction Relation:** behavioral satisfaction, i.e.,  $\models_{(\Gamma, \Sigma)} = \models_{\Sigma}^{\Gamma}$ .

**Theorem 51 Satisfaction Condition:** *Given  $\phi: (\langle \cdot, \Sigma_1 \rangle) \rightarrow (\langle \cdot, \Sigma_2 \rangle)$  a hidden signature morphism,  $d = (\forall X) t = t' \text{ if } t_1 = t'_1, \dots, t_n = t'_n$  a  $\Sigma_1$ -equation, and  $A$  a hidden  $\Sigma_2$ -algebra, then  $A \models_{\Sigma_2}^{\Gamma_2} \phi(e)$  iff  $A|\phi \models_{\Sigma_1}^{\Gamma_1} e$ .*

**Proof:** See [32, 55]. □

### C.2 The Second Institution

Our second institution views the declaration of a behavioral operation as a new kind of sentence, rather than part of a hidden signature. The notion of model also changes, adding an equivalence

relation as in [1]. This is natural for modern software engineering, since languages like Java provide classes with an operation denoted `equals` which serves this purpose. Sentences in [1] are pairs  $\langle e, \Delta \rangle$ , where  $\Delta$  is a set of terms (pretty much like a cobasis over the derived signature), which are satisfied by  $(A, \sim)$  iff  $(A, \sim)$  satisfies  $e$  as in our case below (actually  $e$  is a first-order formula in their framework) and  $\sim \subseteq \equiv_{\Delta}$ . Fix a data algebra  $D$ , and proceed as follows:

**Signatures:** The category **Sign** has hidden signatures over  $D$  as objects, with its morphisms  $\phi: \Sigma_1 \rightarrow \Sigma_2$  the identity on the visible signature  $\Psi$ , and taking hidden sorts to hidden sorts.

**Sentences:** Given a hidden signature  $\Sigma$ , let **Sen**( $\Sigma$ ) be the set of all  $\Sigma$ -equations unioned with  $\Sigma$ . If  $\phi: \Sigma_1 \rightarrow \Sigma_2$  is a hidden signature morphism, then **Sen**( $\phi$ ) is the function taking a  $\Sigma_1$ -equation  $e = (\forall X) t = t'$  **if**  $t_1 = t'_1, \dots, t_n = t'_n$  to the  $\Sigma_2$ -equation  $\phi(e) = (\forall X') \phi(t) = \phi(t')$  **if**  $\phi(t_1) = \phi(t'_1), \dots, \phi(t_n) = \phi(t'_n)$ , where  $X'$  is the set  $\{x : \phi(s) \mid x : s \in X\}$ , and taking  $\sigma : s_1 \dots s_n \rightarrow s$  to  $\phi(\sigma) : \phi(s_1) \dots \phi(s_n) \rightarrow \phi(s)$ . Then **Sen**: **Sign**  $\rightarrow$  **Set** is indeed a functor.

**Models:** Given a hidden signature  $\Sigma$ , let **Mod**( $\Sigma$ ) be the category of pairs  $(A, \sim)$  where  $A$  is a hidden  $\Sigma$ -algebra and  $\sim$  is an equivalence relation on  $A$  which is identity on visible sorts, with morphisms  $f: (A, \sim) \rightarrow (A', \sim')$  with  $f: A \rightarrow A'$  a  $\Sigma$ -homomorphism such that  $f(\sim) \subseteq \sim'$ . If  $\phi: \Sigma_1 \rightarrow \Sigma_2$  is a hidden signature morphism, then **Mod**( $\phi$ ), often denoted  $\downarrow_{\phi}$ , is defined as  $(A, \sim) \downarrow_{\phi} = (A \downarrow_{\phi}, \sim \downarrow_{\phi})$  on objects, where  $A \downarrow_{\phi}$  is the ordinary many sorted algebra reduct and  $(\sim \downarrow_{\phi})_s = \sim_{\phi(s)}$  for all sorts  $s$  of  $\Sigma_1$ , and as  $f \downarrow_{\phi}: (A, \sim) \downarrow_{\phi} \rightarrow (A', \sim') \downarrow_{\phi}$  on morphisms. Notice that indeed  $f \downarrow_{\phi} (\sim \downarrow_{\phi}) \subseteq \sim' \downarrow_{\phi}$ , so **Mod** is well defined.

**Satisfaction Relation:** A  $\Sigma$ -model  $(A, \sim)$  satisfies a conditional  $\Sigma$ -equation  $(\forall X) t = t'$  **if**  $t_1 = t'_1, \dots, t_n = t'_n$  iff for each  $\theta: X \rightarrow A$ , if  $\theta(t_1) \sim \theta(t'_1), \dots, \theta(t_n) \sim \theta(t'_n)$  then  $\theta(t) \sim \theta(t')$ . Also  $(A, \sim)$  satisfies a  $\Sigma$ -sentence  $\gamma \in \Sigma$  iff  $\gamma$  is congruent for  $\sim$ .

**Theorem 52 Satisfaction Condition:** Let  $\phi: \Sigma_1 \rightarrow \Sigma_2$  be a morphism of hidden signatures, let  $e$  be a  $\Sigma_1$ -sentence and let  $(A, \sim)$  be a model of  $\Sigma_2$ . Then  $(A, \sim) \models_{\Sigma_2} \phi(e)$  iff  $(A, \sim) \downarrow_{\phi} \models_{\Sigma_1} e$ .

**Proof:** See [32, 55]. □

This institution justifies our belief that asserting an operation behavioral is a kind of sentence, not a kind of syntactic declaration as in the “extended hidden signatures” of [17]<sup>15</sup>. Coinduction now appears in the following elegant guise:

**Proposition 53** Given a hidden subsignature  $\cdot$  of  $\Sigma$ , a set of  $\Sigma$ -equations  $E$  and a hidden  $\Sigma$ -algebra  $A$ , then

- $(A, \sim) \models_{\Sigma} E, \cdot$  implies  $(A, \equiv_{\Sigma}^{\Gamma}) \models_{\Sigma} E, \cdot$ .
- $(A, \equiv_{\Sigma}^{\Gamma}) \models_{\Sigma} \cdot$ .
- $A \models_{\Sigma}^{\Gamma} E$  iff  $(A, \equiv_{\Sigma}^{\Gamma}) \models_{\Sigma} E$  iff  $(A, \equiv_{\Sigma}^{\Gamma}) \models_{\Sigma} E, \cdot$ .

## D A More Categorical Institution for Algebra

This section develops universal algebra in a much more abstract categorical language than is usual, with satisfaction interpreted as injectivity; we show that this forms an institution. Interestingly, the satisfaction condition becomes “almost equivalent” to the definition of adjoint functor, thus strengthening our belief in the essentiality of the original definition of institution in [29]. We assume the reader familiar with basic notions of factorization systems [36, 48].

<sup>15</sup>However, the most recent version of [21] treats coherence assertions as sentences.

**Definition 54** If  $\mathcal{A}$  is a category and  $\mathcal{C}$  is a class of morphisms in  $\mathcal{A}$ , then an object  $D$  is  $\mathcal{C}$ -**injective** iff for any morphism  $c: A \rightarrow B$  in  $\mathcal{C}$  and any morphism  $f: A \rightarrow D$  there are some morphisms  $g: B \rightarrow D$  such that  $f = c; g$ . ■

**Definition 55** If  $\langle \mathcal{E}_A, \mathcal{M}_A \rangle$  and  $\langle \mathcal{E}_B, \mathcal{M}_B \rangle$  are factorization systems for categories  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, then a functor  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$  is called  $\mathcal{E}$ -**preserving** iff  $\mathcal{F}(\mathcal{E}_A) \subseteq \mathcal{E}_B$ . ■

We begin with the observation that satisfaction of equations in the framework of universal algebra is equivalent to injectivity. Let us consider that  $\mathcal{A}$  is the category of universal or many sorted  $\Sigma$ -algebras over a (many sorted) signature  $\Sigma$ . Each equation  $(\forall X) t = t'$  generates a congruence relation on  $T_\Sigma(X)$  (the term algebra over variables in  $X$ ), which implicitly gives a surjective morphism  $e: T_\Sigma(X) \rightarrow \bullet$ . It can be readily seen that an algebra  $D$  satisfies  $(\forall X) t = t'$  if and only if it is  $\{e\}$ -injective. Conversely, each surjective morphism  $e$  of free algebra source generates an infinite set  $E$  of equations over variables in that free algebra, namely all pairs in its kernel. It can also be readily seen that an algebra is  $\{e\}$ -injective if and only if it satisfies all equations in  $E$ . Therefore, satisfaction of equations and  $\mathcal{C}$ -injectivity, where  $\mathcal{C}$  contains only surjective morphisms with free sources, are equivalent concepts in the framework of universal and/or many sorted algebra.

It can be relatively easily shown [53] that, given a set of surjective morphisms of not necessarily free sources,  $\mathcal{C}$ -injectivity is actually equivalent with  $\mathcal{C}'$ -injectivity, where  $\mathcal{C}'$  can be obtained from  $\mathcal{C}$  and contains only some special morphisms of free source, intuitively representing conditional equations. The institution that follows is therefore taking into consideration conditional equations.

**Definition 56** If  $\mathcal{C}$  is a class of morphisms and  $\mathcal{Q}$  is a class of objects in  $\mathcal{A}$ , then let  $\mathcal{C}^*$  be the class of all objects in  $\mathcal{A}$  which are  $\mathcal{C}$ -injective, and let  $\mathcal{Q}_\mathcal{C}^*$  be the class of all morphisms in  $\mathcal{C}$  such that each object in  $\mathcal{Q}$  is  $\mathcal{Q}_\mathcal{C}^*$ -injective, ■

We will often say that the objects in  $\mathcal{C}^*$  “satisfy” the “formulas” in  $\mathcal{C}$ .

**Fact 57** *Given a class of morphisms  $\mathcal{C}$  in  $\mathcal{A}$ , the pair of operators  $(-\mathcal{C}^*, \mathcal{C}_-^*)$  is a Galois connection between classes of morphisms in  $\mathcal{C}$  and classes of objects in  $\mathcal{A}$ .*

We can now introduce the following more or less standard notion:

**Definition 58** Given a class  $\mathcal{C}$  of morphisms and  $\mathcal{Q}$  a class of objects in  $\mathcal{A}$ , let  $\mathcal{Q}_\mathcal{C}^\bullet$  be the class of objects  $(\mathcal{Q}_\mathcal{C}^*)^*$ . Then  $\mathcal{Q}$  is  $\mathcal{C}$ -*injectively definable* iff  $\mathcal{Q} = \mathcal{Q}_\mathcal{C}^\bullet$ . ■

We next show that the natural injectivity-based logic informally described above can be organized as an institution:

**Signatures:** Let **Sign** be the category having small categories admitting factorization systems as objects and  $\mathcal{E}$ -preserving left adjoint functors as morphisms.

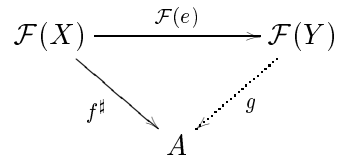
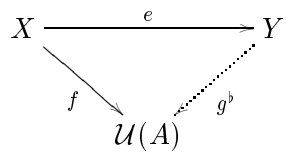
**Sentences:** Let **Sen**: **Sign**  $\rightarrow$  **Set** be defined as **Sen**( $\mathcal{A}$ ) =  $\mathcal{E}_\mathcal{A}$ . Notice that **Sen** is indeed well defined.

**Models:** Let **Mod**: **Sign**  $\rightarrow$  **Cat**<sup>op</sup> be defined as **Mod**( $\mathcal{A}$ ) =  $\mathcal{A}$  and **Mod**( $\mathcal{F}$ ) is a right adjoint of  $\mathcal{F}$ . Suppose that the right adjoints are chosen such that **Mod** is a functor.

**Theorem 59 Satisfaction Condition:** *Given an  $\mathcal{E}$ -preserving left adjoint functor  $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$  of  $\mathcal{U}: \mathcal{A} \rightarrow \mathcal{B}$ , an object  $A \in |\mathcal{A}|$  and a morphism  $e \in \mathcal{E}_\mathcal{B}$ , then  $A \models_{\mathcal{A}} \mathcal{F}(e)$  iff  $\mathcal{U}(A) \models_{\mathcal{B}} e$ .*



**Proof:** The proof follows immediately from properties of adjoint functors, and so we leave it as an exercise. The following diagrams may help visualize this proof:



□

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
2.1	Indexed Categories . . . . .	4
2.2	Functor Categories and Kan Extensions . . . . .	4
2.3	Twisted Relations . . . . .	5
<b>3</b>	<b>Institutions</b>	<b>6</b>
3.1	Some Basics of Institutions . . . . .	8
3.2	Inclusive Institutions . . . . .	9
<b>4</b>	<b>Institution Morphisms and Comorphisms</b>	<b>11</b>
4.1	Duality of Institution Morphisms and Comorphisms . . . . .	15
4.2	Kan Extensions of Institutions . . . . .	15
<b>5</b>	<b>Theoroidal Morphisms</b>	<b>16</b>
5.1	Simple Thoroidal Morphisms . . . . .	18
<b>6</b>	<b>Forward Morphisms</b>	<b>20</b>
<b>7</b>	<b>Semi-Natural Institution Morphisms and Comorphisms</b>	<b>20</b>
<b>8</b>	<b>Summary and Further Research</b>	<b>22</b>
<b>A</b>	<b>Partial Equational Logic</b>	<b>27</b>
A.1	Partial Algebra . . . . .	27
A.2	Partial Satisfaction . . . . .	27
A.3	A Partial Equational Logic Institution . . . . .	28
<b>B</b>	<b>Supersorted Order Sorted Equational Logic</b>	<b>28</b>
B.1	Supersorted Signatures . . . . .	28
B.2	Super Satisfaction . . . . .	29
B.3	The Supersorted Order Sorted Equational Logic Institution . . . . .	29
B.4	Forgetting the Errors . . . . .	29
<b>C</b>	<b>Two Hidden Equational Logic Institutions</b>	<b>30</b>
C.1	The First Institution . . . . .	30
C.2	The Second Institution . . . . .	30
<b>D</b>	<b>A More Categorical Institution for Algebra</b>	<b>31</b>