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A First Digit Theorem for Square-Free Integer Powers

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Abstract

For any fixed integer power, it is shown that the first digits of square-free integer powers follow a generalized Benford law (GBL) with size-dependent exponent that converges asymptotically to a GBL with inverse power exponent. In particular, asymptotically as the power goes to infinity the sequences of square-free integer powers obey Benford's law. Moreover, we show the existence of a one-parametric size-dependent exponent function that converge to these GBL's and determine an optimal value that minimizes its deviation to two minimum estimators of the size-dependent exponent over the finite range of square-free integer powers less than $10^{s\cdot m}, m = 4,...,10$, where $s = 1,2,3,4,5,10$ is a fixed integer power.

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1. Introduction

It is well-known that the first digits of many numerical data sets are not uniformly distributed. Newcomb [14] and Benford [3] observed that the first digits of many series of real numbers obey *Benford's law*

$$P^B(d) = \log_{10}(1+d) - \log_{10}(d), \quad d = 1,2,...,9 \tag{1.1}$$

The increasing knowledge about Benford's law and its applications has been collected in various bibliographies, the most recent being Beebe [2] and Berger and Hill [4]. It is also known that for any fixed power exponent $s \geq 1$, the first digits of integer powers, follow asymptotically a *Generalized Benford law* (GBL) with exponent $\alpha = s^{-1} \in (0,1)$ such that (see Hürlimann [7])

$$P_\alpha^{GB}(d) = \frac{(1+d)^\alpha - d^\alpha}{10^\alpha - 1}, \quad d = 1,2,...,9. \tag{1.2}$$

Clearly, the limiting case $\alpha \to 0$ respectively $\alpha \to 1$ of (1.2) converges weakly to Benford's law respectively the uniform distribution.

We study the distribution of first digits of square-free integer powers. The method consists to fit the GBL to samples of first digits using two size-dependent goodness-of-fit measures, namely the ETA measure (derived from the mean absolute deviation) and the WLS measure (weighted least square measure). In Section 2, we determine the minimum ETA and WLS estimators of the GBL over finite ranges of square-free powers up to $10^{s \cdot m}, m \geq 4$, $s \geq 1$ a fixed power exponent. Computations illustrate the convergence of the size-dependent GBL with minimum ETA and WLS estimators to the GBL with exponent s^{-1}. Moreover, we show the existence of a one-parametric size-dependent exponent function that converge to these GBL's and determine an optimal value that minimizes its deviation to the minimum ETA and WLS estimators. A mathematical proof of the asymptotic convergence of the finite sequences to the GBL with inverse power exponent follows in Section 3.

2. Size-dependent GBL for square-free integer powers

To investigate the optimal fitting of the GBL to first digit sequences of square-free integer powers, it is necessary to specify goodness-of-fit (GoF) measures according to which optimality should hold. First of all, a reasonable GoF measure for the fitting of first-digit distributions should be size-dependent. This has been observed by Furlan [5], Section II.7.1, pp.70-71, who defines the ETA measure, and by Hürlimann [8], p.8, who applies the probability weighted least squares (WLS) measure used earlier by Leemis et al. [12] (chi-square divided by sample size). Let $\{x_n\} \subset [1,\infty), n \geq 1$, be an integer sequence, and let d_n be the (first) significant digit of x_n. The number of x_n's, $n = 1,...,N$, with significant digit $d_n = d$ is denoted by $X_N(d)$. Then, Furlan's *ETA measure* for the GBL is defined to be

$$ETA_N(\alpha) = \frac{9}{2 \cdot N} \cdot MAD_N(\alpha), \quad MAD_N(\alpha) = \frac{1}{9} \cdot \sum_{d=1}^{9} \left| P_\alpha^{GB}(d) - \tfrac{X_N(d)}{N} \right|, \tag{2.1}$$

where $MAD_N(\alpha)$ is the *mean absolute deviation* measure. The latter measure is also used to assess conformity to Benford's law by Nigrini [15] (see also Nigrini [16], Table 7.1, p.160). The *WLS measure* for the GBL is defined by (e.g. [12])

$$WLS_N(\alpha) = \frac{1}{N} \cdot \sum_{d=1}^{9} \frac{(P_\alpha^{GB}(d) - \tfrac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)}. \tag{2.2}$$

Consider now the sequence of square-free integer powers $\{n_f^s\}, n_f^s < 10^{s \cdot m}$, for a fixed power exponent $s = 1,2,3,...$, and arbitrary square-free numbers n_f below $10^m, m \geq 4$. Denote by $I_k^s(d)$ the number of square-free powers below $10^k, k \geq 1$, with first digit d. This number is defined recursively by the relationship

$$I_{k+1}^s(d) = S(\sqrt[s]{(d+1) \cdot 10^k}) - S(\sqrt[s]{d \cdot 10^k}) + I_k^s(d), \quad k = 1,2,..., \tag{2.3}$$

where the counting function $S(n)$ is given by (e.g. Pawlewicz [18], Theorem 1)

$$S(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \mu(k) \cdot \left\lfloor \frac{n}{k^2} \right\rfloor, \tag{2.4}$$

where $\mu(k)$ is the Möbius function such that $\mu(k) = 0$ if p^2 divides k and $\mu(k) = (-1)^e$ if k is a square-free number with e distinct prime factors, and $\lfloor \cdot \rfloor$ denotes the integer-part function. Recent algorithms to efficiently compute these arithmetic functions are contained in Pawlewicz [18] and Auil [1].

Therefore, with $N = S(10^m)$ one has $X_N(d) = I_{s \cdot m}^s(d)$ in (2.1)-(2.2). A list of the $I_{s \cdot m}^s(d), m = 4,...,10, s = 1,2,3,4,5,10$, together with the sample size $N = S(10^m)$, is provided in Table A.1 of the Appendix. Based on this we have calculated the optimal parameters which minimize the ETA (or equivalently MAD) and WLS measures, the so-called minimum ETA (or minimum MAD) and minimum WLS estimators. Together with their GoF measures, these optimal estimators are reported in Table 2.1 below. Note that the minimum WLS is a critical point of the equation

$$\frac{\partial}{\partial\alpha}WLS_N(\alpha)=\frac{1}{N}\cdot\sum_{d=1}^{9}\frac{\partial P_\alpha^{GB}(d)}{\partial\alpha}\cdot\frac{P_\alpha^{GB}(d)^2-(\frac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)^2}=0,$$
$$\frac{\partial P_\alpha^{GB}(d)}{\partial\alpha}=\frac{(1+d)^\alpha\{\ln(\frac{1+d}{10})10^\alpha-\ln(1+d)\}-d^\alpha\{\ln(\frac{d}{10})10^\alpha-\ln(d)\}}{(10^\alpha-1)^2}. \quad (2.5)$$

For comparison, the ETA and WLS measures for the size-dependent GBL exponent

$$\alpha_{LL}(s\cdot m)=s^{-1}\cdot\{1-c\cdot 10^{-m}\}, \quad (2.6)$$

with $c=1$, called LL estimator, are listed. This type of estimator is named in honour of Luque and Lacasa [13] who introduced it in their GBL analysis for the prime number sequence. Through calculation one observes that the LL estimator minimizes the absolute deviations between the LL estimator and the ETA (resp. WLS) estimators over the finite ranges of square-free powers $[1,10^{s\cdot m}]$, $m=4,...,10$, $s=1,2,3,4,5,10$. In fact, if one denotes the ETA and WLS estimators of the sequence $\{n_f^s\}, n_f^s<10^{s\cdot m}$, by $\alpha_{ETA}(s\cdot m)$ and $\alpha_{WLS}(s\cdot m)$, then one has uniformly over the considered finite ranges (consult the columns "Δ to LL estimate" in Table 2.1 in units of $10^{-(m-3)}$)

$$|\alpha_{WLS}(s\cdot m)-\alpha_{LL}(s\cdot m)|\le 1.96\cdot 10^{-(m-3)},$$
$$|\alpha_{ETA}(s\cdot m)-\alpha_{LL}(s\cdot m)|\le 2.53\cdot 10^{-(m-3)}. \quad (2.7)$$

Table 2.1 displays exact results obtained on a computer with single precision, i.e. with 15 significant digits. The ETA (resp. WLS) measures are given in units of $10^{-(m+7)}$ (resp. $10^{-(2m+4)}$). Taking into account the decreasing units, one observes that the optimal ETA and WLS measures decrease with increasing sample size.

Table 2.1: GBL fit for first digit of square-free powers: ETA vs. WLS criterion

s=1	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.9989269	0.9998597	0.010	0.000	23589	24297	**23536**	2263	**2211**	2259
5	1.0001437	1.0002391	0.015	0.025	8013	7766	**7613**	2128	**2115**	2120
6	1.0000393	1.0000925	0.040	0.094	1236	1214	**1185**	479.5	**470.5**	486.2
7	1.0000029	1.0000232	0.030	0.233	276.3	275.7	**271.9**	227.5	**227.1**	250.0
8	1.0000003	1.0000011	0.031	0.109	33.11	32.82	**32.66**	36.47	**36.42**	36.75
9	0.9999999	0.9999999	0.069	0.104	4.867	4.825	**4.804**	7.488	**7.462**	7.469
10	1.0000000	0.9999999	0.342	0.563	0.418	0.339	**0.326**	0.665	**0.600**	0.627
s=2	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.4998130	0.4954987	0.001	0.045	45748	45600	**42296**	7545	**7543**	8777
5	0.4999157	0.4994783	0.008	0.052	5652	5571	**5240**	1350	**1346**	1473
6	0.5001028	0.5001498	0.103	0.150	875	743	**683**	272.5	**201.9**	216.6
7	0.5000182	0.5000184	0.183	0.185	220.5	192.3	**192.2**	177.8	**155.7**	155.7
8	0.5000031	0.5000041	0.315	0.411	55.13	46.47	**44.17**	95.58	**89.00**	89.61
9	0.5000002	0.4999999	0.171	0.097	7.255	7.343	**7.206**	22.135	**21.941**	22.42
10	0.5000000	0.4999999	0.285	0.578	2.033	2.054	**1.990**	15.594	**15.540**	16.033

s=3	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.3321846	0.3353504	0.011	0.021	48745	50462	**46805**	10247	**10162**	10847
5	0.3341766	0.3347193	0.085	0.139	11481	10177	**9342**	3766	**3275**	3477
6	0.3333554	0.3333366	0.022	0.004	1508	1508	**1505**	955.2	**951.7**	954.1
7	0.3333469	0.3333720	0.136	0.387	281.4	262.7	**240.4**	263.2	**250.6**	293.8
8	0.3333364	0.3333380	0.309	0.469	59.26	53.74	**51.27**	137.89	**131.35**	133.11
9	0.3333338	0.3333343	0.457	0.959	8.909	8.156	**7.329**	27.433	**26.006**	27.73
10	0.3333335	0.3333336	1.951	2.308	1.957	1.677	**1.625**	11.456	**8.850**	8.937

s=4	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.2462798	0.2505388	0.037	0.006	47287	51150	**46963**	9747	**8804**	10056
5	0.2502442	0.2513916	0.025	0.139	9426	9198	**8576**	2469	**2427**	3338
6	0.2501231	0.2502287	0.123	0.229	1231	1100	**1067**	596.0	**490.6**	567.7
7	0.2500166	0.2500123	0.167	0.123	242.6	208.3	**205.7**	202.8	**183.6**	184.9
8	0.2500056	0.2500018	0.560	0.179	37.28	34.64	**31.59**	79.65	**57.94**	68.00
9	0.2499995	0.2499989	0.544	1.070	7.554	6.745	**5.980**	19.206	**17.158**	19.08
10	0.2499999	0.2500000	0.609	0.252	0.767	0.801	**0.736**	2.809	**2.552**	2.640

s=5	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.2016366	0.2040166	0.017	0.040	38498	35357	**31523**	5066	**4875**	5269
5	0.1997147	0.1995436	0.028	0.045	4312	3471	**3085**	680.7	**624.9**	645
6	0.1999168	0.1998905	0.083	0.109	940	792	**757**	338.9	**291.0**	295.8
7	0.2000150	0.2000089	0.150	0.089	196.2	179.5	**171.4**	151.5	**135.8**	138.4
8	0.1999998	0.1999991	0.022	0.093	18.69	18.42	**17.55**	14.57	**14.54**	14.89
9	0.1999997	0.1999992	0.267	0.755	12.556	11.993	**11.067**	49.591	**49.096**	50.75
10	0.2000001	0.2000001	0.501	0.639	2.091	2.014	**1.993**	21.817	**21.642**	21.655

s=10	parameters		Δ to LL estimate		ETA GoF measures			WLS GoF measures		
m=	WLS	ETA	WLS	ETA	LL	WLS	**ETA**	LL	**WLS**	ETA
4	0.1084757	0.1078678	0.085	0.079	44303	33110	**31367**	12124	**7085**	7111
5	0.0999163	0.0999959	0.008	0.000	5865	6040	**5860**	1705	**1700**	1705
6	0.0999481	0.0999054	0.052	0.094	1551	1487	**1434**	818.3	**799.5**	812.2
7	0.1000192	0.1000119	0.192	0.119	248.8	220.4	**212.9**	289.0	**263.2**	266.9
8	0.1000013	0.1000002	0.134	0.024	46.86	48.36	**46.37**	120.76	**119.50**	120.34
9	0.0999994	0.0999997	0.624	0.289	10.394	9.661	**9.479**	36.219	**33.497**	34.28
10	0.1000002	0.1000003	1.541	2.526	2.772	2.474	**2.361**	23.292	**21.631**	22.310

3. Asymptotic counting function for square-free integer powers

The following is a slight extension of the argument by Luque and Lacasa [13], Section 5(a). It is well-known that a random process with uniform density x^{-1} generates data that are Benford distributed. Similarly, a sequence of numbers generated by a power-law density $x^{-\alpha}, \alpha \in (0,1)$, has a GBL first-digit distribution $P_{1-\alpha}^{GB}(d)$ with exponent $1-\alpha$. From such a density it is possible to derive a counting function $C(N)$ for that sequence in the interval $[1, N]$. However, assuming a local density of the form $x^{-\alpha(x)}$ such that $C(N) \sim \int_2^N x^{-\alpha(x)} dx$ is not appropriate in general. Indeed, the square-free power relation over an interval $[1, N^s]$ that belongs to (2.6), namely

$$\alpha(N^s) = \frac{s-1+\alpha(N)}{s}, \quad \alpha(N) = \frac{c}{N}, \tag{3.1}$$

does not behave smoothly in $[1, N^s]$, which should be the case for such an approximation. This drawback can be overcome. Denote by $Q_s(N^s)$ the counting function for square-free powers in $[1, N^s]$. Instead of $\int_2^{N^s} x^{-\alpha(N^s)} dx$ define

$$Q_s(N^s) = \frac{6}{\pi^2 \cdot s} \cdot \int_2^{N^s} x^{-\alpha(N^s)} dx, \tag{3.2}$$

where the integral pre-factor is chosen to fulfill the asymptotic limiting value for the square-free number counting function, that is (note that $n_f^s < N^s$ if, and only if, one has $n_f < N$)

$$\lim_{N \to \infty} \frac{Q_s(N^s)}{N} = \frac{6}{\pi^2}. \tag{3.3}$$

In fact, two improved asymptotic expansions of $S(N)$ are known, namely

$$S(N) = \frac{6}{\pi^2} N + O\left(\sqrt{N}\right), \text{ and } S(N) = \frac{6}{\pi^2} N + O\left(N^{\frac{17}{54}+\varepsilon}\right). \tag{3.4}$$

The first one is classical and proved in Hardy and Wright [6], p.269, and Jameson [9], Section 2.5, for example. The second improved estimate is due to Jia [11] (see also Pappalardi [17]). However, it suffices to use the simple estimate (3.3), which is obtained as follows. From (3.2) one gets for arbitrary $s = 1,2,...$

$$Q_s(N^s) = \frac{6}{\pi^2 \cdot s} \cdot \int_2^{N^s} x^{-\alpha(N^s)} dx = \frac{6}{\pi^2} \cdot \frac{1}{s \cdot (1-\alpha(N^s))} \cdot N^{s \cdot (1-\alpha(N^s))}. \tag{3.5}$$

With (3.1) this transforms to

$$Q_s(N^s)=\frac{6}{\pi^2}\cdot\frac{1}{1-\alpha(N)}\cdot N^{1-\alpha(N)}=\frac{6}{\pi^2}\cdot N\cdot\frac{N}{N-c}\cdot\exp\left(-c\frac{\ln(N)}{N}\right), \tag{3.6}$$

which is independent of s and simply denoted by $Q(N)$. The equality $Q_s(N^s)=Q(N)$ reflects the fact that there are as many square-free powers in $[1,N^s]$ as there are square-free numbers in $[1,N]$. Now, what is a good value of $c\in[1,N)$? Clearly, the factor

$$f_N(c)=\tfrac{N}{N-c}\cdot\exp\left(-c\tfrac{\ln(N)}{N}\right) \tag{3.7}$$

converges to 1 as $N\to\infty$ for any fixed c. Its derivative with respect to c satisfies the property

$$\tfrac{\partial}{\partial c}f_N(c)<0,\quad \forall\, c\in[1,\tfrac{\ln(N)-1}{\ln(N)}N)\subseteq[1,N),\quad \forall\, N\geq 4, \tag{3.8}$$

which implies the following min-max property of (3.7) at $c=1$:

$$\min_{N\geq 10^4}\{\max_{c\in[1,\frac{\ln(N)-1}{\ln(N)}N)} f_N(c)\}=f_{10^4}(1)=0.99918. \tag{3.9}$$

The size-dependent exponent (3.1) with $c=1$ not only minimizes the absolute deviations between the LL estimator and the ETA (resp. WLS) estimators over the finite ranges of square-free powers $[1,10^{s\cdot m}]$, $m=4,...,10$, $s=1,2,3,4,5,10$, as shown in Section 2, but it turns out to be uniformly best with maximum error less than 10^{-3} against the asymptotic estimate, at least if $N\geq 10^4$. Moreover, the following limiting asymptotic result has been obtained.

First Digit Square-Free Integer Power Theorem (*GBL for square-free integer powers*). The asymptotic distribution of the first digit of square-free integer power sequences $n_f^s<10^{s\cdot m}$, $m\geq 4$, for fixed $s=1,2,3,...$, as $m\to\infty$, is given by

$$\lim_{m\to\infty}\frac{I_{s\cdot m}^s(d)}{S(10^m)}=\lim_{m\to\infty}P_{\alpha(s\cdot m)}^{GB}(d)=P_{s^{-1}}^{GB}(d),\quad d=1,...,9,\quad \alpha(s\cdot m)=\frac{1}{s}\left(1-\frac{1}{10^m}\right). \tag{3.10}$$

Table 3.1 compares the new counting function $Q(N)=Q_s(N^s),\ \forall\, s=1,2,...$, with the exact and asymptotic counting functions $S(N)$ and $S_{as}(N)=\dfrac{6}{\pi^2}N$.

Table 3.1: Comparison of square-free number counting functions for $N = 10^m$

m	S(N)	Q(N)	6N/π²	Q(N)/S(N)
1	7	5	6	0.7142857142857
2	61	58	60	0.9508196721311
3	608	604	607	0.9934210526316
4	6'083	6'074	6'079	0.9985204668749
5	60'794	60'786	60'792	0.9998684080666
6	607'926	607'919	607'927	0.9999884854407
7	6'079'291	6'079'261	6'079'271	0.9999950652140
8	60'792'694	60'792'699	60'792'710	1.0000000822467
9	607'927'124	607'927'089	607'927'101	0.9999999424273
10	6'079'270'942	6'079'271'005	6'079'271'018	1.0000000103631
11	60'792'710'280	60'792'710'170	60'792'710'185	0.9999999981906
12	607'927'102'274	607'927'101'837	607'927'101'854	0.9999999992812
13	6'079'271'018'294	6'079'271'018'522	6'079'271'018'540	1.0000000000375
14	60'792'710'185'947	60'792'710'185'383	60'792'710'185'402	0.9999999999907
15	607'927'101'854'103	607'927'101'854'006	607'927'101'854'026	0.9999999999998

Concluding Remark 3.1. As proved by Jameson [10] the proportion of odd square-free numbers is asymptotically equal to $4/\pi^2$, from which it follows that the ratio of odd to even square-free numbers is 2:1. The interested reader might investigate the corresponding sequences of odd and even square-free integer powers and derive similar GBL results.

Appendix: Tables of first digits for square-free integer powers

Based on the recursive relation (2.3)-(2.4), the computation of $I^s_{s \cdot m}(d)$, $m = 4,...,10$, is straightforward, at least if a table of the Möbius function is available (e.g. sequence A008683 in OEIS founded by Sloane [19]). These numbers are listed in Table A.1. The entry $s \to \infty$ corresponds to the limiting Benford law as the power goes to infinity.

Table A.1: First digit distribution of square-free powers up to $10^{s \cdot m}$, $m = 4,...,10$

s=1 / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	676	6'753	67'540	675'491	6'754'775	67'547'507	675'474'599
2	677	6'759	67'556	675'452	6'754'706	67'547'406	675'474'562
3	677	6'745	67'532	675'495	6'754'719	67'547'498	675'474'519
4	677	6'768	67'561	675'458	6'754'749	67'547'370	675'474'546
5	671	6'743	67'539	675'463	6'754'764	67'547'480	675'474'534
6	679	6'762	67'566	675'513	6'754'770	67'547'491	675'474'499
7	678	6'749	67'533	675'432	6'754'684	67'547'455	675'474'641
8	672	6'758	67'547	675'486	6'754'746	67'547'422	675'474'533
9	676	6'757	67'552	675'501	6'754'781	67'547'495	675'474'509

s=2 / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	1'171	11'652	116'439	1'164'549	11'645'617	116'456'692	1'164'566'728
2	884	8'934	89'367	893'612	8'936'003	89'360'203	893'603'026
3	753	7'526	75'336	753'361	7'533'536	75'334'407	753'342'957
4	671	6'640	66'369	663'684	6'637'031	66'370'825	663'708'054
5	600	5'997	59'991	600'024	6'000'331	60'003'694	600'037'812
6	553	5'517	55'184	551'776	5'517'885	55'179'201	551'791'957
7	510	5'137	51'367	513'610	5'135'979	51'359'553	513'595'378
8	481	4'835	48'249	482'421	4'823'838	48'237'994	482'379'468
9	460	4'556	45'624	456'254	4'562'474	45'624'555	456'245'562

s=3 / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	1'367	13'670	136'857	1'368'703	13'687'419	136'874'710	1'368'747'899
2	971	9'615	96'052	960'208	9'601'504	96'014'609	960'145'043
3	765	7'647	76'437	764'375	7'643'723	76'436'951	764'369'702
4	642	6'445	64'525	645'443	6'454'686	64'548'120	645'481'279
5	559	5'632	56'413	564'238	5'642'255	56'422'496	564'225'563
6	506	5'055	50'454	504'532	5'045'423	50'454'044	504'540'015
7	459	4'590	45'848	458'528	4'585'135	45'850'705	458'505'935
8	416	4'213	42'169	421'720	4'217'192	42'172'165	421'722'747
9	398	3'927	39'171	391'544	3'915'357	39'153'324	391'532'759

s=4 / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	1'477	14'761	147'762	1'477'917	14'779'258	147'793'000	1'477'928'516
2	999	9'920	99'105	990'957	9'909'733	99'097'825	990'978'621
3	776	7'677	76'658	766'577	7'665'808	76'658'390	766'584'635
4	630	6'344	63'402	633'807	6'337'591	63'376'082	633'760'811
5	546	5'449	54'461	544'694	5'447'186	54'471'736	544'717'879
6	472	4'797	48'035	480'351	4'803'264	48'032'531	480'324'229
7	431	4'319	43'139	431'315	4'313'052	43'130'290	431'303'473
8	393	3'924	39'258	392'585	3'925'720	39'257'307	392'573'396
9	359	3'603	36'106	361'088	3'611'082	36'109'963	361'099'382

s=5 / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	1'541	15'462	154'569	1'545'536	15'455'439	154'554'413	1'545'542'924
2	1'016	10'087	100'847	1'008'552	10'085'379	100'853'684	1'008'538'705
3	769	7'664	76'685	766'804	7'668'233	76'682'538	766'824'201
4	619	6'254	62'587	625'898	6'259'311	62'593'259	625'933'405
5	533	5'325	53'271	532'603	5'325'760	53'257'492	532'573'123
6	468	4'664	46'567	465'699	4'656'829	46'568'456	465'684'982
7	414	4'153	41'502	415'167	4'151'661	41'516'413	415'164'378
8	378	3'752	37'555	375'533	3'755'169	37'551'865	375'518'563
9	345	3'433	34'343	343'499	3'434'913	34'349'004	343'490'661

s=∞ / first digit	6'083	60'794	607'926	6'079'291	60'792'694	607'927'124	6'079'270'942
1	1'831	18'301	183'004	1'830'049	18'300'424	183'004'300	1'830'042'905
2	1'071	10'705	107'050	1'070'510	10'705'062	107'050'653	1'070'506'474
3	760	7'596	75'954	759'539	7'595'362	75'953'647	759'536'431
4	590	5'892	58'914	589'144	5'891'421	58'914'225	589'142'226
5	482	4'814	48'136	481'366	4'813'641	48'136'427	481'364'248
6	407	4'070	40'699	406'989	4'069'876	40'698'769	406'987'673
7	353	3'526	35'255	352'550	3'525'487	35'254'878	352'548'758
8	311	3'110	31'097	310'971	3'109'700	31'097'006	310'970'043
9	278	2'782	27'817	278'173	2'781'721	27'817'220	278'172'183

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