ASYMPTOTIC BEHAVIOR OF RECURSIONS VIA FIXED POINT THEORY

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ABSTRACT. In this paper we provide a formulation of initial value problems for (explicit and implicit) difference equations in terms of abstract equations in sequence spaces. They will be solved using appropriate fixed point theorems and we obtain quantitative attractivity properties.

1. INTRODUCTION

At first glance, it seems to be an almost trivial observation that (nonautonomous) difference equations or recursions like

$$x_{k+1} = f_k(x_k) \tag{1.1}$$

can be formulated as operator equations in appropriate sequence spaces. Nevertheless, the obvious advantage of such a reformulation is based on the fact that a large variety of fixed point theorems or other tools from nonlinear analysis can be employed in order to study asymptotic properties for (1.1), instead of, e.g., Lyapunov or Gronwall techniques. Hence, a dynamical problem reduces to a fixed point problem in an infinite dimensional space. The naïve approach, though, of characterizing a recursion (1.1) by the operator equation

$$S^+x = F(x)$$

with the forward shift operator $(S^+x)_k := x_{k+1}$ and the substitution operator $\big(F(x)\big)_k := f_k(x_k)$, is of little use, since initial conditions are not taken into consideration and the typically non-expansive operator S^+ is technically subtle to handle, i.e., fixed point theorems for non-expansive maps are sophisticated.

Therefore, this paper features an alternative way, inspired by the pioneering work of Petropoulou and Siafarikas. Their "functional analytic method" is based on the fact that (1.1) (and more general equations) allow a characterization as operator equations in a separable Hilbert space, thus essentially the space of square summable sequences ℓ^2, as well as in subspaces of ℓ^2. This method has been successfully applied to investigate the asymptotic behavior of linear and nonlinear ordinary difference equations (cf. (1) and (2; 3; 4), resp.), delay difference (cf. (5; 1)) and partial difference equations (cf. (5; 4)). One of their preferred tools is a fixed point theorem for holomorphic mappings due to Earle and Hamilton (6) (see also (7, p. 111, Theorem 4.6)).

In the present paper we overcome the deficit that the topology of the sequence spaces under consideration is given by an inner product. Consequently, for instance we can also use spaces of merely convergent or even exponentially bounded sequences. Thus, we obtain criteria for the existence of sub-exponentially decaying solutions. One frequently encounters such a situation in critical stability problems (e.g., for reduced equations on center manifolds) or within the framework of ℓ^p-stability (cf. (8)).

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We have subdivided this article into essentially three parts. Sections 2–3 are fundamental for our work and contain some basic results on sequence spaces, difference equations and the following crucial result:

> Theorem 3.3: An initial value problem for a difference equation is equivalent to a fixed point equation in a sequence space.

Criteria based on the contraction mapping principle are presented in Section 4, which provide assumptions guaranteeing that all solutions of a given equation are in a certain space. After all, Section 5 contains some further global criteria using fixed point theorems of Krasnoselskii, Reinermann or Goebel-Kirk type.

Compared to classical methods and techniques in stability theory, the presented approach features some advantages:

- The verification of attractivity properties for given solutions becomes simple and technically transparent. Indeed, our proofs typically consist of two steps: One shows that a nonlinear operator is well-defined on an ambient space, and one deduces a structural property guaranteeing the existence of fixed points, like for instance, contractivity, non-expansiveness, complete or strong continuity. In addition, this yields information on the domain of attraction.
- As demonstrated in (2; 3; 5; 4; 1), the method easily extends for further classes of discrete equations (delay difference, partial difference equations).
- While (8) obtains criteria for ℓ^p-stability in terms of a Lyapunov function, we tackle the problem directly and impose conditions depending only on the right-hand side of the equation, which are therefore easy to check.

Indicating a general tendency, our approach seems to be better suited for nonautonomous equations. On the other hand, it turned out that the methodology exploited in this paper has disadvantages, which should not be concealed:

- For scalar explicit equations in $\mathbb{R}$ traditional approaches often yield better results. This should not surprise; keeping in mind that we lift the problem into an infinite-dimensional space, it is quite clear that important properties of the reals (e.g., compactness criteria or the order-structure) get lost.
- The present approach requires a certain uniformity of, e.g., Lipschitz or boundedness constants in the time variable, which is a technical issue and not intrinsic for the problem.
- Properties of the right-hand side defining the difference equations are strongly related to the obtained fixed point operator. Thus, there are no smoothing properties of, e.g., integral operators yielding compactness or other convenient attributes.

Let us close the introductory paragraphs by indicating some perspectives. We restricted ourselves to tools from metric fixed point theory in linear spaces. As a matter of course, also other techniques from nonlinear analysis seem appropriate to solve our nonlinear equation encountered in Theorem 3.3; for instance local implicit and inverse function theorems, nonlinear alternatives or topological methods. We postpone the use of these methods to later papers. Finally, it is worth to point out that also for ordinary differential equations, fixed point methods have been applied to stability problems ((9; 10)).

Now we provide our terminology and some standard notions from geometry in Banach spaces. The real field is denoted by $\mathbb{R}$ and we write $\mathbb{C}$ for the complex numbers; $\mathbb{Z}$ is the ring of integers, $\mathbb{N}$ the positive integers and a *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$; particularly $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}$, and $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : k \leq \kappa\}$.

Throughout this paper, $\mathcal{X}$ is a real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) Banach space with norm $\|\cdot\|_{\mathcal{X}}$ (or simply $\|\cdot\|$, if no confusion can arise). Writing $\mathcal{Y}$ for another Banach space, the space of linear bounded mappings between $\mathcal{X}$ and $\mathcal{Y}$ is $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, we abbreviate $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$ and $I_{\mathcal{X}}$ is the identity on $\mathcal{X}$.

We write $B_r(x)$ for the open ball in $\mathcal{X}$ with center $x \in \mathcal{X}$ and radius $r \geq 0$; $\bar{B}_r(x)$ stands for the corresponding closed ball. We write Ω° for the interior and $\bar{\Omega}$ for the topological closure of a subset $\Omega \subseteq \mathcal{X}$.

With a mapping $f : \Omega \to \mathcal{Y}$ we write $\operatorname{Lip} f$ for its *Lipschitz* constant and $\operatorname{Lip}_1 f$ for the Lipschitz constant w.r.t. the first argument, if f depends on more than one argument.

Some of the fixed point theorems we are about to use rely heavily on geometrical properties of Banach spaces. Hence, the following notions are crucial for our later considerations; as a reference we recommend and use (11).

Uniform convexity is a key ingredient to derive fixed point results for non-expansive maps. The *modulus of convexity* for $\mathcal{X}$ is the function $\delta_{\mathcal{X}} : [0, 2] \to [0, 1]$ given by

$$\delta_{\mathcal{X}}(t) := \inf \left\{ 1 - \tfrac{\|x+y\|}{2} : x, y \in \bar{B}_1(0), \|x - y\| \geq t \right\}$$

(cf., e.g., (11, p. 64, Definition 2.3)) and $\mathcal{X}$ is called *uniformly convex*, if $\delta_{\mathcal{X}}(t) > 0$ for $t > 0$.

Writing $r(\Omega)$ for the *Chebyshev radius* of $\Omega \subseteq \mathcal{X}$ (cf. (11, p. 112)), the *normal structure coefficient* $N(\mathcal{X})$ of $\mathcal{X}$ is defined as

$$N(\mathcal{X}) := \inf \left\{ \tfrac{\operatorname{diam} \Omega}{r(\Omega)} : \Omega \subseteq \mathcal{X} \text{ convex, closed, bounded with } \operatorname{diam} \Omega > 0 \right\}$$

(cf. (11, p. 114, Definition 2.1)), where $\operatorname{diam} \Omega := \sup_{x,y \in \Omega} \|x - y\|$.

2. Sequence spaces

In the remaining paper, let $\Omega \subseteq \mathcal{X}$ be a subset with $0 \in \Omega$. To consolidate notation, we first define the space $\ell(\mathbb{I}, \Omega)$ of all sequences $\phi = (\phi_k)_{k \in \mathbb{I}}$ with values $\phi_k \in \Omega$ and then define various subspaces of $\ell(\mathbb{I}, \Omega)$.

2.1. Bounded sequences.

A real sequence $\omega = (\omega_k)_{k \in \mathbb{I}}$ with positive values is called a *weight sequence*, if

$$\Upsilon(\omega) := \sup_{k \in \mathbb{I}} \tfrac{\omega_k}{\omega_{k+1}} < \infty.$$

With a positive sequence ω, we define the Banach space of ω*-bounded sequences*

$$\ell^\infty_\omega(\mathbb{I}, \mathcal{X}) := \left\{ \phi \in \ell(\mathbb{I}, \mathcal{X}) : \sup_{k \in \mathbb{I}} \omega_k^{-1} \|\phi_k\| < \infty \right\}, \quad \|\phi\|_{\ell^\infty_\omega(\mathbb{I}, \mathcal{X})} := \sup_{k \in \mathbb{I}} \omega_k^{-1} \|\phi_k\|;$$

for simplicity reasons we often write $\|\cdot\|_\omega$ instead of $\|\cdot\|_{\ell^\infty_\omega(\mathbb{I}, \mathcal{X})}$. Obviously, the Banach space of *bounded sequences* $\ell^\infty(\mathbb{I}, \mathcal{X})$ corresponds to the special case $\omega_k = 1$. Moreover, considering

$$\ell_0(\mathbb{I}, \mathcal{X}) := \left\{ \phi \in \ell(\mathbb{I}, \mathcal{X}) : \lim_{k \to \infty} \|\phi_k\| = 0 \right\}$$

as normed subspace of $\ell^\infty(\mathbb{I}, \mathcal{X})$ yields another Banach space. Since $\ell_0(\mathbb{I}, \mathcal{X})$ possesses a Schauder basis, we arrive at

Lemma 2.1. *Let $\dim \mathcal{X} < \infty$. Then a set $K \subseteq \ell_0(\mathbb{I}, \mathcal{X})$ is relatively compact, if and only if K is bounded and*

$$\lim_{n \to \infty} \sup_{\phi \in K} \sup_{k > n} \|\phi_k\| = 0.$$

Proof. Using the canonical unit vectors $e_n = (\delta_{n,k})_{k\in\mathbb{I}}$ as Schauder basis of $\ell_0(\mathbb{I},\mathbb{R})$, the claim follows from (11, p. 34, Theorem 4.1). □

2.2. **Summable sequences.** With a real $p \geq 1$ we define the Banach spaces of *p-summable sequences*

$$\ell^p(\mathbb{I},\mathcal{X}) := \left\{\phi \in \ell(\mathbb{I},\mathcal{X}) : \sum_{k\in\mathbb{I}} \|\phi_k\|^p < \infty\right\}, \quad \|\phi\|_{\ell^p(\mathbb{I},\mathcal{X})} := \sqrt[p]{\sum_{k\in\mathbb{I}} \|\phi_k\|^p}; \tag{2.1}$$

since we do not want to overextend our notation, we usually write $\|\cdot\|_p$ rather than $\|\cdot\|_{\ell^p(\mathbb{I},\mathcal{X})}$. Compactness in $\ell^p(\mathbb{I},\mathcal{X})$ can be characterized similarly to Lemma 2.1 as follows:

Lemma 2.2. *Let $\dim\mathcal{X} < \infty$. Then a set $K \subseteq \ell^p(\mathbb{I},\mathcal{X})$ is relatively compact, if and only if K is bounded and*

$$\lim_{n\to\infty} \sup_{\phi\in K} \sqrt[p]{\sum_{k>n} \|\phi_k\|^p} = 0.$$

Proof. Again, the canonical unit vectors $e_n = (\delta_{n,k})_{k\in\mathbb{I}}$ form a Schauder basis of $\ell^p(\mathbb{I},\mathbb{R})$ and, thus, the proof follows from (11, p. 34, Theorem 4.1). □

If $\mathcal{X}$ is uniformly convex, then also $\ell^p(\mathbb{I},\mathcal{X})$ is uniformly convex for $p \in (1,\infty)$ (cf. (12, p. 63, Theorem 2.4.16)), whereas $\ell^p(\mathbb{I},\mathcal{X})$ are not uniformly convex for $p \in \{1,\infty\}$. If $\dim\mathcal{X} < \infty$ we note that $\mathcal{X}$ is isomorphic to $\mathbb{F}^N$ and we equip $\mathcal{X}$ with the Euclidean norm; then $\ell^p(\mathbb{I},\mathcal{X})$ becomes uniformly convex.

The modulus of convexity $\delta_{\ell^p(\mathbb{I},\mathcal{X})}$ can be obtained from

$$\delta_{\ell^p(\mathbb{I},\mathcal{X})}(t) = 1 - \sqrt[p]{1 - \left(\tfrac{t}{2}\right)^p} \quad \text{for all } p \in [2,\infty),$$
$$2 = \left(1 - \delta_{\ell^p(\mathbb{I},\mathcal{X})}(t) + \tfrac{t}{2}\right)^p + \left(1 - \delta_{\ell^p(\mathbb{I},\mathcal{X})}(t) - \tfrac{t}{2}\right)^p \quad \text{for all } p \in (1,2)$$

(cf. (11, p. 64, Example 6)), while the corresponding normal structure coefficient is given by $N(\ell^p(\mathbb{I},\mathcal{X})) = \min\left\{2^{1-1/p}, 2^{1/p}\right\}$ for $p \geq 1$ (cf. (11, p. 128, Theorem 6.3)).

Let $\mathcal{Y}$ stand for one of the sets $\ell^\infty_\omega(\mathbb{I},\Omega)$, $\ell_0(\mathbb{I},\Omega)$ or $\ell^p(\mathbb{I},\Omega)$, denoting the respective subsets of sequences with values in Ω. We endow $\mathcal{Y}$ with the canonical metric topology and remark that $\mathcal{Y}$ is a complete metric space, if Ω is closed. For positive valued sequences $\omega \in \ell^p(\mathbb{I},\mathbb{R})$ we have the inclusions

$$\ell^\infty_\omega(\mathbb{I},\Omega) \subseteq \ell^p(\mathbb{I},\Omega) \hookrightarrow \ell_0(\mathbb{I},\Omega) \hookrightarrow \ell^\infty(\mathbb{I},\Omega)$$

and actually each of the embeddings is norm one.

3. Preliminaries

Unless otherwise noted, we assume $\mathbb{I}$ is a discrete interval which is unbounded above. Since we are interested in asymptotic behavior, this is a reasonable assumption. We pick $\kappa \in \mathbb{I}$ and suppose $\Omega \subseteq \mathcal{X}$ to be a set with $0 \in \Omega$.

3.1. **Difference equations.** To denote difference equations (the notions *recursion* or *iteration* are also frequently used) we use the notation

$$x_{k+1} = f_k(x_k, x_{k+1}) \tag{3.1}$$

with the *right-hand side* $f_k : \Omega\times\Omega \to \Omega$, $k \in \mathbb{I}$. A sequence $\phi = (\phi_k)_{k\in\mathbb{Z}^+_\kappa}$ in Ω satisfying $\phi_{k+1} = f_k(\phi_k, \phi_{k+1})$ for $k \in \mathbb{Z}^+_\kappa$ is called a *(forward) solution* of (3.1). Analogously, a *backward solution* has this property with $\mathbb{Z}_\kappa^+$ replaced by $\mathbb{Z}_\kappa^-$. We say that (3.1) is *well-posed* on $\Omega_0 \subseteq \Omega$, if for all $\kappa \in \mathbb{I}$, $\xi \in \Omega_0$ there exists a unique solution ϕ with $\phi(\kappa) = \xi$. In this case, let $\varphi(\kappa, \xi)$ denote the *general solution* of (3.1), i.e. $\varphi(\kappa, \xi) \in \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$ solves (3.1) and satisfies the initial condition $\varphi(\kappa, \xi)_\kappa = \xi$ for $\kappa \in \mathbb{Z}$, $\xi \in \Omega_0$.

If f does not depend on its third argument, we denote (3.1) as *explicit* difference equation, remark that it is trivially well-posed on Ω and its general solution can be obtained by

$$\varphi(\kappa, \xi)_k := \begin{cases} \xi & \text{for } k = \kappa \\ f_{k-1} \circ \ldots \circ f_\kappa(\xi) & \text{for } k > \kappa \end{cases}. \tag{3.2}$$

3.2. **Attractivity notions.** In order to introduce an appropriate notion of attractivity, let $\mathcal{Y}$ be a subspace of $\ell_0(\mathbb{I}, \mathcal{X})$. Differing from the standard terminology, we say a difference equation (3.1) is *$\mathcal{Y}$-attractive*, if for each $\kappa \in \mathbb{I}$ there exists a $\rho > 0$ and a solution $\phi \in \ell(\mathbb{Z}_\kappa^+, \Omega)$ such that for all $\xi \in B_\rho(0) \cap \Omega$ the following holds:

$$\phi(\kappa) = \xi, \qquad \phi \in \mathcal{Y};$$

in addition, equation (3.1) is called *uniformly $\mathcal{Y}$-attractive*, if $\rho > 0$ can be chosen independently of $\kappa \in \mathbb{I}$, and *globally $\mathcal{Y}$-attractive* when $\phi \in \mathcal{Y}$ holds for all initial values $\xi \in \Omega$. This paper provides criteria for global $\mathcal{Y}$-attractivity.

Concerning these attractivity notions it is worth to point out that (3.1) is not assumed to possess the trivial solution, i.e., 0 needs not to be a fixed point of $f_k(0, \cdot)$. Hence, $\mathcal{Y}$-attractivity is a property of the difference equation (3.1) and not (necessarily) of its solutions. Nevertheless, this notion of attractivity can easily be attached to individual solutions of (3.1). Thereto, let $\phi^* \in \ell(\mathbb{I}, \Omega)$ be a given reference solution of (3.1). In order to determine attractivity properties of ϕ^* it is convenient to work with the *difference equation of perturbed motion*

$$x_{k+1} = f_k(x_k + \phi_k^*, x_{k+1} + \phi_{k+1}^*) - f_k(\phi_k^*, \phi_{k+1}^*). \tag{3.3}$$

Clearly, ϕ^* is (uniformly, globally) $\mathcal{Y}$-attractive in the standard terminology, if and only if the zero solution of (3.3) has this property. In particular, our methods apply to (3.3).

We have abstract formulations of certain classical attractivity notions for difference equations (cf., e.g., (13) or (14, p. 240, Definition 5.4.1)), namely attractivity for $\mathcal{Y} = \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$, exponential stability for $\mathcal{Y} = \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{X})$ with $\omega_k = \gamma^k$ and $\gamma \in (0, 1)$, and also ℓ^p-stability for $\mathcal{Y} = \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$.

The notion of ℓ^p-stability has been introduced to difference equations in (8). As the following example demonstrates, it lies in between attractivity and exponential stability.

Example 3.1. Consider the explicit equation $x_{k+1} = \frac{k}{k+1} x_k$ for $\mathbb{I} = \mathbb{N}$. Its general solution is given by $\varphi(\kappa, \xi)_k = \frac{\kappa}{k}\xi$ for $\kappa \in \mathbb{I}$ and $\xi \in \mathbb{R}$. Hence, the trivial solution is $\ell^p(\mathbb{Z}_\kappa^+, \mathbb{R})$-stable for $p > 1$, but not $\ell^1(\mathbb{Z}_\kappa^+, \mathbb{R})$- or exponentially stable. Moreover, (8) provides a similar example of a linear difference equation which is asymptotically stable, but not ℓ^p-stable for any $p \geq 1$.

3.3. **Operator theoretical setting.** Let $\kappa \in \mathbb{I}$. It is crucial for our functional analytical approach to introduce the operators:

- the linear *embedding operator* $E^+ : \mathcal{X} \to \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$, $E^+\xi := (\xi, 0, 0, \ldots)$,
- the linear *right shift operator* $S_\kappa^+ : \ell(\mathbb{Z}_\kappa^+, \mathcal{X}) \to \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$, $S_\kappa^+\phi := (0, \phi_\kappa, \phi_{\kappa+1}, \ldots)$,
- the nonlinear *substitution operator* $F_f : \ell(\mathbb{I}, \Omega) \to \ell(\mathbb{I}, \mathcal{X})$, $F_f(\phi) := \big(f_k(\phi_k, \phi_{k+1})\big)_{k \in \mathbb{I}}$,

- $G_f : \ell(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell(\mathbb{Z}_\kappa^+, \Omega)$, given by

$$G_f(\phi, \xi) := E^+\xi + S_\kappa^+ F_f(\phi). \tag{3.4}$$

The substitution operator G_f depends linearly on the functions f_k. Moreover, if $f_k : \Omega \times \Omega \to \Omega$ is a linear mapping, then G_f becomes affine linear.

Lemma 3.1. *Let $\kappa \in \mathbb{I}$ and $\mathcal{Y}$ be one of the spaces $\ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{X})$, $\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ or $\ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$. Then one has the inclusions $E^+ \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $S_\kappa^+ \in \mathcal{L}(\mathcal{Y})$ with norm*

$$\|E^+\|_{\mathcal{L}(\mathcal{X}, \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{X}))} = \omega_\kappa^{-1}, \qquad \|S_\kappa^+\|_{\mathcal{L}(\ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{X}))} \leq \Upsilon(\omega), \tag{3.5}$$

$$\|E^+\|_{\mathcal{L}(\mathcal{X}, \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X}))} = 1, \qquad \|S_\kappa^+\|_{\mathcal{L}(\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X}))} \leq 1,$$

$$\|E^+\|_{\mathcal{L}(\mathcal{X}, \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X}))} = 1, \qquad \|S_\kappa^+\|_{\mathcal{L}(\ell^p(\mathbb{Z}_\kappa^+, \mathcal{X}))} \leq 1. \tag{3.6}$$

If $\dim \mathcal{X} < \infty$, then $E^+ \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is compact.

Proof. The proof of the norm estimates is left to the reader. In case $\dim \mathcal{X} < \infty$, the operator E^+ is finite dimensional, hence compact. □

Let $D \subseteq \mathcal{Y}$ be a nonempty subset of a Banach space $\mathcal{Y}$. For a self-mapping $G : D \to D$ we define its *iterates* recursively by

$$G^0(x) := x, \qquad G^{n+1} := G(G^n(x)) \quad \text{for all } x \in D,\ n \in \mathbb{Z}_0^+.$$

Proposition 3.2. *Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi^0 = (\phi_k^0)_{k \in \mathbb{Z}_\kappa^+}$ be a sequence in Ω. Then the iterates $G_f^n(\cdot, \xi)$ of $G_f(\cdot, \xi) : \ell(\mathbb{Z}_\kappa^+, \Omega) \to \ell(\mathbb{Z}_\kappa^+, \Omega)$ have the representation $G_f^n(\phi^0, \xi) = (\phi_k^n)_{k \in \mathbb{Z}_\kappa^+}$ with*

$$\phi_\kappa^n = \xi, \qquad \phi_{k+1}^{n+1} = f_k(\phi_k^n, \phi_{k+1}^n) \quad \textit{for all } n \in \mathbb{Z}_0^+,\ k \in \mathbb{Z}_\kappa^+.$$

In particular, if (3.1) *is explicit, then*

$$G_f^n(\phi^0, \xi)_k = \begin{cases} f_{k-1} \circ \ldots \circ f_\kappa(\xi), & \textit{if } \kappa \leq k < n + \kappa, \\ f_{k-1} \circ \ldots \circ f_{k-n}(\phi_{k-n}^0), & \textit{if } k \geq n + \kappa. \end{cases}$$

Remark 3.1. For explicit equations (3.1) one has $G_f^n(\phi^0, \xi)_k = \varphi(\kappa, \xi)_k$ for $\kappa \leq k < n + \kappa$ and, thus, iterating the operator $G_f(\cdot, \xi)$ yields a successive approximation of solutions to (3.1).

Proof. The proof is an easy induction argument. □

The basic tool for our whole analysis is given in

Theorem 3.3. *Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi = (\phi_k)_{k \in \mathbb{Z}_\kappa^+}$ be a sequence in Ω. Then ϕ is a solution of* (3.1) *with $\phi_\kappa = \xi$, if and only if ϕ solves the fixed point equation*

$$\phi = G_f(\phi, \xi). \tag{3.7}$$

Proof. Let $\phi \in \ell(\mathbb{Z}_\kappa^+, \Omega)$ be a solution of (3.1) with $\phi_\kappa = \xi$. Then we have:

$$\phi_\kappa = \xi = \left(E^+\xi\right)_\kappa = \left(E^+\xi + S_\kappa^+ F_f(\phi)\right)_\kappa$$

and for $k > \kappa$,

$$\phi_k = f_{k-1}(\phi_{k-1}, \phi_k) = \left(F_f(\phi)\right)_{k-1} = \left(E^+\xi + S_\kappa^+ F_f(\phi)\right)_k.$$

The converse direction can be shown similarly. □

Remark 3.2. While the above Theorem 3.3 is formulated for forward solutions of (3.1), we point out that a dual theory holds for backward solutions $\phi \in \ell(\mathbb{Z}_\kappa^-, \mathcal{X})$. More precisely, if $\mathbb{I}$ is unbounded below, then a sequence ϕ with $\phi_\kappa = \xi$ solves (3.1) on $\mathbb{Z}_\kappa^-$, if and only if the fixed point relation

$$\phi = E^-\xi + S_\kappa^- F_f(\phi)$$

holds, where we have used

- the linear embedding operator $E^- : \mathcal{X} \to \ell(\mathbb{Z}_\kappa^-, \mathcal{X})$, $E^-\xi := (\dots, 0, 0, \xi)$,
- the linear *left shift operator* $S_\kappa^- : \ell(\mathbb{Z}_\kappa^-, \Omega) \to \ell(\mathbb{Z}_\kappa^-, \Omega)$, $S_\kappa^-\phi := (\dots, \phi_{\kappa-1}, \phi_\kappa, 0)$.

Remark 3.3. If the right-hand side f_k does not depend on its first variable, then (3.1) is a purely algebraic problem and the fixed point equation (3.7) degenerates to the infinite algebraic system $\phi_k = f_{k-1}(\phi_k)$ for $\kappa < k$.

We close this section with certain frequently used assumptions and a convenient terminology on the right-hand side of (3.1) guaranteeing well-definedness, (Lipschitz-) continuity or compactness of G_f, respectively.

Thereto, let $\mathcal{Y}$ be a subset of $\ell(\mathbb{I}, \mathcal{X})$. We say the difference equation (3.1) or the function $f_k : \Omega \times \Omega \to \mathcal{X}$ is $\mathcal{Y}$*-admissible*, if there exists a sequence $\phi^* \in \mathcal{Y}$ such that $G_f(\phi^*, 0) \in \mathcal{Y}$.

Hypothesis 1. *We say a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies the assumption*

$(B)_f$ *with $\alpha \in [0, \infty)$ and sequences a, b, c from subsets of $\ell(\mathbb{I}, \mathbb{R})$, if the estimate*

$$\|f_k(x, y)\| \le a_k + \max\{b_k \|x\|^\alpha, c_{k+1} \|y\|^\alpha\}$$

for all $k \in \mathbb{I}$, $x, y \in \Omega$ holds true,

$(L)_f$ *with sequences L, l from subsets of $\ell(\mathbb{I}, \mathbb{R})$, if the estimate*

$$\|f_k(x, y) - f_k(\bar{x}, \bar{y})\| \le \max\{L_k \|x - \bar{x}\|, l_{k+1} \|y - \bar{y}\|\}$$

for all $k \in \mathbb{I}$, $x, y \in \Omega$ holds true.

4. Contraction-like Criteria

Throughout this section, we suppose $\mathcal{Y}$ is another Banach space and consider an abstract mapping $G : D \to \mathcal{Y}$ defined on a nonempty subset $D \subseteq \mathcal{Y}$.

For the sake of completeness and consistency we formulate a slight generalization of the well-known Banach contraction mapping principle.

Proposition 4.1. *Let $n \in \mathbb{N}$. If D is closed and $G^n : D \to D$ is contractive, then G has a unique fixed point.*

Proof. See, for instance, (15, p. 17, (6.3)). □

Lemma 4.2 (well-definedness on ℓ_ω^∞)**.** *Let $\kappa \in \mathbb{I}$ and $\bar{\omega}$ be a weight sequence. If the function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha \in [0, \infty)$ and $a \in \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, $b, c \in \ell_{\frac{\bar{\omega}}{\omega^\alpha}}^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f : \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell_{\bar{\omega}}^\infty(\mathbb{Z}_\kappa^+, \mathcal{X})$ is well-defined and satisfies*

$$\|G_f(\phi, \xi)\|_{\bar{\omega}} \le \max\left\{\|\xi\| \bar{\omega}_\kappa^{-1}, \Upsilon(\bar{\omega}) \|a\|_\omega + \max\left\{\Upsilon(\bar{\omega}) \|b\|_{\frac{\bar{\omega}}{\omega^\alpha}}, \|c\|_{\frac{\bar{\omega}}{\omega^\alpha}}\right\} \|\phi\|_\omega^\alpha\right\}$$

for all $\xi \in \Omega$ and $\phi \in \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \Omega)$.

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi \in \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega)$. Provided the involved quantities exist, we make use of the relation

$$\|G_f(\phi,\xi)\|_{\bar\omega} \overset{(3.4)}{=} \max\left\{\|\xi\|\,\bar\omega_\kappa^{-1}, \sup_{k\in\mathbb{Z}^+_\kappa} \|f_k(\phi_k,\phi_{k+1})\|\,\bar\omega_{k+1}^{-1}\right\}$$

which easily follows from the definition of G_f. From Hypothesis $(B)_f$ we obtain

$$\begin{aligned} &\|f_k(\phi_k,\phi_{k+1})\|\,\bar\omega_{k+1}^{-1} \\ &\quad\leq\quad a_k\bar\omega_{k+1}^{-1} + \max\left\{b_k\frac{\omega_k^\alpha}{\bar\omega_{k+1}}\left(\|\phi_k\|\,\omega_k^{-1}\right)^\alpha, c_{k+1}\frac{\omega_{k+1}^\alpha}{\bar\omega_{k+1}}\left(\|\phi_{k+1}\|\,\omega_{k+1}^{-1}\right)^\alpha\right\} \\ &\quad\leq\quad a_k\bar\omega_{k+1}^{-1} + \max\left\{b_k\frac{\omega_k^\alpha}{\bar\omega_{k+1}}, c_{k+1}\frac{\omega_{k+1}^\alpha}{\bar\omega_{k+1}}\right\}\|\phi\|_\omega^\alpha \\ &\quad\leq\quad \Upsilon(\bar\omega)\,\|a\|_\omega + \max\left\{\Upsilon(\bar\omega)\,\|b\|_{\frac{\bar\omega}{\omega^\alpha}}, \|c\|_{\frac{\bar\omega}{\omega^\alpha}}\right\}\|\phi\|_\omega^\alpha \quad \text{for all } k\in\mathbb{Z}^+_\kappa. \end{aligned}$$

Then, passing over to the least upper bound over all integers $k \in \mathbb{Z}^+_\kappa$ shows that the mapping $G_f : \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega) \times \Omega \to \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \mathcal{X})$ is well-defined satisfying the claimed norm-estimate. □

Lemma 4.3 (Lipschitz condition on ℓ^∞_ω). *Let $\kappa \in \mathbb{I}$ and $\bar\omega$ be a weight sequence. If the function $f_k : \Omega\times\Omega \to \mathcal{X}$ satisfies $(L)_f$ with $L, l \in \ell^\infty_{\frac{\bar\omega}{\omega}}(\mathbb{Z}^+_\kappa, \mathbb{R})$, then the operator $G_f : \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega) \times \Omega \to \ell(\mathbb{Z}^+_\kappa, \mathcal{X})$ fulfills $G_f(\phi,\xi) - G_f(\bar\phi,\xi) \in \ell^\infty_{\bar\omega}(\mathbb{Z}^+_\kappa, \mathcal{X})$ for all $\phi,\bar\phi \in \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega)$ and*

$$\operatorname{Lip}_1 G_f \leq \max\left\{\Upsilon(\bar\omega)\,\|L\|_{\frac{\bar\omega}{\omega}}, \|l\|_{\frac{\bar\omega}{\omega}}\right\}. \tag{4.1}$$

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi, \bar\phi \in \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega)$. First of all, by definition of the operator G_f one has (provided it exists)

$$\left\|G_f(\phi,\xi) - G_f(\bar\phi,\xi)\right\|_{\bar\omega} \overset{(3.4)}{=} \sup_{k\in\mathbb{Z}^+_\kappa}\left\|f_k(\phi_k,\phi_{k+1}) - f_k(\bar\phi_k,\bar\phi_{k+1})\right\|\bar\omega_{k+1}^{-1}.$$

Now, for $k \in \mathbb{Z}^+_\kappa$ we derive from Hypothesis $(L)_f$ that

$$\begin{aligned} &\left\|f_k(\phi_k,\phi_{k+1}) - f_k(\bar\phi_k,\bar\phi_{k+1})\right\|\bar\omega_{k+1}^{-1} \\ \leq\quad & \max\left\{L_k\left\|\phi_k - \bar\phi_k\right\|\bar\omega_{k+1}^{-1}, l_{k+1}\left\|\phi_{k+1} - \bar\phi_{k+1}\right\|\bar\omega_{k+1}^{-1}\right\} \\ \leq\quad & \max\left\{L_k\frac{\omega_k}{\bar\omega_{k+1}}, l_{k+1}\frac{\omega_{k+1}}{\bar\omega_{k+1}}\right\}\left\|\phi - \bar\phi\right\|_\omega \leq \max\left\{\Upsilon(\bar\omega)\,\|L\|_{\frac{\bar\omega}{\omega}}, \|l\|_{\frac{\bar\omega}{\omega}}\right\}\left\|\phi - \bar\phi\right\|_\omega, \end{aligned}$$

which yields our claim by passing over to the supremum over $k \in \mathbb{Z}^+_\kappa$. □

This yields a prototype result on global attractivity.

Theorem 4.4. *Let $\kappa \in \mathbb{I}$, ω be a weight sequence and Ω be closed. If the right-hand side $f_k : \Omega\times\Omega \to \Omega$ of* (3.1) *is $\ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega)$-admissible and satisfies $(L)_f$ with sequences $L, l \in \ell^\infty(\mathbb{Z}^+_\kappa, \mathbb{R})$ so that*

$$\max\left\{\Upsilon(\omega)\,\|L\|_{\ell^\infty(\mathbb{Z}^+_\kappa,\mathbb{R})}, \|l\|_{\ell^\infty(\mathbb{Z}^+_\kappa,\mathbb{R})}\right\} < 1, \tag{4.2}$$

then the difference equation (3.1) *is well-posed on Ω with $\varphi(\kappa,\xi) \in \ell^\infty_\omega(\mathbb{Z}^+_\kappa, \Omega)$ for all $\xi \in \Omega$. In particular,* (3.1) *is globally $\ell^\infty_\omega(\mathbb{Z}^+_\kappa, \mathcal{X})$-attractive.*

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and choose $\phi \in \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \Omega)$. We want to apply Proposition 4.1 with $n = 1$ and the closed set $D = \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \Omega)$ to the mapping $G_f(\cdot, \xi)$. Since Lemma 4.3 is applicable with $\omega = \bar{\omega}$, it remains to show that $G_f(\cdot, \xi) : D \to D$ is well-defined. Thereto, we obtain from the triangle inequality (note that (3.1) is $\ell_\omega^\infty(\mathbb{Z}_\kappa^+, \Omega)$-admissible) and our assumptions,

$$\|G_f(\phi, \xi)\|_\omega \overset{(4.1)}{\leq} \|G_f(\phi^*, \xi)\|_\omega + \|\phi - \phi^*\|_\omega ,$$

thus $G_f(\phi, \xi) \in \ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{X})$. Moreover, since f_k has values in Ω, we have $G_f(\phi, \xi) \in D$. Our assumptions with (4.1) guarantee that $G_f(\cdot, \xi)$ is a contraction and its unique fixed point, by Theorem 3.3, is the solution of (3.1). □

Corollary 4.5. *If the equation* (3.1) *is explicit, then assumption* (4.2) *can be replaced by*

$$\sup_{k \geq \kappa} \omega_{k+1}^{-1} \prod_{j=k}^{k+n} L_j < 1 \quad \textit{for one } n \in \mathbb{N}.$$

Proof. With the explicit representation in Proposition 3.2, the nth iterate G_f^n is easily seen to be a contraction and Proposition 4.1 implies our claim. □

Lemma 4.6 (Lipschitz condition on ℓ_0)**.** *Let $\kappa \in \mathbb{I}$. If a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(L)_f$ with $L, l \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f : \ell_0(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$ fulfills $G_f(\phi, \xi) - G_f(\bar{\phi}, \xi) \in \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ for $\phi, \bar{\phi} \in \ell_0(\mathbb{Z}_\kappa^+, \Omega)$ and*

$$\operatorname{Lip}_1 G_f \leq \max \left\{ \|L\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})}, \|l\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})} \right\}. \tag{4.3}$$

Proof. The proof is analogous to Lemma 4.3 and omitted. □

Theorem 4.7. *Let $\kappa \in \mathbb{I}$ and Ω be closed. If the right-hand side $f_k : \Omega \times \Omega \to \Omega$ of* (3.1) *is $\ell_0(\mathbb{Z}_\kappa^+, \Omega)$-admissible and satisfies $(L)_f$ with sequences $L, l \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$ so that*

$$\max \left\{ \|L\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})}, \|l\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})} \right\} < 1, \tag{4.4}$$

then the difference equation (3.1) *is well-posed on Ω with $\varphi(\kappa, \xi) \in \ell_0(\mathbb{Z}_\kappa^+, \Omega)$ for all $\xi \in \Omega$. In particular,* (3.1) *is globally $\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$-attractive.*

Proof. Using Lemma 4.6 one proceeds as in Theorem 4.4. □

Lemma 4.8 (Lipschitz condition on ℓ^p)**.** *Let $\kappa \in \mathbb{I}$ and $p \geq 1$. If a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(L)_f$ with $L, l \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f : \ell^p(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$ fulfills $G_f(\phi, \xi) - G_f(\bar{\phi}, \xi) \in \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$ for all $\phi, \bar{\phi} \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$ and*

$$\operatorname{Lip}_1 G_f \leq \|l\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})} + \|L\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})} . \tag{4.5}$$

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi, \bar{\phi} \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$. We derive from Hypothesis $(L)_f$ that

$$\begin{aligned} & \left\| f_k(\phi_k, \phi_{k+1}) - f_k(\bar{\phi}_k, \bar{\phi}_{k+1}) \right\|^p \\ \leq \; & \max \left\{ L_k \left\| \phi_k - \bar{\phi}_k \right\|, l_{k+1} \left\| \phi_{k+1} - \bar{\phi}_{k+1} \right\| \right\}^p \\ \leq \; & L_k^p \left\| \phi_k - \bar{\phi}_k \right\|^p + l_{k+1}^p \left\| \phi_{k+1} - \bar{\phi}_{k+1} \right\|^p \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \end{aligned}$$

and the elementary inequality $(t + s)^{1/p} \leq t^{1/p} + s^{1/p}$ for reals $s, t \geq 0$ yields

$$\left\| G_f(\phi, \xi) - G_f(\bar{\phi}, \xi) \right\|_p \overset{(3.4)}{=} \sqrt[p]{\sum_{k=\kappa}^{\infty} \left\| f_k(\phi_k, \phi_{k+1}) - f_k(\bar{\phi}_k, \bar{\phi}_{k+1}) \right\|^p}$$

$$\leq \sqrt[p]{\sum_{k=\kappa}^{\infty} L_k^p \left\|\phi_k - \bar{\phi}_k\right\|^p + \sum_{k=\kappa}^{\infty} l_{k+1}^p \left\|\phi_{k+1} - \bar{\phi}_{k+1}\right\|^p}$$
$$\leq (\|l\|_1 + \|L\|_1) \left\|\phi - \bar{\phi}\right\|_p .$$

Hence, we are done. □

Theorem 4.9. *Let $\kappa \in \mathbb{I}$, $p \geq 1$ and Ω be closed. If the right-hand side $f_k : \Omega \times \Omega \to \Omega$ of* (3.1) *is $\ell^p(\mathbb{Z}_\kappa^+, \Omega)$-admissible and satisfies $(L)_f$ with sequences $L, l \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$ so that*

$$\|l\|_{\ell^\infty(\mathbb{Z}_\kappa^+,\mathbb{R})} + \|L\|_{\ell^\infty(\mathbb{Z}_\kappa^+,\mathbb{R})} < 1, \tag{4.6}$$

then the difference equation (3.1) *is well-posed on Ω with $\varphi(\kappa, \xi) \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$ for all $\xi \in \Omega$. In particular,* (3.1) *is globally $\ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$-attractive.*

Proof. Having Lemma 4.8, one proceeds as in Theorem 4.4. □

Corollary 4.10. *If the equation* (3.1) *is explicit, then the respective assumption* (4.4) *or* (4.6) *can be replaced by*

$$\sup_{k \geq \kappa} \prod_{j=k}^{k+n} L_j < 1 \quad \textit{for one } n \in \mathbb{N}.$$

Proof. See the proof of Corollary 4.5. □

Corollary 4.11. *Under the assumptions of Theorem 4.4, 4.7 or 4.9 the general solution $\varphi(\kappa, \cdot)_k : \Omega \to \Omega$ of* (3.1) *is globally Lipschitz for all $k \in \mathbb{Z}_\kappa^+$.*

Proof. Let $\kappa \in \mathbb{I}$, $\xi, \bar{\xi} \in \Omega$ and denote by $\mathcal{Y}$ one of the sets $\ell_\omega^\infty(\mathbb{Z}_\kappa^+, \mathcal{Y})$, $\ell_0(\mathbb{Z}_\kappa^+, \mathcal{Y})$ or $\ell^p(\mathbb{Z}_\kappa^+, \mathcal{Y})$. We choose $\phi \in \mathcal{Y} \cap \ell(\mathbb{Z}_\kappa^+, \Omega)$ and obtain from Lemma 3.1,

$$\left\|G_f(\phi, \xi) - G_f(\phi, \bar{\xi})\right\|_{\mathcal{Y}} \stackrel{(3.4)}{=} \left\|E^+(\xi - \bar{\xi})\right\|_{\mathcal{Y}} \leq \left\|E^+\right\| \left\|\xi - \bar{\xi}\right\| .$$

Consequently $G_f(\phi, \cdot) : \Omega \to \mathcal{Y}$ satisfies a Lipschitz condition. Then, by the uniform contraction principle (cf. (15, p. 17, (6.2))) also the fixed point $\varphi(\kappa, \cdot) : \Omega \to \mathcal{Y}$ is globally Lipschitz. By properties of the evaluation map, this carries over to $\varphi(\kappa, \cdot)_k$ for every $k \in \mathbb{Z}_\kappa^+$. □

Example 4.1 (neural model of Cowen and Stein). We study a discrete counterpart of a model from (16) in form of the difference equation

$$x_{k+1}^i = \alpha_i x_k^i + g(\gamma_i - \sum_{j=1}^{\infty} \beta_{ij} x_{k+1}^j) \quad \text{for all } i \in \mathbb{N} \tag{4.7}$$

with parameters $\alpha_i \geq 0$, $\gamma_i, \beta_{ij} \in \mathbb{R}$ satisfying

$$\ell := \sup_{i \in \mathbb{N}} \Big(\alpha_i + \sum_{j=1}^{\infty} \tfrac{1+e^{\gamma_i}}{1+e^{\gamma_j}} \left|\beta_{ij}\right|\Big) < 1;$$

here, the function $g : \mathbb{R} \to (0, 1)$ is given by $g(x) := \frac{1}{1+e^x}$ and we have

$$|g'(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}. \tag{4.8}$$

We write (4.7) as recursion of the form (3.1) in $\Omega = \ell_\omega^\infty(\mathbb{N}, [0, \infty))$, with

$$f(x, y)_i := \alpha_i x_i + g(\gamma_i - \sum_{i=1}^{\infty} \beta_{ij} y_i) \quad \text{for all } i \in \mathbb{N}$$

and show that $f : \Omega \times \Omega \to \Omega$ is well-defined. Thereto, define the sequence $\bar{\omega}_i := \frac{1}{1+e^{\gamma_i}}$, and for $x, \bar{x}, y, \bar{y} \in \Omega$ we get

$$\begin{aligned} |f(x,y)_i - f(\bar{x},\bar{y})_i| &\leq \alpha_i |x_i - \bar{x}_i| + \left| g(\gamma_i - \sum_{i=1}^{\infty} \beta_{ij} y_i) - g(\gamma_i - \sum_{i=1}^{\infty} \beta_{ij} \bar{y}_i) \right| \\ &\overset{(4.8)}{\leq} \alpha_i |x_i - \bar{x}_i| + \sum_{j=1}^{\infty} |\beta_{ij}| \, |y_j - \bar{y}_j| \quad \text{for all } i \in \mathbb{N}. \end{aligned}$$

Multiplication with $\bar{\omega}_i^{-1}$ and passing over to the supremum over $i \in \mathbb{N}$ gives us

$$\|f(x,y) - f(\bar{x},\bar{y})\|_{\bar{\omega}} \leq \ell \max\left\{ \|x - \bar{x}\|_{\bar{\omega}}, \|y - \bar{y}\|_{\bar{\omega}} \right\};$$

since we evidently have $f(x,y)_i \geq 0$ and $f(0,0) \in \ell^\infty_{\bar{\omega}}(\mathbb{N}, \mathbb{R})$, this implies $f(x,y) \in \Omega$. Hence, due to the contraction condition $\ell < 1$, equation (4.7) possesses a unique equilibrium $\phi^* \in \Omega$, i.e., $f(\phi^*, \phi^*) = \phi^*$. Every solution of (4.7) approaches this equilibrium exponentially, which can be seen as follows: Consider the corresponding equation of perturbed motion (3.3). Its right-hand side satisfies the assumptions of Theorem 4.4 with $\omega = (\gamma^k)_{k \in \mathbb{Z}_0^+}$ for $\gamma \in (\ell, 1)$.

5. Further global Criteria

Let $\mathcal{Y}$ be a Banach space and consider a map $G : D \to \mathcal{Y}$ defined on a nonempty set $D \subseteq \mathcal{Y}$. Then G is called *completely continuous*, if G is continuous and $G(S) \subseteq \mathcal{Y}$ is relatively compact for every bounded $S \subseteq D$. We present a generalization of Schauder's fixed point theorem due to Krasnoselskii.

Proposition 5.1. *Let $C \subseteq \mathcal{Y}$ be bounded, closed and convex. If $G_0 : C \to \mathcal{Y}$ is contractive and $G_1 : C \to \mathcal{Y}$ is completely continuous, then the sum $G_0 + G_1 : C \to C$ has a fixed point.*

Proof. See for instance (15, p. 70, (9.19)), or (17, p. 163, Theorem 5.2.9) in connection with (18, p. 496, Example 11.7). □

Lemma 5.2 (well-definedness on ℓ^p)**.** *Let $\kappa \in \mathbb{I}$ and $p, q \geq 1$. If a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha \geq \frac{p}{q}$ and $a \in \ell^q(\mathbb{Z}_\kappa^+, \mathbb{R})$, $b, c \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f : \ell^q(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$ is well-defined and satisfies*

(5.1)

$$\|G_f(\phi,\xi)\|_{\ell^q(\mathbb{Z}_\kappa^+,\mathbb{R})} \leq \|\xi\| + \|a\|_{\ell^q(\mathbb{Z}_\kappa^+,\mathbb{R})} + \left(\|b\|_{\ell^\infty(\mathbb{Z}_\kappa^+,\mathbb{R})} + \|c\|_{\ell^\infty(\mathbb{Z}_\kappa^+,\mathbb{R})} \right) \|\phi\|^\alpha_{\ell^{\alpha q}(\mathbb{Z}_\kappa^+,\mathbb{R})}$$

for all $\xi \in \Omega$ and $\phi \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$.

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$ be given. Then Hypothesis $(B)_f$, Lemma 3.1 and Minkowski's inequality imply

$$\begin{aligned} \|G_f(\phi,\xi)\|_q &\overset{(3.4)}{\leq} \|E^+\xi\|_q + \|F_f(\phi, S_\kappa^+\phi)\|_q \overset{(3.6)}{=} \|\xi\| + \sqrt[q]{\sum_{k=\kappa}^{\infty} \|f_k(\phi_k, \phi_{k+1})\|^q} \\ &\leq \|\xi\| + \sqrt[q]{\sum_{k=\kappa}^{\infty} \left(a_k + \max\left\{ b_k \|\phi_k\|^\alpha; c_{k+1} \|\phi_{k+1}\|^\alpha \right\}\right)^q} \\ &\leq \|\xi\| + \|a\|_q + (\|b\|_1 + \|c\|_1) \sqrt[q]{\sum_{k=\kappa}^{\infty} \|\phi_k\|^{\alpha q}} \end{aligned}$$

$$= \quad \|\xi\| + \|a\|_q + (\|b\|_1 + \|c\|_1)\|\phi\|_{\alpha q}^{\alpha},$$

where (2.1) guarantees $\phi \in \ell^{\alpha q}(\mathbb{Z}_\kappa^+, \mathcal{X})$. Hence, Lemma 5.2 is verified. □

Lemma 5.3 (complete continuity on ℓ^p)**.** *Let $\kappa \in \mathbb{I}$, $p, q \geq 1$ and $\dim \mathcal{X} < \infty$. If a continuous function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha \geq \frac{p}{q}$ and $a \in \ell^q(\mathbb{Z}_\kappa^+, \mathbb{R})$, $b, c \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f(\cdot, \xi) : \ell^p(\mathbb{Z}_\kappa^+, \Omega) \to \ell^q(\mathbb{Z}_\kappa^+, \mathcal{X})$ is completely continuous for all $\xi \in \Omega$.*

Proof. Using Lemma 2.2, the proof follows from (19). □

Theorem 5.4. *Let $p \geq 1$, $\dim \mathcal{X} < \infty$ and $\Omega \subseteq \mathcal{X}$ be closed and convex. Assume the right-hand side $f_k : \Omega \times \Omega \to \Omega$ of* (3.1) *allows the decomposition*

$$f_k(x, y) = g_k(x, y) + h_k(x, y)$$

into functions $g_k, h_k : \Omega \times \Omega \to \mathcal{X}$ with the following properties:

(i) *$G_g(0, 0) \in \ell^p(\mathbb{I}, \mathcal{X})$ and g_k satisfies $(L)_g$ with $L, l \in \ell^\infty(\mathbb{I}, \mathbb{R})$,*
(ii) *h_k is continuous, satisfies $(B)_h$ with $\alpha = 1$, $a \in \ell^p(\mathbb{I}, \mathbb{R})$, $b, c \in \ell_0(\mathbb{I}, \mathbb{R})$ and*

$$\|b\|_{\ell_0(\mathbb{I},\mathbb{R})} + \|c\|_{\ell_0(\mathbb{I},\mathbb{R})} + \|l\|_{\ell^\infty(\mathbb{I},\mathbb{R})} + \|L\|_{\ell^\infty(\mathbb{I},\mathbb{R})} < 1.$$

Then the difference equation (3.1) *is globally $\ell^p(\mathbb{I}, \mathcal{X})$-attractive.*

Proof. Let $\xi \in \Omega$. Thanks to our assumptions, we can decompose the operator $G_f : \ell^p(\mathbb{I}, \Omega) \times \Omega \to \ell(\mathbb{I}, \mathcal{X})$ as follows

$$G_f(\phi, \xi) = G_g(\phi, 0) + G_h(\phi, \xi) \quad \text{for all } \phi \in \ell^p(\mathbb{I}, \Omega)$$

and show that Proposition 5.1 is applicable. Above all, we know from Lemma 4.8 that $G_g(\cdot, 0)$ is a contraction. On the other hand, Lemma 5.3 implies that $G_h(\cdot, \xi)$ is completely continuous. Now we define

$$\sigma := \|b\| + \|c\| + \|L\| + \|l\|, \qquad \rho := \frac{\|G_g(0,0)\|_p + \|a\|_p}{1 - \sigma}$$

and choose real constants $R > \rho$, $r \in (0, (1 - \sigma)(R - \rho)]$ so large that $\xi \in \bar{B}_r(0)$. Then $C := \bar{B}_R(0) \cap \ell(\mathbb{I}, \Omega)$ is a bounded, closed and convex subset of $\ell^p(\mathbb{I}, \mathcal{X})$. It remains to verify that the operator $G_f(\cdot, \xi) : C \to C$ is well-defined for $\xi \in \bar{B}_r(0) \cap \Omega$. This follows from the estimate (see Lemma 4.8 and 5.2)

$$\begin{aligned} \|G_f(\phi, \xi)\| &\leq \|G_g(\phi, 0) - G_g(0, 0)\| + \|G_g(0, 0)\| + \|G_h(\phi, \xi)\| \\ &\overset{(4.5)}{\leq} (\|l\| + \|L\|) \|\phi\| + \|G_g(0, 0)\| + \|G_h(\phi, \xi)\| \\ &\overset{(5.1)}{\leq} \|\xi\| + \|G_g(0, 0)\| + \|a\| + \sigma \|\phi\| \leq R \end{aligned}$$

for all $\phi \in C$, $\xi \in \bar{B}_r(0) \cap \Omega$. Then Proposition 5.1 implies the assertion. □

Lemma 5.5 (well-definedness on ℓ_0)**.** *Let $\kappa \in \mathbb{I}$. If a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha > 0$, $a \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$ and $b, c \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f : \ell_0(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ is well-defined and satisfies*
(5.2)

$$\|G_f(\phi, \xi)\|_{\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})} \leq \|\xi\| + \|a\|_{\ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})} + \max\left\{\|b\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})}, \|c\|_{\ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})}\right\} \|\phi\|_{\ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})}^{\alpha}$$

for all $\xi \in \Omega$ and $\phi \in \ell_0(\mathbb{Z}_\kappa^+, \Omega)$.

Proof. The proof is straight-forward and essentially the same as of Lemma 5.2; thus we omit it. □

Lemma 5.6 (complete continuity on ℓ_0)**.** *Let $\kappa \in \mathbb{I}$ and $\dim \mathcal{X} < \infty$. If a continuous function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha > 0$ and $a, b, c \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f(\cdot, \xi) : \ell_0(\mathbb{Z}_\kappa^+, \Omega) \to \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ is completely continuous for all $\xi \in \Omega$.*

Proof. From the above Lemma 5.5 we know that $G_f : \ell_0(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ is well-defined. Let $\xi \in \Omega$. In case of ℓ^p-spaces the continuity of such substitution operators is shown in (19, Theorem 1.1). The interested reader may check that the corresponding arguments also hold in our present ℓ_0-setting, yielding that $G_f(\cdot, \xi)$ is continuous. It remains to verify that $G_f(\cdot, \xi)$ maps bounded subsets $S \subseteq \ell_0(\mathbb{Z}_\kappa^+, \Omega)$ into relatively subsets of $\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$. Since S is bounded, there exists an $R \geq 0$ such that $\|\phi_k\| \leq \|\phi\|_{\ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})} \leq R$ for all $\phi \in S$. W.l.o.g. we assume in Hypothesis $(B)_f$ that the sequences $a, b, c \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$ are non-increasing. Consequently, for each $\phi \in S$ we have

$$\begin{aligned}\|G_f(\phi, \xi)_k\| &\overset{(3.4)}{=} \|f_k(\phi_k, \phi_{k+1})\| \leq a_k + \max\left\{b_k \|\phi_k\|^\alpha, c_{k+1} \|\phi_{k+1}\|^\alpha\right\} \\ &\leq a_n + \max\{b_n, c_n\} R^\alpha \quad \text{for all } k > n \geq \kappa\end{aligned}$$

and consequently

$$\sup_{\phi \in S} \sup_{k > n} \|G_f(\phi, \xi)_k\| \xrightarrow[n \to \infty]{} 0.$$

Thus, Lemma 2.1 implies that $G_f(S, \xi) \subseteq \ell_0(\mathbb{Z}_\kappa^+, \mathcal{X})$ is relatively compact and we have established Lemma 5.6. □

Theorem 5.7. *Let $\dim \mathcal{X} < \infty$ and $\Omega \subseteq \mathcal{X}$ be closed and convex. Assume the right-hand side $f_k : \Omega \times \Omega \to \Omega$ of* (3.1) *allows the decomposition*

$$f_k(x, y) = g_k(x, y) + h_k(x, y)$$

into functions $g_k, h_k : \Omega \times \Omega \to \mathcal{X}$ with the following properties:

(i) *$G_g(0, 0) \in \ell_0(\mathbb{I}, \mathcal{X})$ and g_k satisfies $(L)_g$ with $L, l \in \ell_0(\mathbb{I}, \mathbb{R})$,*

(ii) *h_k is continuous, satisfies $(B)_h$ with $\alpha > 0$, $a, b, c \in \ell_0(\mathbb{I}, \mathbb{R})$ and*

$$\|b\|_{\ell_0(\mathbb{I}, \mathbb{R})} + \|c\|_{\ell_0(\mathbb{I}, \mathbb{R})} + \max\left\{\|l\|_{\ell_0(\mathbb{I}, \mathbb{R})}, \|L\|_{\ell_0(\mathbb{I}, \mathbb{R})}\right\} < 1.$$

Then the difference equation (3.1) *is globally $\ell_0(\mathbb{I}, \mathcal{X})$-attractive.*

Proof. Let $\xi \in \Omega$. We decompose the operator $G_f : \ell^p(\mathbb{I}, \Omega) \times \Omega \to \ell(\mathbb{I}, \mathcal{X})$

$$G_f(\phi, \xi) = G_g(\phi, 0) + G_h(\phi, \xi) \quad \text{for all } \phi \in \ell^p(\mathbb{I}, \Omega)$$

and apply Proposition 5.1. From Lemma 4.8 we get that $G_g(\cdot, 0)$ is a contraction and Lemma 5.3 implies the complete continuity of $G_h(\cdot, \xi)$. The remaining arguments are similar to the proof of Theorem 5.4. □

As application of Theorem 5.4 we investigate a linearly implicit partial difference equation.

Example 5.1 (discrete reaction-diffusion equation). Let $\mathbb{I}$ be a discrete interval, $\kappa \in \mathbb{I}$ and n^-, n^+ be integers satisfying $N := n^+ - n^- - 2 > 0$. We define $\mathbb{J} := \{n^- + 1, \ldots, n^+ - 1\}$, the finite dimensional space $\mathcal{X} := \ell(\mathbb{J}, \mathbb{R}) \cong \mathbb{R}^N$, choose $\upsilon \in \mathcal{X}$ and consider the discrete reaction-diffusion equation

$$u_{k+1,n} = \alpha u_{k+1,n-1} + \beta u_{k+1,n} + \gamma u_{k+1,n+1} + F(k, n, u_{k,n^-+1}, \ldots, u_{k,n^+-1}),$$

(5.3) $\quad$ for $n \in \mathbb{J}$, $k \in \mathbb{Z}_\kappa^+$

equipped with the initial boundary conditions $u_{\kappa,n} = \upsilon_n$ for all $n \in \mathbb{J}$,

$$u_{k,n^-} = 0, \qquad u_{k,n^+} = 0 \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

under the following assumptions:

(i) There exists a closed convex neighborhood $\Omega \subseteq \mathbb{R}^N$ of 0 such that
$$\alpha u_{n-1} + \beta u_n + \gamma u_{n+1} + F(k,n,u) \in \Omega \quad \text{for all } u = (u_{n^-+1}, \ldots, u_{n^+-1}) \in \Omega$$
and $k \in \mathbb{Z}_\kappa^+$, $n \in \mathbb{J}$,

(ii) the scalars $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $\alpha\gamma > 0$ and
$$\left|\beta + 2\sqrt{\alpha\gamma}\,\operatorname{sgn}\alpha \cos\left(\tfrac{j\pi}{N+1}\right)\right| < 1 \quad \text{for all } j \in \{1, \ldots, N\},$$

(iii) the nonlinearity $F : \mathbb{I} \times \mathbb{J} \times \mathcal{X} \to \mathbb{R}$ is continuous and linearly bounded
$$|F(k,n,u)| \leq a_k + b_k \sum_{j \in \mathbb{J}} |u_j| \quad \text{for all } n \in \mathbb{N},\ u = (u_{n^-+1}, \ldots, u_{n^+-1}) \in \Omega \tag{5.4}$$
with positive sequences $a \in \ell^p(\mathbb{I}, \mathbb{R})$, $b \in \ell_0(\mathbb{I}, \mathbb{R})$.

The partial difference equation (5.3) can be written as an ordinary difference equation in the space $\mathcal{X}$, namely
$$x_{k+1} = Ax_{k+1} + F_k(x_k), \tag{5.5}$$
with the linear operator $A \in \mathcal{L}(\mathcal{X})$, $A := \operatorname{tridiag}(\alpha, \beta, \gamma)$ and the substitution operator $F_k : \mathcal{X} \to \mathcal{X}$,
$$(F_k(x))_n := f(k,n,x) \quad \text{for all } n \in \mathbb{J}.$$
The assumption (ii) yields $|\lambda| < 1$ for all $\lambda \in \sigma(A)$ and from (20, p. 6, Technical lemma 1) we know that there exists a norm on $\mathcal{X}$ such that $\|A\| < 1$. Thus, (5.5) can be rewritten as $x_{k+1} = [I - A]^{-1} F_k(x_k)$ and therefore the solutions of (5.3) and (5.5) are uniquely determined and depend continuously on their initial conditions. Finally, Theorem 5.4 is applicable, since there exists a $\bar\kappa \in \mathbb{Z}_\kappa^+$ such that $\sup_{k \geq \bar\kappa} b_k < 1 - \|A\|$. Hence, (5.5) is globally $\ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$-attractive for $\kappa \geq \bar\kappa$.

We call $G : D \to \mathcal{Y}$ *strongly continuous*, if for every sequence $(x_n)_{n\in\mathbb{N}}$ in D with weak limit $x \in D$ one has the limit relation $\lim_{n\to\infty} \|G(x_n) - G(x)\|_{\mathcal{Y}} = 0$. Then a result of Reinermann reads as follows:

Proposition 5.8. *Let $\mathcal{Y}$ be a uniformly convex Banach space and assume $C \subseteq \mathcal{Y}$ is bounded, closed, convex. If $G_0 : C \to \mathcal{Y}$ is non-expansive and $G_1 : C \to \mathcal{Y}$ is strongly continuous, then the sum $G_0 + G_1 : C \to C$ has a fixed point.*

Proof. See (18, p. 501, Theorem 11.B). □

Lemma 5.9 (strong continuity on ℓ^p)**.** *Let $\kappa \in \mathbb{I}$, $p > 1$ and $\dim \mathcal{X} < \infty$. If a function $f_k : \Omega \times \Omega \to \mathcal{X}$ satisfies $(B)_f$ with $\alpha \geq 1$, $a \in \ell^p(\mathbb{Z}_\kappa^+, \mathbb{R})$, $b, c \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, and $(L)_f$ with $L, l \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the operator $G_f(\cdot, \xi) : \ell^p(\mathbb{Z}_\kappa^+, \Omega) \to \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$ is strongly continuous for all $\xi \in \Omega$.*

Proof. Let $\kappa \in \mathbb{I}$. We subdivide the proof into two steps:

(I) Consider the linear substitution operators $T_1, T_2 : \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X}) \to \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$,
$$(T_1\phi)_k := L_k \phi_k, \qquad (T_2\phi)_k := l_{k+1}\phi_{k+1}$$
which, due to $l, L \in \ell_0(\mathbb{Z}_\kappa^+, \mathbb{R})$ are well-defined by Lemma 5.2 and completely continuous by Lemma 5.3. Thus, (21, Prop. VI-3.3) implies that T_1, T_2 are strongly continuous, i.e., for every sequence $(\phi^n)_{n\in\mathbb{N}}$ in $\ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$ with $\phi^n \rightharpoonup \phi$ for $n \to \infty$ we have
$$\|T_1\phi^n - T_1\phi\|_p^p = \sum_{k=\kappa}^{\infty} L_k^p \|\phi_k^n - \phi_k\|^p \xrightarrow[n\to\infty]{} 0,$$

$$\|T_2\phi^n - T_2\phi\|_p^p = \sum_{k=\kappa}^{\infty} l_{k+1}^p \left\|\phi_{k+1}^n - \phi_{k+1}\right\|^p \xrightarrow[n\to\infty]{} 0.$$

(II) Let $\xi \in \Omega$, $\phi \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$ and $(\phi^n)_{n\in\mathbb{N}}$ be a sequence in $\ell^p(\mathbb{Z}_\kappa^+, \Omega)$ with $\phi^n \rightharpoonup \phi$ for $n \to \infty$. Then, as in the proof of Lemma 4.8, one has

$$\|G_f(\phi^n, \xi) - G_f(\phi, \xi)\|_p \le \sqrt[p]{\sum_{k=\kappa}^{\infty} L_k^p \|\phi_k^n - \phi_k\|^p} + \sqrt[p]{\sum_{k=\kappa}^{\infty} l_{k+1}^p \left\|\phi_{k+1}^n - \phi_{k+1}\right\|^p} \xrightarrow[n\to\infty]{} 0$$

by step (I). This implies our claim. □

Theorem 5.10. *Let $p > 1$, $\dim \mathcal{X} < \infty$ and $\Omega \subseteq \mathcal{X}$ be closed and convex. Assume the right-hand side $f_k : \Omega \times \Omega \to \Omega$ of* (3.1) *allows the decomposition*

$$f_k(x, y) = g_k(x, y) + h_k(x, y)$$

into functions $g_k, h_k : \Omega \times \Omega \to \mathcal{X}$ with the following properties:

(i) *f_k satisfies $(B)_f$ with $\alpha = 1$, $a \in \ell^p(\mathbb{I}, \mathbb{R})$ and $b, c \in \ell^\infty(\mathbb{I}, \mathbb{R})$*

$$\|b\|_{\ell^\infty(\mathbb{I},\mathbb{R})} + \|c\|_{\ell^\infty(\mathbb{I},\mathbb{R})} < 1,$$

(ii) *g_k is $\ell^p(\mathbb{I}, \mathcal{X})$-admissible and satisfies $(L)_g$ with $L, l \in \ell^\infty(\mathbb{I}, \mathbb{R})$ and*

$$\|l\|_{\ell^\infty(\mathbb{I},\mathbb{R})} + \|L\|_{\ell^\infty(\mathbb{I},\mathbb{R})} \le 1,$$

(iii) *h_k is $\ell^p(\mathbb{I}, \mathcal{X})$-admissible and satisfies $(L)_h$ with $L, l \in \ell_0(\mathbb{I}, \mathbb{R})$ (here L, l may be different from (ii)).*

Then the difference equation (3.1) *is globally $\ell^p(\mathbb{I}, \mathcal{X})$-attractive.*

Proof. Due to $p > 1$ the Banach space $\ell^p(\mathbb{I}, \mathcal{X})$ is uniformly convex. We choose $\xi \in \Omega$, decompose $G_f : \ell^p(\mathbb{I}, \Omega) \times \Omega \to \ell(\mathbb{I}, \mathcal{X})$

$$G_f(\phi, \xi) = G_g(\phi, \xi) + G_h(\phi, 0) \quad \text{for all } \phi \in \ell^p(\mathbb{I}, \Omega).$$

and verify the assumptions of Proposition 5.8. Since the mappings g_k, h_k are $\ell^p(\mathbb{I}, \mathcal{X})$-admissible, using Lemma 4.8 it is not difficult to see that $G_g(\cdot, \xi), G_h(\cdot, 0) : \ell^p(\mathbb{I}, \Omega) \to \ell^p(\mathbb{I}, \Omega)$ are well-defined. Thanks again to Lemma 4.8, the mapping $G_g(\cdot, \xi)$ is non-expansive, and Lemma 5.9 guarantees that $G_h(\cdot, 0)$ is strongly continuous. The remaining argument is similarly to the proof of Theorem 5.4. □

Our next example addresses the roughness of the stability result derived in Example 3.1.

Example 5.2. Consider the scalar implicit difference equation

$$x_{k+1} = \tfrac{k}{k+1} x_k + h_k(x_{k+1}) \tag{5.6}$$

with $\mathbb{I} = \mathbb{N}$. To mimic the notation of Theorem 5.10 we define $g_k(x) := \frac{k}{k+1}x$, suppose there exists a $R > 0$ such that

$$|g_k(x) + h_k(y)| \le R \quad \text{for all } x, y \in [-R, R]$$

and set $\Omega := [-R, R]$. Under the additional assumptions

$$\sum_{k\in\mathbb{N}} |h_k(0)|^p < \infty, \qquad \lim_{k\to\infty} \operatorname{Lip} h_k|_\Omega = 0$$

it is easy to see that Theorem 5.10 is applicable to (5.6) and we obtain that all solutions starting in $[-R, R]$ are ℓ^p-summable for $p > 1$. The same result also holds for the explicit version of (5.6) given by $x_{k+1} = \frac{k}{k+1} x_k + h_k(x_k)$.

Now suppose there exists a nonnegative sequence $(\ell_n)_{n\in\mathbb{N}}$ so that

$$\|G^n(x) - G^n(\bar{x})\|_{\mathcal{Y}} \le \ell_n \|x - \bar{x}\|_{\mathcal{Y}} \quad \text{for all } n \in \mathbb{N},\ x, \bar{x} \in D;$$

in case $\lambda := \sup_{n\in\mathbb{N}} \ell_n < \infty$ we denote G as *uniformly λ-Lipschitz*.

Proposition 5.11. *Let $C \subseteq \mathcal{Y}$ be bounded, closed, convex. A uniformly λ-Lipschitz map $G : C \to C$ possesses a fixed point, if one of the conditions holds:*

(i) $\lambda < \sqrt{N(\mathcal{Y})}$,

(ii) *$\mathcal{Y}$ is uniformly convex and λ is less than the unique solution $h \in \left[\frac{1}{2}, \infty\right)$ of $h\left(1 - \delta_{\mathcal{Y}}\left(h^{-1}\right)\right) = 1$,*

(iii) *$\mathcal{Y}$ is uniformly convex and* $\lim_{n\to\infty} \ell_n = 1$.

Proof. (i) See (11, p. 151, Theorem 3.2).
(ii) See (11, p. 142, Theorem 1.2).
(iii) See (22, Theorem 3). □

In the remaining part of the paper we deal with the explicit version of (3.1), namely the equation

$$x_{k+1} = f_k(x_k) \tag{5.7}$$

with a function $f_k : \Omega \to \Omega$.

Lemma 5.12 (Lipschitz condition on ℓ^p)**.** *Let $\kappa \in \mathbb{I}$, $n \in \mathbb{N}$ and $p \ge 1$. If a function $f_k : \Omega \to \mathcal{X}$ satisfies $(L)_f$ with $L, l \in \ell^\infty(\mathbb{Z}_\kappa^+, \mathbb{R})$, then the iterates $G_f^n : \ell^p(\mathbb{Z}_\kappa^+, \Omega) \times \Omega \to \ell(\mathbb{Z}_\kappa^+, \mathcal{X})$ fulfill $G_f^n(\phi, \xi) - G_f^n(\bar{\phi}, \xi) \in \ell^p(\mathbb{Z}_\kappa^+, \mathcal{X})$ for all $\phi, \bar{\phi} \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$ and*

$$\operatorname{Lip}_1 G_f^n \le \sup_{k\ge\kappa} \prod_{j=k}^{k+n} L_j. \tag{5.8}$$

Proof. Let $\kappa \in \mathbb{I}$, $\xi \in \Omega$ and $\phi, \bar{\phi} \in \ell^p(\mathbb{Z}_\kappa^+, \Omega)$. We only establish (5.8), since the other assertions are immediate from Lemma 4.8. By Proposition 3.2 and Hypothesis $(L)_f$ one has

$$\begin{aligned}
&\left\|G_f^n(\phi, \xi) - G_f^n(\bar{\phi}, \xi)\right\|_p \\
&\overset{(3.4)}{=} \sqrt[p]{\sum_{k=n+\kappa}^{\infty} \left\|f_{k-1} \circ \ldots \circ f_{k-n}(\phi_{k-n}) - f_{k-1} \circ \ldots \circ f_{k-n}(\bar{\phi}_{k-n})\right\|^p} \\
&\le \sqrt[p]{\sum_{k=n+\kappa}^{\infty} L_{k-1}^p \cdot \ldots \cdot L_{k-n}^p \left\|\phi_{k-n} - \bar{\phi}_{k-n}\right\|^p} \\
&\le \sup_{k\ge\kappa} \prod_{j=k}^{k+n} L_j \sqrt[p]{\sum_{k=\kappa}^{\infty} \|\phi_k - \bar{\phi}_k\|^p} = \sup_{k\ge\kappa} \prod_{j=k}^{k+n} L_j \|\phi - \bar{\phi}\|_p
\end{aligned}$$

and Lemma 5.12 is established. □

Theorem 5.13. *Let $p \ge 1$, $\dim \mathcal{X} < \infty$ and $\Omega \subseteq \mathcal{X}$ be closed and convex. Assume the right-hand side $f_k : \Omega \to \Omega$ of* (5.7) *satisfies $(B)_f$ with with $\alpha = 1$, $a \in \ell^p(\mathbb{I}, \mathbb{R})$, $b \in \ell^\infty(\mathbb{I}, \mathbb{R})$,*

$$\|b\|_{\ell^\infty(\mathbb{I},\mathbb{R})} < 1,$$

and $(L)_f$ with $L \in \ell^\infty(\mathbb{I}, \mathbb{R})$, such that one of the following conditions

(i) $\sup_{n\in\mathbb{N}} \ell_n < \sqrt{\min\left\{2^{1-\frac{1}{p}}, 2^{\frac{1}{p}}\right\}}$,

(ii) $\sup_{n\in\mathbb{N}} \ell_n < \sqrt[p]{\frac{3}{2}}$ *and* $p > 1$,

(iii) $\lim_{n\to\infty} \ell_n = 1$ *and* $p > 1$

holds, where $\ell_n := \sup_{k\geq\kappa} \prod_{j=k}^{k+n} L_j$. *Then the explicit difference equation* (5.7) *is globally* $\ell^p(\mathbb{I}, \mathcal{X})$*-attractive.*

Proof. Let $\xi \in \Omega$ be given, define $\rho := \frac{\|a\|}{1-\|b\|}$ and choose real constants $R > \rho$, $r \in (0, (1 - \|b\|)(R - \rho))$ so large that $\xi \in \bar{B}_r(0)$. Then the set $C := \bar{B}_R(0) \cap \ell(\mathbb{I}, \Omega)$ is bounded, closed and convex in $\ell^p(\mathbb{I}, \mathcal{X})$. We have to show that $G_f(\cdot, \xi) : C \to C$ is well-defined. This follows from Lemma 5.2, since we have

$$\|G_f(\phi, \xi)\|_p \overset{(5.1)}{\leq} \|\xi\| + \|a\|_p + \|b\| \, \|\phi\|_p \leq R \quad \text{for all } \phi \in C$$

and $\xi \in \bar{B}_r(0) \cap \Omega$. In addition, by Lemma 5.12 the mapping $G_f(\cdot, \xi)$ is uniformly λ-Lipschitz with $\lambda := \sup_{n\in\mathbb{N}} \ell_n$. Then Proposition 5.11 yields the assertion. □

REFERENCES

[1] E. Petropoulou, P. Siafarikas, Existence of complex ℓ_2 solutions of linear delay systems of difference equations, J. Difference Equations and Applications 11(1) (2005) 49–62.

[2] E. Petropoulou, P. Siafarikas, Bounded solutions and asymptotic stability of nonlinear difference equations in the complex plane, Archivum Math. (Brno) 36(2) (2000) 139–158.

[3] E. Petropoulou, P. Siafarikas, Bounded solutions and asymptotic stability of nonlinear difference equations in the complex plane. II, Computers & Math. with Appl. 42(3–5) (2001) 427–452.

[4] E. Petropoulou, P. Siafarikas, A functional-analytic method fo the study of difference equations, Advances in Diff. Equations 3 (2004) 237–248.

[5] E. Petropoulou, P. Siafarikas, Solutions of nonlinear delay and advanced partial difference equations in the space $\ell^1_{\mathbb{N}\times\mathbb{N}}$, Computers & Math. with Appl. 45(6–9) (2003) 905–934.

[6] C. Earle, R. Hamilton, A fixed point theorem for holomorphic mappings, in: Global Analysis, 16, Proceedings of Symposia in Pure Mathematics, AMS, Providence, RI, 1970, pp. 61–65.

[7] S. Reich, D. Shoikhet, Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach spaces, Imperial College Press, London, 2005.

[8] S. Gordon, Stability and summability of solutions of difference equations, Math. Syst. Theory 5 (1971) 56–75.

[9] T.A. Burton, T. Furomochi, A note on stability by Schauder's theorem, Funkcialaj Ekvacioj 44 (2001) 73–82.

[10] T.A. Burton, T. Furomochi, Krasnoselskii's fixed point theorem and stability, Nonlinear Analysis (TMA) 49 (2002) 445–454.

[11] J. Ayerbe, T. Domínguez Benavides, G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Vol. 99 of Operator Theory: Advances and Applications, Birkhäuser, Basel-Boston-Berlin, 1997.

[12] V. Istrăţescu, Strict convexity and complex strict convexity, Vol. 89 of Lecture Notes in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1984.

[13] W.G. Kelley, A.C. Peterson, Difference Equations, 2nd Ed., Academic Press, San Diego, 2001.
[14] R. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.
[15] J. Dugundji, A. Granas, Fixed Point Theory, PNW - Polish Scientific Publishers, Warszawa, 1982.
[16] M. Oguytöreli, On the neural equations of Cowan and Stein, Utilitas Math. 2 (1972) 305–317.
[17] V. Istrăţescu, Fixed Point Theory. An Introduction, D. Reidel Publishing Company, Dortrecht, Boston, London, 1981.
[18] E. Zeidler, Nonlinear Functional Analysis and its Applications I (Fixed-Points Theorems), Springer, Berlin-Heidelberg-New York, 1993.
[19] F. Dal, G. Guseinov, Properties of discrete composition operators, J. Difference Equations and Applications 11(1) (2005) 21–27.
[20] G. Iooss, Bifurcation of Maps and Applications, North-Holland, Amsterdam, 1979.
[21] J. Conway, A course in functional analysis, 2nd Edition, Vol. 96 of Graduate Texts in Mathematics, Springer, New York, 1990.
[22] K. Goebel, W. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proceedings of the AMS 35 (1972) 171–174.

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