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INVESTMENT UNDER UNCERTAINTY

BY ROBERT E. LUCAS, JR. AND EDWARD C. PRESCOTT

This paper determines the time series behavior of investment, output, and prices in a competitive industry with a stochastic demand. It is shown, first, that the equilibrium development for the industry solves a particular dynamic programming problem (maximization of "consumer surplus"). This problem is then studied to determine the characteristics of the equilibrium paths.

I. INTRODUCTION

EXPLANATORY VARIABLES in empirical studies of the demand for investment goods fall into three broad classes: variables measuring anticipated, future demand—sales, profits, stock price indexes; variables measuring past decisions, the effects of which persist into the present—lagged capital stock and investment rates; and variables measuring current market opportunities—interest rates, factor prices, and, again, profits.¹ Existing investment theory has concerned itself largely with the latter two classes of variables, first by rationalizing the role of prices in determining a long-run "desired" capital stock using a static, profit maximizing hypothesis, later by discovering the optimizing basis for a staggered approach to the desired stock in "costs of adjustment" which penalize rapid change.² In the present paper, an uncertain future is introduced into an adjustment-cost type model of the firm, and the optimal response to this uncertainty is studied in an attempt to integrate variables of the first category into the existing theory.³

Briefly, we shall be concerned with a competitive industry in which product demand shifts randomly each period, and where factor costs remain stable. In this context, we attempt to determine the competitive equilibrium time paths of capital stock, investment rates, output, and output price for the industry as a whole and for the component firms. From the viewpoint of firms in this industry, forecasting future demand means simply forecasting future output prices. The usual way to formulate this problem is to postulate some forecasting rule for firms, which in turn generates some pattern of investment behavior, which in turn, in conjunction with industry demand, generates an actual price series.

¹ For reviews of the empirical investment literature, see Eisner and Strotz [7], or, more recently, Schramm [18].

² We refer to the body of theory that stems from the work of Haavelmo [10], Eisner and Strotz [7], and Jorgenson [12]. More recent contributions are by Gould [8], Lucas [13, 14], and Treadway [19].

³ A recent paper by Pashigian [16], discusses one-period equilibrium in a competitive industry under supply and demand uncertainty, a problem closely related to that studied here.

Typically the forecasting rule postulated takes the form of anticipated prices being a fixed function of past prices—"adaptive expectations." But it is clear that if the underlying disturbance (in our case, the demand shift) has a regular stochastic character (such as a Markov process), forecasting in this manner is adaptive only in a restrictive and not very interesting sense. Except for an unlikely coincidence, price forecasts and actual prices will have different probability distributions, and this difference will be persistent, costly to forecasters, and readily correctible.

To avoid this difficulty, we shall, in this paper, go to the opposite extreme, assuming that the actual and anticipated prices have the *same* probability distribution, or that price expectations are *rational*.⁴ Thus we surrender, in advance, any hope of shedding light on the *process* by which firms translate current information into price forecasts. In return, we obtain an operational investment theory linking current investment to observable current and past explanatory variables, rather than to "expected" future variables which must, in practice, be replaced by various "proxy variables."

In the next section, our basic model of the industry is stated. In Section 3, we examine formally the investment decision from the viewpoint of the individual firm, clarifying the role of observable securities prices as evaluators of additions to capital stock. In Sections 4 and 5 competitive *industry* equilibrium is defined, and it is shown that equilibrium in this sense exists and is unique. In Sections 6 and 7, we investigate the long-run behavior of industry equilibrium under alternative assumptions on the nature of the demand shift process. Conclusions are summarized in Section 8.

2. STATEMENT OF THE MODEL

Consider an industry consisting of many small firms, each producing a single output, q_t , by means of a single input, capital, k_t . We assume that production takes place under constant returns to scale so that with appropriate choice of units we may use k_t also to denote production at full capacity. Thus the production function is:

$$(1) \quad 0 \leq q_t \leq k_t.$$

Denote gross investment by x_t . Investment and capacity are related in the non-linear way:

$$(2) \quad k_{t+1} = k_t h(x_t/k_t),$$

where h is bounded, continuously differentiable, increasing, and strictly concave.⁵ Assume that $\delta = h^{-1}(1)$ exists and satisfies $0 < \delta < 1$, so that $k_t \delta$ is the investment

⁴ This term is taken from Muth [15], who applied it to the case where the expected and actual price (both random variables) have a common *mean value*. Since Muth's discussion of this concept applies equally well to our assumption of a common *distribution* for these random variables, it seems natural to adopt the term here.

⁵ These assumptions on h imply that $kh(x/k)$, regarded as a function of the two variables x and k , is increasing in both arguments and concave. To see the latter property, let (k^0, x^0) be a convex combination of (k^1, x^1) and (k^2, x^2) and let $\phi = \theta k^0/k^1$, $0 \leq \phi \leq 1$. Then $x^0/k^0 = \phi(x^0/k^0) + (1 - \phi)(x^1/k^1)$ and

rate which will just maintain the stock k_t . Assume also that $h(0) > 0$, so that even with no investment in period t , some capital will remain in $t + 1$ if $k_t > 0$.

Given an initial stock k_0 , (2) cannot be solved for k_t as a linear function of k_0, x_0, \dots, x_{t-1} , as is possible with conventional depreciation hypotheses, so (2) requires some explanation. One entirely plausible possibility is that the true relationship between physical investment and plant capacity is, in fact, non-linear, so that a given quantity of, say, machines, makes a better plant (yields more productive services) the longer the period over which they are assembled. Alternatively, one may assume that the relation between capacity and physical investment is linear, and regard x_t as the dollar value of investment. Then if investment costs per unit of capacity are a strictly convex function of physical investment per unit of capacity, (2) is implied.⁶

Denote the product price by p_t . Then ex post, the present value of the firm, using the discount factor $\beta = 1/(1 + r)$, where $r > 0$ is the cost of capital, is

$$(3) \quad V = \sum_{t=0}^{\infty} \beta^t [p_t q_t - x_t].$$

It is evident that the allocation across firms of a given industry stock of capital is immaterial, so that in the following the notation k_t, x_t , and q_t will be used interchangeably for industry and firm variables. (Alternatively, think of a competitive industry with one firm.)

To complete the formulation of the problem of the individual firm, one must assume a particular stochastic structure for the price sequence $\{p_t\}$ and postulate an objective function for the firm. This will be done in Section 4, in the course of defining industry equilibrium. At this point, however, we already have enough information for a suggestive, if formal, examination of the decision problem faced by the individual firm. This is undertaken in the following section.

3. THE INVESTMENT DECISION OF AN INDIVIDUAL FIRM

In this section, we suppose that the objective of the firm is the maximization of the mean value of the present value expression given in (3), with the stochastic behavior of prices somehow specified. Since we have omitted variable factors of production in our formulation, the firm's supply decision is trivial: produce at full capacity. Hence the only current decision is the choice of an investment level, in

using the concavity of h ,

$$\begin{aligned} k^\theta h(x^\theta/k^\theta) &\geq \phi k^\theta h(x^0/k^0) + (1 - \phi)k^\theta h(x^1/k^1) \\ &= \theta k^\theta h(x^0/k^0) + (1 - \theta)k^1 h(x^1/k^1). \end{aligned}$$

⁶ The strict concavity of the function h gives rise to the "adjustment costs" referred to in the introduction. The main function of introducing such costs, to anticipate the development somewhat, is to assure that the model reflects observed, gradual changes in capital stocks, as opposed to immediate passage to a long-run equilibrium level. This feature is shared by the adjustment costs imposed in [7], [8], [13], [14], and [19], although there are differences in form among all of these treatments and the present one. These differences will not be critical to any of the arguments which follow.

which the known (unit) cost of a unit of investment goods is compared to an expected, marginal return. To compute the latter, one must solve the present value maximization problem, but it will turn out (predictably) that "solving" this problem amounts to finding an appropriate "shadow price" to use in evaluating an addition to capital stock. From our (economists') point of view, this difficult task cannot be avoided, and we undertake it in the following sections, but from a firm's point of view it can be, and indeed is, avoided.

From the entrepreneur's viewpoint, the objective is to maximize the value of all *claims* to the income stream (3), and this value is the quantity of capital held times the value per unit of capital as *observed* on securities markets. Similarly, the appropriate valuation to place on an addition to next period's capital stock (current investment) is the price per unit, reflected in securities prices, expected to prevail next period. That is, the burden of evaluating the income stream produced by the firm is borne not by firms but by traders in the firms' securities.

This fact considerably simplifies the firm's decision problem (correspondingly complicating household's). In our formulation, additional simplification arises from the evident fact that the value placed on the claim to a unit of capital will not depend on the distribution of capital across firms. It is thus treated by each firm as a parametric market price. Denote by w_t the current value of a unit of capital, and by w_t^* the (undiscounted) value per unit expected to prevail next period. For a firm beginning period t with capital stock k_t , an investment x will lead to a next period value of $\beta k_t h(x/k_t) w_t^*$ (from (2)). The cost of this investment is $-x$. Hence the firm must solve:

$$\max_{x \geq 0} [-x + \beta k_t h(x/k_t) w_t^*].$$

The current value of the firm is, of course,

$$(4) \quad w_t k_t = p_t k_t - x + \beta k_t h(x/k_t) w_t^*.$$

If the maximum problem is correctly solved, and if there is agreement on w_t^* , we have a second condition:

$$(5) \quad 0 \geq -1 + \beta h'(x/k_t) w_t^*, \quad \text{with equality if } x > 0.$$

Solving (5) for x as a function of k_t , β , and w_t^* gives the investment demand function as seen by the firm. From the point of view of an outside observer (ourselves), however, (4) and (5) are solved *jointly* for x and w_t^* as functions of k_t , p_t , and w_t . This yields the observed investment function:

$$(6) \quad x_t = k_t g(w_t - p_t), \quad g'(\cdot) > 0.$$

We remark at this point that (6) is essentially the function used by Grunfeld in his empirical study of the investment behavior of several United States corporations [9]. Grunfeld's justification for using w_t as an "explanatory" variable was that it served as a proxy for the firm's own estimation of future income streams. Our argument goes somewhat further, to assert that the firm need not even form its own estimation of the future, beyond forecasting the value placed on assets in its

industry next period. (Of course, entrepreneurs, in common with other agents in securities markets, form judgments on the income streams of their own and other firms. The point is that these judgments are apart from, and irrelevant to, the investment goods demand decision.)

By aggregating (6) across firms (which leads to an equation of exactly the same form) one obtains an industry demand function for investment goods, given the value per unit, w_t . But this aggregate function tells us *nothing* about the development through time of capital stock, output, and prices, since the time path of w_t is, as yet, unknown. Equation (6) is a consistency requirement which must hold at each point in time, but it is not a theory of capital accumulation in the industry. To obtain such a theory, we must determine how investment and the price w_t are jointly determined on the basis of available information on current and future industry demand. This problem is studied in the following sections.

4. INDUSTRY EQUILIBRIUM: DEFINITION

To answer the questions raised in the preceding section—that is, to obtain a theory of the development of the industry through time—we begin with a mathematical formulation of the stochastically shifting industry demand. Given this formulation, we propose a definition of industry equilibrium involving in addition to a market clearing requirement, a precise specification of firms' objective functions. In the next section, we show that there is a unique equilibrium development for the industry. Characterization of this development is then resumed in Sections 6 and 7.

The industry demand function is assumed to have the usual properties and is assumed to be subject to random shifts. Specifically,

$$(7) \quad p_t = D(q_t, u_t),$$

where $\{u_t\}$ is a Markov process taking values on the real line E , with a transition function $p(\cdot, \cdot)$ defined on $E^2 = E \times E$. For fixed u , $p(\cdot, u)$ is a probability distribution function on E ; and $p(u, \cdot)$ is a Baire function on E .⁷ Thus if A is a Borel subset of E , the probability that $u_{t+1} \in A$ conditional on $u_t = u$ is given by

$$\int_A p(dz, u),$$

where the notation indicates that the integration is with respect to the distribution $p(\cdot, u)$, u fixed. For given u_t , D is a continuous, strictly decreasing function of q_t , taking on the finite value $\bar{p}(u_t) \leq \bar{p} < \infty$ at $q_t = 0$, and with $\int_0^{\bar{p}} D(z, u_t) dz$ bounded uniformly in u_t and q . Assume that D is continuous and increasing in u_t , so that an increase in u_t means a shift to the right of the demand curve.⁸

⁷ Baire functions are all members of the class of functions consisting of all continuous functions, pointwise limits of sequences of continuous functions, pointwise limits of sequences of this larger class, and so on. Alternatively Baire functions are measurable with respect to the Borel sets.

⁸ Further restrictions will be imposed, as needed, on $p(\cdot, \cdot)$ in Sections 6 and 7.

To complete the formulation of the problem, we must specify the price-forecasting policies of firms, define what is meant by an investment-output policy, and postulate an objective function for firms. If the industry were in short-run (one period) equilibrium at each point in time, price and output would be determined for that period as functions of the shift variable u_t . From the vantage point of time 0, the price at time t will then depend on the initial state of the industry, (k_0, u_0) , and on the realization u_1, \dots, u_t of the $\{u_t\}$ process between 0 and t . Hence it is natural to define an anticipated price process, for given (k_0, u_0) , as a sequence $\{p_t\}$ of functions of (u_1, \dots, u_t) , or functions with domain E^t . Similarly, one may think of an investment-output plan as a sequence $\{q_t, x_t\}$ of functions on E^t , or as a contingency plan indicating in advance how the firm will react to any possible realization of the $\{u_t\}$ process. Specifically, let L be the set of all sequences $x = \{x_t\}$, $t = 0, 1, \dots$, where x_0 is a number and, for $t \geq 1$, x_t is a bounded Baire function on E^t , bounded in the sense that for all $x \in L$,

$$\|x\| = \sup_t \sup_{(u_1, \dots, u_t) \in E^t} |x_t(u_1, \dots, u_t)|$$

is finite. We restrict the sequences $\{p_t\}$, $\{q_t\}$, and $\{x_t\}$ to be elements of L^+ : elements of L with non-negative terms for all (t, u_1, \dots, u_t) . Then for any price-output-investment sequence, present value, V , is a well defined random variable with a finite mean. We shall take as the objective of the firm the maximization of the mean value of V with respect to the investment-output policy, given an anticipated price sequence.

It remains for us to link the anticipated price sequence to the actual price sequence—also a sequence of functions on E^t . Typically, this is done by postulating the *process* by which firms actually develop forecasts from actual past values and other information. In this paper, we shall not take this route but rather postulate a property of the *outcome* of this (unspecified) process. Specifically, we assume that expectations of firms are rational, or that the anticipated price at time t is the same function of (u_1, \dots, u_t) as is the actual price. That is, we assume that firms know the true distribution of prices for all future periods.

These considerations lead to the following notion of competitive equilibrium for the industry under study.⁹

⁹ At this point, we make two rather defensive observations on the scope of the definition we are using. First, we regard firms as mean value maximizers, with respect to the distribution of the demand shift variable, u_t . If this distribution is interpreted as reflecting relative frequencies of certain physical events (such as rainfall), then this assumption implies risk-neutral preferences on the part of shareholders. Alternatively, by imagining this industry as set in an economy characterized by a "state-preference" model (see [1, 4, and 11]), one can reinterpret this distribution as giving the structure of market interest rates, and our "cost of capital" r as the certainty interest rate. In this case, (8) implies nothing about risk preferences.

Second, now that the precise content of the "rationality" of expectations is made clear (in the following definition), we add a final comment on its "reasonableness": we can think of no objection to this assumption which is not better phrased as an objection to our hypothesis that the stochastic component of demand has a regular, stationary structure. If the demand shift assumption is reasonable, then expectations rational in our sense are surely more plausible than any simple, adaptive scheme. If the demand assumption is unreasonable, then adopting an alternative expectations hypothesis will certainly not improve matters.

DEFINITION: An *industry equilibrium* (for fixed initial state (k, u)) is an element $\{q_t^0, x_t^0, p_t^0\}$ of $L^+ \times L^+ \times L^+$ such that (7) is satisfied for all (t, u_1, \dots, u_t) and such that

$$(8) \quad E \left\{ \sum_{t=0}^{\infty} \beta^t [p_t^0 q_t^0 - x_t^0] \right\} \geq E \left\{ \sum_{t=0}^{\infty} \beta^t [p_t^0 q_t - x_t] \right\}$$

for all $\{q_t, x_t\} \in L^+ \times L^+$ satisfying (1) and (2). (In (8), the expectation of the t th term in the series is taken with respect to the joint distribution of (u_1, \dots, u_t) .)

In the next section, we show that the industry has a unique equilibrium in the sense of the above definition, and, further, that this equilibrium can be obtained by solving a particular dynamic programming problem. Subsequent sections are devoted to developing various properties of this equilibrium.

5. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

In this section, we show that the industry described above has exactly one competitive equilibrium development through time. The device employed to do this involves first showing that a competitive equilibrium development will lead the industry to maximize a certain "consumer surplus" expression, and then showing that the latter maximum problem can be solved using the techniques of dynamic programming.

Define the function $s(q, u), q \geq 0, u \in E$, by

$$s(q, u) = \int_0^q D(z, u) dz,$$

so that for fixed $u, s(q, u)$ is a continuously differentiable, increasing, strictly concave, positive, and bounded function of q , and for fixed q, s is increasing in u . Thus $s(q_t, u_t)$ is the area under the industry's demand curve at an output of q_t and with the state of demand u_t . Then define the discounted "consumer surplus," S , for the industry by:

$$S = E \left\{ \sum_{t=0}^{\infty} \beta^t [s(q_t, u_t) - x_t] \right\}.$$

The quantity S is used in applied cost-benefit work as a measure of the dollar value to society of a policy $\{q_t, x_t\}$. For our purposes, however, the welfare significance of S is not important. We are interested only in using the connection between the maximization of S and competitive equilibrium in order to determine the properties of the latter.

Associated with the problem of maximizing the quantity S is the functional equation

$$(9) \quad v(k, u) = \sup_{x \geq 0} \left\{ s(k, u) - x + \beta \int v \left[kh \left(\frac{x}{k} \right), z \right] p(dz, u) \right\}.$$

The main result of this section, linking (9) to the determination of industry equilibrium, is the following theorem.

THEOREM 1: *The functional equation (9) has a unique, bounded solution v on $(0, \infty) \times E$, and for all (k, u) , the right side of (9) is attained by a unique $x(k, u)$. In terms of this function, the unique industry equilibrium, given k_0 and u_0 , is given by:*

$$(10) \quad x_t = x(k_t, u_t),$$

$$(11) \quad k_{t+1} = k_t h\left(\frac{x(k_t, u_t)}{k_t}\right),$$

$$(12) \quad q_t = k_t, \quad \text{and}$$

$$(13) \quad p_t = D(q_t, u_t),$$

for $t = 0, 1, 2, \dots$, and all realizations of the $\{u_t\}$ process.

The remainder of this section is devoted to the proof of Theorem 1, and to obtaining some properties of the functions $v(k, u)$ and $x(k, u)$. The reader interested primarily in the nature of the solutions to the difference equations (10)–(13) (that is, in the characterization of the development of the industry) should proceed directly to Section 6.

The proof of Theorem 1 involves two distinct steps. First, one must connect the determination of industry equilibrium to the problem of maximizing consumer surplus. Second, the latter problem must be shown to lead to the functional equation (9). Both parts turn out to be complicated, although the outlines of each are familiar.

For the first part, we have the following lemma.

LEMMA 1: *Suppose for given (k_0, u_0) the problem of maximizing S , subject to (1)–(2), over all $\{q_t, x_t\} \in L^+ \times L^+$ is solved by the sequence $\{q_t^0, x_t^0\}$. Then $\{q_t^0, x_t^0, p_t^0\}$, where p_t^0 is obtained from q_t^0 using (7), is an industry equilibrium. Conversely, suppose $\{q_t^0, x_t^0, p_t^0\}$ is an industry equilibrium. Then $\{q_t^0, x_t^0\}$ maximizes S , subject to (1)–(2), over $L^+ \times L^+$.*

PROOF: The proof is an application of Theorems 1 and 2 of [5] and of [17]. The basic space used in [5] corresponds to our $L \times L$. The economy in our application consists of a single consumer whose preferences, given by S , are defined on $L^+ \times L^+$ (X in [5]), and a single firm, whose production possibility set $Y \subset L^+ \times L^+$ consists of all elements satisfying (1) and (2). Pareto optimality in this economy is then equivalent to maximizing S over Y .

Assumptions I–V of [5] (the convexity of X , the concavity of S , the continuity of S , the convexity of Y , and the existence of an interior point of Y) are readily verified. If $k_0 > 0$, the hypotheses of the Remark [5, p. 591] are valid. If $k_0 = 0$, the unique industry equilibrium occurs when output and investment are zero for all t ; in the following, we assume $k_0 > 0$.

To prove the second part of the Lemma, one applies Theorem 1 of [5].

To prove the first part of the Lemma, we use Theorem 2 of [5], plus the Remark, which states that if $\{q_t^0, x_t^0\}$ maximizes S over Y , there exists a continuous linear form α on $L \times L$ such that $\{q_t^0, x_t^0\}$ maximizes $\alpha(\{q_t, x_t\})$ over Y , and such that $\{q_t^0, x_t^0\}$ maximizes S over all elements of X which satisfy

$$\alpha(\{q_t, x_t\}) \leq \alpha(\{q_t^0, x_t^0\}).$$

It then follows from Theorems 1 and 2 of [17] that there exist elements $\{\alpha_t\}, \{\gamma_t\}$ of L^+ such that $\{q_t^0, x_t^0\}$ maximizes

$$(14) \quad E \left\{ \sum_{t=0}^{\infty} \beta^t [\alpha_t q_t - \gamma_t x_t] \right\}$$

over Y , and such that $\{q_t^0, x_t^0\}$ maximizes S over all elements of X which satisfy

$$(15) \quad E \left\{ \sum_{t=0}^{\infty} \beta^t [\alpha_t q_t - \gamma_t x_t] \right\} \leq E \left\{ \sum_{t=0}^{\infty} \beta^t [\alpha_t q_t^0 - \gamma_t x_t^0] \right\}$$

where the expectations in (14) and (15) are with respect to the distributions P_t of $(u_1, \dots, u_t), t = 1, 2, \dots$

We next show that for some constant $\lambda > 0$

$$(16) \quad \gamma_t \leq \lambda$$

almost everywhere, with equality a.e. on any Borel set A such that $E\{\gamma_t I_A x_t^0\} > 0$. (I_A is the function taking the value 1 on $A \subset E^t$ and 0 elsewhere.) To verify (16), consider first the case $E\{\gamma_t I_A x_t^0\} = 0$ for all t, A . If (16) is false in this case, then for all $\lambda > 0$ one can find t, A such that $E\{\gamma_t I_A\} > \lambda E\{I_A\}$. Then S can always be improved, subject to (15), by increasing q_s^0 over some $B \subset E^s$ with $E\{I_B\} > 0$ and increasing x_t^0 over $A \subset E^t$ with $E\{\gamma_t I_A\}/E\{I_A\}$ sufficiently large, which is a contradiction.

For the case where $E\{\gamma_t I_A x_t^0\} > 0$ for some t, A , define λ by $E\{\gamma_t I_A\} = \lambda E\{I_A\}$. Now if s, B satisfy $E\{\gamma_s I_B\} > \lambda E\{I_B\}$, S can be increased, subject to (15), by increasing x_s^0 on B and decreasing x_t^0 on A . Similarly if $E\{\gamma_s I_B\} < \lambda E\{I_B\}$, S can be improved, subject to (15), by decreasing x_s^0 on B and increasing x_t^0 on A , unless $x_s^0 = 0$ almost everywhere on B . In the latter case, however, $E\{\gamma_s I_B x_s^0\} = 0$. Hence (16) is proved.

In view of (16) and the fact that $\{q_t^0, x_t^0\}$ maximizes the expression (14) over Y , we have

$$\sum_{t=0}^{\infty} \beta^t E\{\alpha_t q_t^0 - \gamma_t x_t^0\} \geq \sum_{t=0}^{\infty} \beta^t E\{\alpha_t q_t - \gamma_t x_t\} \geq \sum_{t=0}^{\infty} \beta^t E\{\alpha_t q_t - \lambda x_t\}$$

for all $\{q_t, x_t\} \in Y$, with equality if $\{q_t, x_t\} = \{q_t^0, x_t^0\}$. Then letting $p_t^0 = \lambda^{-1} \alpha_t$, we have shown that $\{q_t^0, x_t^0\}$ maximizes

$$\sum_{t=0}^{\infty} \beta^t E\{p_t^0 q_t - x_t\}$$

over Y , or that condition (8) of the definition of industry equilibrium is satisfied.

To show that (7) is satisfied (almost everywhere) by $\{q_t^0, x_t^0, p_t^0\}$, observe that if $\{q_t^0, x_t^0\}$ maximizes S over all elements of X satisfying (15), it also maximizes S over the subset of X on which $x_t = 0$ whenever $x_t^0 = 0$. Over this set, (16) (and the fact that x_t^0 is clearly 0 whenever $y_t = 0$) implies that the constraint (15) is equivalent to

$$\sum_{t=0}^{\infty} \beta^t E\{p_t^0 q_t - x_t\} \leq \sum_{t=0}^{\infty} \beta^t E\{p_t^0 q_t^0 - x_t^0\}.$$

Since s is strictly increasing in q , this constraint will be binding, so that $\{q_t^0\}$ maximizes

$$\sum_{t=0}^{\infty} \beta^t E\{s(q_t, u_t) - p_t^0 q_t\}.$$

Then since $q_t^0 > 0$, we have

$$p_t^0 = s_1(q_t^0, u_t) = D(q_t^0, u_t),$$

which proves that (7) is satisfied (almost everywhere) and completes the proof of Lemma 1.

We now turn to the study of the problem of maximizing S subject to (1) and (2), the solution to which will be, by the Lemma just proved, the unique industry equilibrium. This will be done by studying the functional equation (9), which is related to the problem of maximizing S by the principle of optimality, a version of which is utilized in Lemma 4, below.

To show the existence and uniqueness of a solution to (9) and to obtain some of its properties, we utilize the method of successive approximation as applied in [2]. As a convenient device in this argument, we employ the operator T , taking bounded Baire functions on $(0, \infty) \times E$ into the same set of functions, defined by

$$(17) \quad Tf(k, u) = \sup_{x \geq 0} \left\{ s(k, u) - x + \beta \int f \left(kh \left(\frac{x}{k} \right), z \right) p(dz, u) \right\}.$$

Then, clearly, solutions to (9) are coincident with solutions to $Tf = f$. The relevant properties of T are given by the next lemma.

LEMMA 2: (i) If f is non-decreasing in k , so is Tf ; (ii) if f is concave in k , so is Tf ; (iii) $Tf = f$ has a unique solution, f^* ; and (iv) for any g , $\lim_{n \rightarrow \infty} T^n g = f^*$.

PROOF: To prove (i), observe that if f is non-decreasing, $s(k, u)$ and $\int f[kh(x/k), z]p(dz, u)$ are non-decreasing functions of k for all fixed x .

To prove (ii), note first that the concavity of f in k implies that f is continuous in k on $(0, \infty)$, so that the expression in brackets on the right of (7) is continuous in k . Further, this expression is finite at $x = 0$ and tends to $-\infty$ as x becomes large (since f is bounded). Hence, for any $k > 0$, $Tf(k, u)$ is attained by some $x \geq 0$.

Let k^0 and k^1 be on $(0, \infty)$, let k^θ be a convex combination, and let x^0 and x^1 attain $Tf(k^0, u)$ and $Tf(k^1, u)$ respectively. Then:

$$\begin{aligned} Tf(k^\theta, u) &= s(k^\theta, u) + \sup_{x \geq 0} \left\{ -x + \beta \int f[k^\theta h(x/k^\theta), z]p(dz, u) \right\} \\ &\geq \theta s(k^0, u) - \theta x^0 + \theta \beta \int f[k^0 h(x^0/k^0), z]p(dz, u) \\ &\quad + (1 - \theta)s(k^1, u) - (1 - \theta)x^1 \\ &\quad + (1 - \theta)\beta \int f[k^1 h(x^1/k^1), z]p(dz, u) \\ &= \theta Tf(k^0, u) + (1 - \theta)Tf(k^1, u). \end{aligned}$$

To prove (iii) and (iv), observe that T is monotone ($f \geq g$ for all (k, u) implies $Tf \geq Tg$) and if a is any constant, $T(f + a) = Tf + a\beta$. By Theorem 5 of [3], these two facts imply (iii) and (iv).

Lemma 2 leads directly to a third lemma.

LEMMA 3: *The functional equation (9) is satisfied by a unique bounded Baire function $v(k, u)$ on $(0, \infty) \times E$. The function $v(k, u)$ is continuous, non-decreasing and concave in k for any fixed u . For any (k, u) , $v(k, u)$ is attained by a unique investment rate $x(k, u)$ and $x(k, u)$ is a Baire function, continuous in k .*

PROOF: Let $g(k, u) = s(k, u)$. Then by Lemma 2, $\lim_{n \rightarrow \infty} T^n g = f^*$, where f^* is the unique bounded solution to (9). Let $v(k, u) = f^*$. Since $s(k, u)$ is increasing in both arguments and strictly concave in k , the limit function v is non-decreasing and concave in k . Since v is concave in k for fixed u , it is continuous in k on $(0, \infty)$.

It follows from these facts that the expression in brackets on the right side of (9) is a continuous, strictly concave function of x , positive at $x = 0$ and negative for x sufficiently large, so that it attains its supremum on $[0, \infty)$ at a unique x , depending on k and u . This defines the policy function $x(k, u)$.

To show that $x(k, u)$ is a Baire function, we must show that for all x^0 ,

$$S(x^0) = \{(k, u) \in (0, \infty) \times E : x(k, u) \leq x^0\}$$

is a Borel set. For $x^0 < 0$, this is trivial. Now denote the expression in brackets on the right of (9) by $H(x, k, u)$, and for $x^0 \geq 0$ and $\varepsilon > 0$, define

$$S(x^0, \varepsilon) = \{(k, u) \in (0, \infty) \times E : H(x^0, k, u) \geq H(x^0 + \varepsilon, k, u)\}.$$

Since H is a Baire function, $-H(x^0, k, u) + H(x^0 + \varepsilon, k, u)$ is a Baire function on $(0, \infty) \times E$, for each $\varepsilon > 0$. Hence $S(x^0, \varepsilon)$ is a Borel set. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0 as n tends to infinity. Then

$$\bigcap_{n=1}^{\infty} S(x^0, \varepsilon_n)$$

is a Borel set, and since this set equals (using the concavity of H in x) the set $S(x^0)$,

$x(k, u)$ is a Baire function. Since the expression in brackets varies continuously with k , $x(k, u)$ is continuous in k .¹⁰

The function $x(k, u)$ defines, for given (k_0, u_0) , a sequence of Baire functions $\{k_t, x_t\}$ as given by (10) and (11). The next lemma shows that this sequence is the (essentially) unique optimal policy.

LEMMA 4: *For any fixed $(k_0, u_0) \in (0, \infty) \times E$, the sequence $\{k_t^0, x_t^0\}$ defined by (10) and (11) is an element of Y , and this policy maximizes S over Y . Further, it is the unique optimal policy in the sense that any other element of Y yielding the same value of S differs from $\{k_t^0, x_t^0\}$ at most over sets of P_t measure zero.*

PROOF: From Lemma 3 and the continuity of h , the terms of the sequence $\{k_t^0, x_t^0\}$ are Baire functions, and they clearly satisfy (1) and (2). To show that $\{k_t^0, x_t^0\} \in Y$, then, we need only show that k_t^0 and x_t^0 are bounded, uniformly in (u_1, \dots, u_t, t) . Since the policy $x_t = 0$ for all t is always feasible, we have $v(k, u) > 0$ for all $k > 0$. Then from (9),

$$-x(k, u) + \beta \int v \left[kh \left(\frac{x(k, u)}{k} \right), z \right] p(dz, u) \geq \beta \int v[kh(0), z] p(dz, u) > 0$$

for all (k, u) , $k > 0$. Let B be a bound for v so that the term on the left, above, is less than

$$-x(k, u) + \beta B.$$

Hence $x(k, u)$ is bounded from above by βB . Then if $k_t > (\beta B/\delta)$, $k_{t+1} < k_t$, so that $\max[(\beta B/\delta), k_0]$ is an upper bound for k_t .

To show that $\{k_t^0, x_t^0\}$ is optimal, it is sufficient to show that it yields the return $v(k_0, u_0)$ [3, Theorem 6, part (f)]. Define the operator F_X , taking bounded Baire functions on $(0, \infty) \times E$ into the same set of functions, by

$$F_X f(k, u) = s(k, u) - x(k, u) + \beta \int f \left[kh \left(\frac{x(k, u)}{k} \right), z \right] p(dz, u).$$

The functional equation $F_X f = f$ has a unique solution, f^* [3, Theorem 5], and this solution, evaluated at (k_0, u_0) , gives the value of S under the policy $\{k_t^0, x_t^0\}$. Since $F_X v = v$, $v = f^*$, which proves that $\{k_t^0, x_t^0\}$ is optimal.

To prove uniqueness, let $\{k_t^1, x_t^1\}$ yield the same value to S as $\{k_t^0, x_t^0\}$, and let A_t be the Borel subset of E^t on which x_t^1 and x_t^0 differ. Let t' be the first period in which the probability of A_t is non-zero. Then the returns from the policies differ by

$$\beta^{t'} E \left\{ -x_t^0 + v \left[k_t^0 h \left(\frac{x_t^0}{k_t^0} \right), u_{t'+1} \right] + x_t^1 - v \left[k_t^0 h \left(\frac{x_t^1}{k_t^0} \right), u_{t'+1} \right] \right\} > 0,$$

¹⁰ A theorem in [4, p. 19], suitably specialized to this application, states that if $f(x, y)$ is continuous, the function $g(y) = \max_{x \in S} f(x, y)$, where S is compact, is continuous. Further, if $g(y)$ is attained at a unique $x(y)$ for each y , $x(y)$ is also continuous. This fact will be used at various points below, without reference.

where the expectation is taken with respect to P_{t+1} . This contradicts the assumed optimality of $\{k_t^1, x_t^1\}$, and completes the proof of the Lemma.

The proof of Theorem 1 now follows directly from Lemmas 1, 3, and 4. In the next two sections, we pursue the study of the unique industry equilibrium as given by (10)–(13).

6. LONG RUN EQUILIBRIUM WITH INDEPENDENT ERRORS

Equations (10) and (11) and $p(\cdot, \cdot)$ determine a Markov process $\{k_t, u_t\}$ taking values on $(0, \infty) \times E$ which governs the development of k_t, x_t , and u_t through time, starting from a given (k_0, u_0) . To determine the long run characteristics of this process, it will be necessary to restrict the $\{u_t\}$ process further. In the present section, we treat the special but interesting case where u_t and u_s are independent for $s \neq t$, or where the transition function $p(z, u)$ does not depend on u .

In this case, the functional equation (9) becomes

$$(18) \quad v(k, u) = \max_{x \geq 0} \left\{ s(k, u) - x + \beta \int v \left[kh \left(\frac{x}{k} \right), z \right] p(dz) \right\},$$

from which it is clear that the optimal investment rate, $x(k, u)$, will not depend on u . It follows from (10) and (11) that the time path of capital stock will be deterministic, following the difference equation

$$(19) \quad k_{t+1} = k_t h \left(\frac{x(k_t)}{k_t} \right),$$

where $x(k_t)$ is the unique investment rate attaining $v(k_t, u)$. Hence we turn our attention to the existence, uniqueness, and stability of stationary solutions to (19), with results summarized in Theorem 2, below.

A capital stock $k^c > 0$ will be a stationary solution to (19) if and only if it is a solution to

$$(20) \quad x(k) = \delta k,$$

since $h(\delta) = 1$. The solutions to (20) are described in the next lemma.

LEMMA 5: Equation (20) has a solution $k^c > 0$ if and only if

$$(21) \quad \int D(0, u)p(du) > \delta + \frac{r}{h'(\delta)}.$$

A positive stationary solution, if it exists, must satisfy

$$(22) \quad \int D(k, u)p(du) = \delta + \frac{r}{h'(\delta)},$$

so there is at most one positive solution to (20).

PROOF: We first show that any solution to (20) satisfies (22). From (18),

$$v(k, u) \geq s(k, u) - \delta k + \beta \int v(k, z)p(dz)$$

(since $x = \delta k$ is always feasible) for all (k, u) with equality if and only if k satisfies (20). Then taking the mean of both sides with respect to $p(u)$ and collecting terms:

$$(23) \quad E\{v(k, u)\} \geq \frac{1}{1-\beta} [E\{s(k, u)\} - \delta k]$$

with equality if and only if k satisfies (20). If k^c satisfies (20), (18) implies

$$-\delta k^c + \beta E\{v(k^c, u)\} \geq -x + \beta E\{v(k^c h(x/k^c))\}$$

for all $x \geq 0$. Now applying (23), which holds with equality at k^c ,

$$\begin{aligned} & -\delta k^c + \frac{\beta}{1-\beta} [E\{s(k^c, u)\} - \delta k^c] \\ & \geq -x + \frac{\beta}{1-\beta} [E\{s(k^c h(x/k^c))\} - \delta k^c h(x/k^c)], \end{aligned}$$

for $x \geq 0$. At $x = \delta k^c$, this inequality holds with equality. At any other value of x , the right side takes on a smaller value, which is to say that δk^c maximizes the expression on the right. Then the first order condition

$$E\{s_1(k^c, u)\} = E\{D(k^c, u)\} = \delta + \frac{r}{h'(\delta)}$$

is satisfied. If (21) does not hold, no $k^c > 0$ satisfies this condition, so the necessity of (21) is proved. Further, we have proved that positive stationary solutions satisfy (22) and, since D is strictly decreasing in k , that there is at most one such solution.

To show that (21) is sufficient for the existence of a positive solution to (20), we must rule out the possibilities that $x(k) > \delta k$ or $x(k) < \delta k$ for all $k > 0$. To rule out the former possibility, recall from Lemma 4 that $x(k)$ is bounded, so that for k sufficiently large, $x(k) < \delta k$. Suppose, contrary to the Lemma, that (22) has a positive solution k^c and that $x(k) < \delta k$ for all $k > 0$. Define the function $z(k)$ by

$$z(k) = \max \left\{ \frac{E\{s(k, u)\} - \delta k}{1-\beta}, 0 \right\}$$

so that from (23), $z(k) < E\{v(k, u)\}$ for all $k > 0$. Now, define the operator F_X on bounded continuous functions on $(0, \infty)$ by

$$F_X y(k) = E\{s(k, u)\} - x(k) + \beta y \left[kh \left(\frac{x(k)}{k} \right) \right].$$

It is readily verified that for any y, z in the domain of F_X , $\|F_X y - F_X z\| \leq \beta \|y - z\|$. Also, $F_X E\{v(k, u)\} = E\{v(k, u)\}$ where v is the solution to (9), and for any y ,

$$(24) \quad \lim_{n \rightarrow \infty} F_X^n y = E\{v(k, u)\}.$$

We next show that for $z(k)$ as defined above, $F_X z(k) < z(k)$ for all $k \leq k^c$.

We have, directly from the definitions of z and F_X ,

$$(25) \quad \begin{aligned} F_X z - z &= \frac{\beta}{1-\beta} \left[E \left\{ s \left[kh \left(\frac{x(k)}{k} \right), u \right] \right\} - E\{s(k, u)\} \right] \\ &\quad - \frac{\beta}{1-\beta} \delta k \left[h \left(\frac{x(k)}{k} \right) - 1 \right] + \delta k - x(k). \end{aligned}$$

The strict concavity of s and h implies

$$(26) \quad s \left[kh \left(\frac{x(k)}{k} \right), u \right] - x(k, u) \leq D(k, u)k \left[h \left(\frac{x(k)}{k} \right) - 1 \right]$$

and

$$(27) \quad h \left(\frac{x(k)}{k} \right) - 1 < h'(\delta) \left[\frac{x(k)}{k} - \delta \right].$$

Combining (25), (26), (27), and (22) then gives, for $k \leq k^c$,

$$(28) \quad F_X z < z.$$

One may also verify that $y(k) < z(k)$ over the interval $(0, k^c]$ implies $F_X y(k) < F_X z(k)$ on this interval. Thus (24) and (28) together imply

$$E\{v(k, u)\} = \lim_{n \rightarrow \infty} F_X^n z(k) < z(k), \quad k \in (0, k^c],$$

which contradicts (23). This completes the proof of Lemma 5.

It can be shown that $v(k) = E\{v(k, u)\}$ is differentiable for $k > 0$.¹¹ Then from (18), $x(k)$ must satisfy

$$(29) \quad -1 + \beta v' \left[kh \left(\frac{x(k)}{k} \right) \right] h' \left[\frac{x(k)}{k} \right] \leq 0$$

with equality if $x(k) > 0$. Inspection of (29) reveals that $x(k)/k$ is a strictly decreasing function of k where $x(k) > 0$, and that $kh(x(k)/k)$ is a strictly increasing function of k . Hence if (20) has a solution, $k^c > 0$. Then the results of this section may be summarized in this theorem.

THEOREM 2: *Under the hypothesis of independence of the $\{u_t\}$ process, there are two possibilities for the behavior of the optimal capital stock, k_t . If (21) holds, and if $k_0 > 0$, k_t will converge monotonically to the stationary value k^c , given implicitly by (22). If (21) fails to hold, or if $k_0 = 0$, capital will converge monotonically to zero.*

The marginal condition (22), which is satisfied by the long-run capital stock, has a familiar interpretation. The left side of (22) is the expected marginal value product of capital. Since in our model the marginal physical product is unity, this becomes simply expected output price. The right side of (22) is a rental price or user cost of capital, equal to the sum of a depreciation cost term, δ , and a capital or interest cost term, $(r/h'(\delta))$.

While capital and investment are varying deterministically as described in Theorem 2, output each period is supplied to the market inelastically, in a quantity

¹¹ If we let $g_n(k) = E\{T^n s(k, u)\}$ and $a_n(k) = g_n(k) - g_{n-1}(k)$, it can be shown that (i) $a_n(k) \leq \beta n B_1$, (ii) $a_n(0) = 0$, (iii) $a'_n(k) > 0$, and (iv) $|a''_n(k)| \leq B_2$ provided $h''(k)$ is bounded, for $k \geq k^* > 0$. Given these conditions, it follows $a'_n(k) \leq (\beta^{n^2}/2) B_1^{1/2} B_2$ proving $g'_n(k)$ converges uniformly for $k \geq k^*$. Since $g_n(k)$ converges to $v(k) = E\{v(k, u)\}$, $v(k)$ is differentiable for all $k \geq k^*$. Since this result holds for all $k^* > 0$, $v'(k)$ exists for all $k > 0$.

given by the historically determined capacity of the industry. Fluctuations in demand then affect price only, in a manner which can be computed from the demand function (7). (It may be noted that if variable inputs were introduced, the short-run supply would be upward sloping but not vertical, so that *both* price and output would vary with demand shifts. Nevertheless, in this case as well the capital-investment path will be deterministic.)

While the case of independent errors discussed in this section may appear to be an unlikely specialization of our general model, it is interesting to note that it corresponds perhaps more closely than any other analytical model of the industry to the familiar geometrical dichotomy between short-run and long-run supply. In the short-run case, capacity is fixed and price and output are determined entirely by the *current* demand function and the short-run supply function. In determining the long-run equilibrium, on the other hand, demand fluctuations play *no* role, with equilibrium capacity determined entirely by average, or normal, demand.

7. LONG RUN EQUILIBRIUM WITH SERIALLY DEPENDENT ERRORS

In the case of a serially independent $\{u_t\}$ process, as studied in the preceding section, a demand shift in period t results in a windfall gain (or loss) for firms in that period, but yields *no* information about what the state of demand in future periods is likely to be. As a result, the current state of demand has no effect on investment policy, and the capital stock of the industry follows a deterministic difference equation. In this section, we shall drop the assumption of independent demand shifts, replacing it with an assumption that these shifts are positively correlated in a particular way, so that, for example, an upward shift in demand not only increases price and profit *this* period but increases the probability that price will continue to be higher than average over the near future.

To study the case of dependent errors, it will be necessary to impose some additional restrictions on the $\{u_t\}$ process, with the general aim of assuring that the distribution of (k_t, u_t) will converge to a stationary distribution which is independent of the initial state of the industry, (k_0, u_0) . First, we require, for all $v \in E$ and all non-degenerate intervals $A \subset E$,

$$(30) \quad \int_A p(du, v) > 0,$$

so that for any u_t and any non-degenerate subinterval of E , u_{t+1} will fall in that subinterval with positive probability. Similarly, it assures that u_{t+s} will fall in that subinterval with positive probability, or that

$$(31) \quad \text{pr}[u_{t+s} \in A | u_t = u_0] > 0$$

for all $u_0 \in E$ and all non-degenerate subintervals A of E . Assumption (30) does not preclude the possibility that the left side of (31) may approach 0 as s approaches infinity. We shall rule out this possibility explicitly by adding

$$(32) \quad \lim_{s \rightarrow \infty} \text{pr}[u_{t+s} \in A | u_t = u_0] > 0$$

for all u_0 , A as above, where the limiting value does not depend on the value of u_0 . That is, u_t has a limiting distribution which does not depend on the initial value u_0 and which assigns positive probability to all non-degenerate intervals of E . It follows from (31) and (32) that for any fixed A the terms of the sequence (31), $s = 1, 2, \dots$, are uniformly bounded away from zero.

We also wish to insure that the relation between current and future demands is always positive (so that a high demand this period *always* signals high demand for the future) by requiring that

$$(33) \quad \text{pr} \{u_{t+1} \geq x|u_t\}$$

be a strictly increasing function of u_t for all $x \in E$ and that

$$(34) \quad \begin{cases} \lim_{u_t \rightarrow \infty} \text{pr} \{u_{t+1} \geq x|u_t\} = 1, \\ \lim_{u_t \rightarrow -\infty} \text{pr} \{u_{t+1} \geq x|u_t\} = 0, \end{cases}$$

for all $x \in E$. An example of a process satisfying the restrictions (30), (32), (33), and (34) is:

$$u_{t+1} = \rho u_t + \varepsilon_t,$$

where $0 < \rho < 1$, and where $\{\varepsilon_t\}$ is a sequence of independent, identically distributed normal random variables. Finally, in addition to the assumption (Section 2) that $D(k, u)$ (and hence $s(k, u)$ also) is an increasing function of u , we add the restrictions that the limits

$$(35) \quad \bar{s}(k) = \lim_{u \rightarrow \infty} s(k, u),$$

$$(36) \quad \underline{s}(k) = \lim_{u \rightarrow -\infty} s(k, u),$$

exist, and convergence is uniform in k .

Our first task, given these restrictions, is to decompose the set of possible states of the (k_t, u_t) process, the set $(0, \infty) \times E$, into transient sets (sets which cannot be entered, and which will be departed with probability ultimately approaching 1) and ergodic sets (sets which once entered cannot be departed, and which contain no transient subsets). To do this, we develop some additional notation and prove some preliminary lemmas.

Since $\int v(k, z)p(dz, u)$ is a bounded, increasing (by (33)) function of u , the limits

$$(37) \quad \bar{v}(k) = \lim_{u \rightarrow \infty} \int v(k, z)p(dx, u),$$

$$(38) \quad \underline{v}(k) = \lim_{u \rightarrow -\infty} \int v(k, z)p(dz, u),$$

exist for all fixed k . We wish to show that convergence is uniform in k and to exhibit functional equations solved by $\bar{v}(k)$ and $\underline{v}(k)$. To this end, we first introduce another lemma.

LEMMA 6: Let $f(k, u)$ be a bounded Baire function on $(0, \infty) \times E$, such that

$$\lim_{u \rightarrow \infty} f(k, u) = \bar{f}(k) \quad \text{and} \quad \lim_{u \rightarrow -\infty} f(k, u) = \underline{f}(k)$$

exist, and convergence is uniform. Then

$$\lim_{u \rightarrow \infty} \int f(k, z)p(dx, u) = \bar{f}(k) \quad \text{and} \quad \lim_{u \rightarrow -\infty} \int f(k, z)p(dx, u) = \underline{f}(k),$$

and convergence is uniform.

PROOF: Let B be a bound for f . Then for $x \in E$,

$$\begin{aligned} & \left| \bar{f}(k) - \int f(k, z)p(dz, u) \right| \\ &= \left| \int_{-\infty}^x [\bar{f}(k) - f(k, z)]p(dz, u) + \int_x^{\infty} [\bar{f}(k) - f(k, z)]p(dz, u) \right| \\ &\leq 2B \int_{-\infty}^x p(dz, u) + \sup_{z \in [x, \infty)} |\bar{f}(k) - f(k, z)| \int_x^{\infty} p(dz, u). \end{aligned}$$

For x sufficiently large, $|\bar{f}(k) - f(k, z)|$ is arbitrarily small, uniformly in k , while $\int_x^{\infty} p(dz, u)$ is bounded by 1. For any fixed x , however large, the first term on the right is made arbitrarily small by choosing u sufficiently large, applying (34). The proof for $\underline{f}(k)$ is similar.

As an application of Lemma 6, we have the next lemma.

LEMMA 7: The function $\bar{v}(k)$, defined in (37), satisfies the functional equation

$$(39) \quad \bar{v}(k) = \sup_{x \geq 0} \left\{ \bar{s}(k) - x + \beta \bar{v} \left[kh \left(\frac{x}{k} \right) \right] \right\}$$

and an analogous functional equation is satisfied by $\underline{v}(k)$.

PROOF: We first show that

$$(40) \quad \lim_{u \rightarrow \infty} v(k, u) = \bar{v}(k)$$

uniformly in k . Define the operator T as in Section 3, and let $0(k, u) = 0$ on $(0, \infty) \times E$. Let $0_n(k, u) = T^n 0(k, u)$. Then, by Lemma 3, $\lim_{n \rightarrow \infty} T^n 0 = v(k, u)$, uniformly in k and u . It will then be sufficient to show that $0_n(k, u)$ converges uniformly to the limit $\bar{0}_n(k)$ as $u \rightarrow \infty$, for all n . For $n = 1$, the proposition is true, since $0_1(k, u) = s(k, u)$, and $s(k, u)$ converges uniformly to $\bar{s}(k)$ by (35). Suppose $0_n(k, u)$ converges uniformly to $\bar{0}_n(k)$. By the definition of the operator T ,

$$0_{n+1}(k, u) = \sup_{x \geq 0} \left\{ s(k, u) - x + \beta \int 0_n \left[kh \left(\frac{x}{k} \right), z \right] p(dz, u) \right\}.$$

Let

$$\bar{O}_{n+1}(k) = \bar{s}(k) + \sup_{x \geq 0} \left\{ -x + \beta \bar{O}_n \left[kh \left(\frac{x}{k} \right) \right] \right\}.$$

Since $\bar{O}_n(k) \geq O_n(k, u)$,

$$\begin{aligned} 0 &\leq \bar{O}_{n+1}(k) - O_{n+1}(k, u) \\ &= \sup_{x \geq 0} \left\{ -x + \beta \bar{O}_n \left[kh \left(\frac{x}{k} \right) \right] \right\} \\ &\quad - \sup_{x \geq 0} \left\{ -x + \beta \int O_n \left[kh \left(\frac{x}{k} \right), z \right] p(dz, u) \right\} + \bar{s}(k) - s(k, u) \\ &\leq \beta \sup_{x \geq 0} \left| \bar{O}_n \left[kh \left(\frac{x}{k} \right) \right] - \int O_n \left[kh \left(\frac{x}{k} \right), z \right] p(dz, u) \right| + \bar{s}(k) - s(k, u). \end{aligned}$$

By Lemma 6, the induction hypothesis, and (35), the term on the right is arbitrarily small, uniformly in $kh(x/k)$, for u sufficiently large. Thus $O_{n+1}(k, u)$ converges uniformly to $\bar{O}_{n+1}(k)$ and $\lim_{n \rightarrow \infty} \bar{O}_n(k)$. This completes the proof that the convergence in (40) is uniform.

Now

$$\bar{v}(k) = \lim_{u \rightarrow \infty} \sup_{x \geq 0} \left\{ s(k, u) - x + \beta \int v[kh(x/k)] p(dz, u) \right\}.$$

By (35), (40), and Lemma 6, the expression in braces converges uniformly in k ; thus, the supremum and limit operation can be interchanged to obtain (39). The proof for $v(k)$ is similar

The limit functions $\bar{v}(k)$ and $v(k)$ share the properties of continuity, monotonicity, and concavity with $\int v(k, z)p(dz, u)$ regarded as a function of k for fixed u . Then, as in Section 4, $\bar{v}(k)$ and $v(k)$ are attained at unique values $\bar{x}(k)$ and $x(k)$ respectively, and these functions are continuous. Also as in Section 4, the functions $(1/k)\bar{x}(k)$ and $(1/k)x(k)$ are decreasing and the functions $kh[(1/k)\bar{x}(k)]$ and $kh[(1/k)x(k)]$ are increasing. Finally,

$$\bar{x}(k) = \lim_{u \rightarrow \infty} x(k, u)$$

and

$$x(k) = \lim_{u \rightarrow -\infty} x(k, u)$$

for all k .

From (33), $x(k, u)$ is an increasing function of u , so that $x(k) > x(k)$ for all k such that $\bar{x}(k) > 0$. By an argument used to prove Lemma 4, $\bar{x}(k) < \delta k$ for k sufficiently large. Let \bar{k} be the least positive solution to $\bar{x}(k) = \delta k$ (or 0 if there is no positive solution) and let \underline{k} be the greatest positive solution to $x(k) = k$ (or 0 if there is no solution). Then since $x(k) < \bar{x}(k)$ if $\bar{x}(k) > 0$, either $\underline{k} = \bar{k} = 0$, or $0 \leq \underline{k} < \bar{k} < \infty$.

We now prove another lemma.

LEMMA 8: (i) For all $k \geq \bar{k}$, there exists $h_0 < 1$ (not dependent on k) such that the event $h[(x(k, u)/k)] \leq h_0$ has positive probability, for all t and all u_0 ; (ii) for all

$0 < k \leq \bar{k}$ there exists $h_1 > 1$ (not dependent on k) such that the event $h[(x(k, u/k)] \geq h_1$ has positive probability, for all t and all u_0 .

PROOF: We prove (i) only, since the proofs of the two parts are essentially the same. For any k , the event $(1/k)x(k) \leq (1/k)x(k, u) \leq (1/2k)[\bar{x}(k) + x(k)]$ has positive probability, by (31) and the definitions of $\bar{x}(k)$ and $x(k)$. For $k \geq \bar{k}$, $(1/k)\bar{x}(k) \leq (1/k)\bar{x}(\bar{k}) = \delta$, so that the occurrence of this event implies

$$\frac{1}{k}x(k, u) \leq \frac{\delta}{2} + \frac{1}{2k}x(\bar{k}) < \delta.$$

Letting $h_0 = h[(\delta/2) + (1/2k)x(\bar{k})] < 1$, (i) is proved.

Using Lemma 8, we can prove the next lemma.

LEMMA 9: The sets $T_1 = (0, \bar{k}) \times E$ and $T_2 = [\bar{k}, \infty) \times E$ are transient, and the set $B = (\bar{k}, \bar{k}) \times E$ contains all ergodic sets.

PROOF: Since $\{k_t, u_t\}$ takes values in $T_1 \cup B \cup T_2$ (for $k_0 > 0$), it is sufficient to show that (i) if $(k_0, u_0) \notin T_i$ ($i = 1, 2$), $(k_t, u_t) \notin T_i$ with probability 1 for all $t = 1, 2, \dots$; and (ii) for all $(k_0, u_0) \in T_1 \cup T_2$,

$$\lim_{t \rightarrow \infty} \text{pr} [(k_t, u_t) \in B] = 1.$$

To prove (i), we show that $\text{pr} [k_{t+1} \geq \bar{k} | (k_t, u_t)] = 0$ for $(k_t, u_t) \in T_1 \cup B$. For $k_t < \bar{k}$, we have

$$k_{t+1} \leq k_t h\left(\frac{1}{k_t}x(k_t)\right) < \bar{k} h\left(\frac{1}{\bar{k}}\bar{x}(\bar{k})\right) = \bar{k},$$

since $kh((1/k)\bar{x}(k))$ is an increasing function of k . A similar argument rules out passage into T_1 from $B \cup T_2$.

To prove (ii), let $(k_0, u_0) \in T_2$. Since $x(k_0, u) \leq \delta k_0$ for all u , we have $k_t \leq k_0$ for all t . Let $h_0 \leq 1$ be the number whose existence was established in Lemma 8. For some $t^*, k_0 h_0^{t^*} < \bar{k}$. Hence if the event $h[(x(k, u)/k)] \leq h_0$ occurs t^* times in t periods, $k_t < \bar{k}$. Since the probability that this event occurs is bounded away from zero, the probability that it will not occur t^* times in t periods goes to zero as t becomes large. A similar argument applies if $(k_0, u_0) \in T_1$.

Next, we have the final lemma.

LEMMA 10: The set $B = (\bar{k}, \bar{k}) \times E$ constitutes the single ergodic set.

PROOF: It has been shown (Lemma 9) that all ergodic sets are contained in B . To show that B is a single ergodic set, it is sufficient to show that for some $t \geq 1$, all $(k_0, u_0) \in B$, and all non-degenerate rectangular subsets A of B , that

$$\text{pr} [(k_t, u_t) \in A | (k_0, u_0)] > 0.$$

Let $A = \{(k, u) \in B: k_a \leq k \leq k_b, u_a \leq u \leq u_b\}$, where $k \leq k_a < k_b \leq \bar{k}$ and $u_a < u_b, u_a, u_b \in E$. Suppose, to be specific, that $k_0 < k_a$. Then for any finite t one can choose $\theta_2 > \theta_1 > 1$ such that $k_0\theta_1^t = k_a$ and $k_0\theta_2^t = k_b$, and $\lim_{t \rightarrow \infty} \theta_1 = 1$. By choosing t large enough, the event $h[x(k, u)/k] \geq \theta_1$ has positive probability for all $k \leq k_a$, so that the event

$$\theta_1 \leq h[x(k, u)/k] \leq \theta_2$$

has positive probability. Hence the probability that this event will occur t consecutive times is positive, as is the probability that $u_a \leq u_t \leq u_b$, for any u_{t-1} . Hence the lemma is proved.

Lemmas 9 and 10 accomplish the division of the set $(0, \infty) \times E$ of possible states of the $\{k_t, u_t\}$ process into two transient sets and a single ergodic set. In doing this, nothing has been assumed to assure that the ergodic set B is non-empty, or equivalently, that $\bar{x}(k) = \delta k$ has a positive solution \bar{k} . The following theorems characterize the long-run behavior of the system under the hypothesis that B is non-empty. We return to the questions of verifying this hypothesis, and of characterizing the system when it is false, below.

THEOREM 3: *If B is not empty, for all (k, u) and all (k_0, u_0) , the limit*

$$\lim_{t \rightarrow \infty} \text{pr} \{k_t \leq k, u_t \leq u | (k_0, u_0)\} = P(k, u)$$

exists, and does not depend on (k_0, u_0) . The function $P(k, u)$ is a probability distribution on $(0, \infty) \times E$, assigning probability zero to all subsets of $T_1 \cup T_2$ and positive probability to all Borel subsets of B with positive area.

PROOF: See [6, Theorem 5.7, p. 214, and surrounding text].

Since under the limiting distribution, k_t lies on a bounded interval with probability 1, the mean of k, k^* say, exists and is positive. Then we have the following "stability" theorem.

THEOREM 4: *If B is not empty, then for any initial state (k_0, u_0) ,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^T k_t = k^*$$

with probability 1.

PROOF: The result follows from [6, Theorem 6.2, p. 220].

To verify the hypotheses of Theorems 3 and 4, we wish to determine when the equation $\bar{x}(k) = \delta k$ has a positive solution. This question is analogous to the question of the existence of a positive solution to $x(k) = \delta(k)$ studied (and solved) in the preceding section. We have the following theorem.

THEOREM 5: The ergodic set E is non-empty ($\bar{x}(k) = \delta k$ has a positive solution) if and only if

$$(41) \quad \lim_{u \rightarrow \infty} D(0, u) > \delta + \frac{r}{h(\delta)}.$$

PROOF: The proof follows that of Lemma 5, with the operator $\int f(u)p(du)$ on functions f replaced by $\lim_{u \rightarrow b} f(u)$. This replacement is justified by Lemma 7.

If (41) fails to hold, capital stock in the industry will go to zero with probability one, since $x(k_t, u_t) < \delta k_t$ for all (k_t, u_t) . Thus Theorems 3, 4, and 5 provide a complete description of the long-run behavior of the industry under the assumptions of this section.

8. CONCLUSIONS

The object of this paper has been the extension of "cost-of-adjustment" type investment theory to situations involving demand uncertainty. In doing so, we have tried to go beyond formulations of the "price expectations affect supply, which in turn affects actual price" variety, to consider the simultaneous determination of anticipated and actual prices. This involves studying the determination of industry equilibrium, in addition to the individual firm's optimizing behavior, a step which radically alters the nature of the problem.

Generally, the equilibrium behavior of capital stock, output, and price through time is similar to the certainty case studied in [14]: the interplay of shifting demand and the costs of varying capacity leads to a difference equation in capital stock. The stationary character of the demand shifts leads capital stock to settle down, either with certainty or "on average," to a long-run equilibrium level, determined by interest rates, adjustment costs, and average demand levels.

An interesting feature of our theory is the role played by securities prices in informing firms of the market (not "shadow") price placed on additions to capital stock. We have found (Section 3) that securities prices as a variable in a firm level investment function have a much stronger justification than simply as a "proxy" variable for future demand.

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