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The q-Numerical Range of a Certain 3×3 Matrix

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Abstract

The paper provides the equation of the boundary of the q-numerical range of a certain 3×3 unitarily irreducible matrix.

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1. Introduction and the main result

In this decade, some authors have dealed with the q-numerical range of a matrix and have obtained interesting results (cf.[8],[9],[10],[4],[5]). Let A be an $n \times n$ complex matrix and $0 \leq q \leq 1$. The q-numerical range of A is defined and denoted by

$$W_q(A) = \{\zeta^* A \xi : \xi, \zeta \in \mathbf{C}^n, \xi^* \xi = \zeta^* \zeta = 1, \zeta^* \xi = q\}. \tag{1.1}$$

In [12] Tsing shows that the range $W_q(A)$ satisfies the formula

$$W_q(A) = \{q\xi^* A\xi + \sqrt{1-q^2}w\sqrt{\xi^* A^* A\xi - \xi^* A\xi} : w \in \mathbf{C}, |w| \leq 1$$

$$\xi \in \mathbf{C}^n, \xi^*\xi = 1\}, \tag{1.2}$$

and using this formula he shows the range $W_q(A)$ is convex. The q-numerical range is unitary similarity invariant, $W_q(A) = W_q(U\,A\,U^*)$, for any unitary matrix U. It is also transpose invariant,$W_q(A) = W_q(A^T)$. From these viewpoints, the set $W_q(A)$ is a natural generalization of the classical numerical range $W(A) = W_1(A)$ which was introduced by Toeplitz [12]. As a consequence of Tarski-Seidenberg theorem (cf.[2]), the boundary of the set $W_q(A)$ lies on an algebraic curve. The range $W_q(A)$ is closedly related with the following homogeneous polynomial:

$$F_A(X, Y, Z, W) = \det(WI_n + X/2(A + A^*) - iY/2(A - A^*) + ZA^*A). \tag{1.3}$$

If the same size square matrices A, B satisfies $F_A = F_B$, then the equation $W_q(A) = W_q(B)$ holds for $0 \leq q \leq 1$. In [10], [4], it is shown that the range $W_q(A)$ is determined by the set

$$W(A, A^*A) = \{(\xi^*A\xi, \xi^*A^*A\xi) \in \mathbf{C} \times \mathbf{R} : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}. \tag{1.4}$$

Especially, if A, B are respective $n\times n$, $m\times m$ matrices satisfying $W(A, A^*A) = W(B, B^*B)$, then the equation $W_q(A) = W_q(B)$ holds for $0 \leq q \leq 1$. The set $W(A, A^*A)$ is called the Davis-Wielandt shell for A (cf. [4). By results of [1] or [3], the Davis-Wielandt shell $W(A, A^*A)$ is convex for $n \geq 3$. If $n = 2$, its boundary surrounds a convex region. If same seize square matrices A, B satisfies the equation

$$F_A(X, Y, Z, 1) = F_B(X, Y, Z, 1)$$

for every $(X, Y, Z) \in \mathbf{R}^3$, then $W(A, A^*A) = W(B, B^*B)$. If A is an $n \times n$ matrix, then the boundary of $W(A, A^*A)$ lies on an algebraic surface of order less than or equal to $n(n - 1)^2$.

By using Tsing's circular union formula (1.2), we can perform a numerical approximation for a rather wide class of matrices A.The equation of the boundary of $W_q(A)$ for a 3×3 unitarily reducible matrix A is known (cf. [7],[5],[6]). However the equation of the boundary of $W_q(A)$ for a unitarily irreducible 3×3 matrix A is not known yet. We will provide a typical example of such an object. Let

$$N = \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

We obtain the exact equation of $\partial W_q(N)$. This example provides a starting point to the general study of the q-numerical of unitarily irreducible 3×3 matrices.

Theorem 1.1 *Suppose that N is a 3×3 nilpotent matrix given by (1.5) and q is a a real number $0 \leq q \leq 1$. (1) If $q = 1$, then the boundary of $W_1(N)$ lies on the sextic algebraic curve*

$$\{x+iy : (x,y) \in \mathbf{R}^2, 11664x^6+28080x^4y^2+21168x^2y^4+4752y^6+7776x^5+15552x^3y^2$$

$$+7776xy^4 - 4833x^4 - 6210x^2y^2 - 1377y^4 - 5704x^3 - 4680xy^2$$

$$-1944x^2 - 648y^2 - 288x - 16 = 0\}.$$

(2) If $q = 0$, then the boundary of $W_0(N)$ is the circle

$$\{x + iy \in \mathbf{C} : |x + iy| = \frac{27}{22}\}.$$

(3) If $0 < q \leq 1/2$ or $13/14 \leq q < 1$, then the boundary of $W_q(N)$ lies on the algebraic curve defined by $K(x, y : q) = 0$ for a polynomial $K(x, y : q)$ of degree 14 defined by (4.6).

(4) If $1/2 < q < 13/14$, then the boundary of $W_q(N)$ consists of two arcs. One part is a circular arc

$$\{x+iy : (x,y) \in \mathbf{R}^2, (x-\frac{2q}{3})^2+y^2 = (\frac{4}{3})^2, \frac{2q}{3}-\frac{4}{3} \leq x \leq \frac{-5 + 2q^2 + 2\sqrt{3}\sqrt{1 - q^2}}{q}\}.$$

Another part lies on the curve $K(x, y : q) = 0$ and satisfies $x = \Re(x + iy) > (-5 + 2q^2 + 2\sqrt{3}\sqrt{1 - q^2})/q$.

We prove this theorem in sections 2-4.

2. Another expression of the q-numerical range

A fundamental principle to perform the computation of the boundary of the range can be found in [13], [10], [4]. Let A be an n-by-n matrix. We introduce a function h_A on the numerical range $W(A)$ by

$$h_A(z) = \max\{t \in \mathbf{R} : (z, t) \in W(A, A^*A)\}. \tag{2.1}$$

By the convexity of $W(A, A^*A)$ for $n \geq 3$ or the convexity of the region surounded by $W(A, A^*A)$, this function is concave on $W(A)$. Hence it is continuous in the interior of $W(A)$. By Schwarz's inequality,

$$|\xi^* A\xi|^2 \leq \langle A\xi, A\xi\rangle = \xi^* A^* A\xi$$

and hence $h(z) - |z|^2 \geq 0$. As the sum of two concave functions, the function $z \mapsto h(z) - |z|^2$ is a non-negative concave function. It follows from the concaveness of the function $s \in [0, \infty) \mapsto \sqrt{s}$ that the relation

$$\begin{aligned}&(h(tz_1 + (1-t)z_2) - |tz_1 + (1-t)z_2|^2)^{1/2}\\ &\geq (t(h(z_1) - |z_1|^2) + (1-t)(h(z_2) - |z_2|^2))^{1/2}\\ &\geq t(h(z_1) - |z_1|^2)^{1/2} + (1-t)(h(z_2) - |z_2|^2)^{1/2}\end{aligned}$$

holds for $z_1, z_2 \in W(A)$, $0 \leq t \leq 1$. Hence the function $(h(z) - |z|^2)^{1/2}$ on $W(A)$ is also concave, and the formula (1.2) is rewritten as

$$W_q(A) = \{qz + \sqrt{1-q^2}w\sqrt{h(z) - |z|^2} : z \in W(A), w \in \mathbf{C}, |w| \leq 1\}. \quad (2.2)$$

We introduce a compact convex set $\Gamma(A)$ by

$$\Gamma(A) = \{(x_1, x_2, u_1, u_2) \in \mathbf{R}^4 : x_1 + ix_2 \in W(A), x_1^2 + x_2^2 + u_1^2 + u_2^2 \leq h(x_1 + ix_2)\}. \quad (2.3)$$

Define an orthogonal projection Π_q of $\mathbf{R}^4$ onto $\mathbf{C} \cong \mathbf{R}^2$ by

$$\Pi_q(x_1, x_2, u_1, u_2) = (qx_1 + \sqrt{1-q^2}u_1) + i(qx_2 + \sqrt{1-q^2}u_2). \quad (2.4)$$

Then the formula (2.2) becomes

$$W_q(A) = \Pi_q(\Gamma(A)). \quad (2.5)$$

3. Boundary of the Davis-Wielandt shell

We shall determine the equation of the boundary of the Davis-Wielandt shell $W(N, N^*N)$. We use the equation

$$\{ux + vy + wz : (x, y, z) \in \mathbf{R}^3, (x + iy, z) \in W(N, N^*N)\}$$

$$= [\lambda_3(u\Re(N) + v\Im(N) + wN^*N), \lambda_1(u\Re(N) + v\Im(N) + wN^*N)], \quad (3.1)$$

for every non-zero vector $(u, v, w) \in \mathbf{R}^3$, where

$$\lambda_1(H) \geq \lambda_2(H) \geq \lambda_3(H)$$

are eigenvalues of a 3×3 Hermitian matrix H. By this equation, every boundary point $(x + iy, z)$ of $W(N, N^*N)$ lies on the dual surface DS of the surface $F_N(X, Y, Z, 1) = 0$ or a tangent plane of the surface DS. By the elimination method, we compte that the dual surface DS is defined by a quartic polynomial

$$L(x, y, z) = 272x^4 + 528x^2y^2 + 256y^4 + 168x^3z + 168x\,y^2\,z + 249x^2\,z^2 + 249y^2\,z^2 + 132z^4$$
$$+32x^3 + 32x\,y^2 - 360x^2\,z - 344y^2\,z - 168x\,z^2 - 233z^3 + 16x^2 + 16y^2$$
$$-32x\,z + 88z^2 - 16z. \tag{3.2}$$

The dual surface $L(x, y, z) = 0$ has a tangent plane

$$L_0 = 3z - 4x - 4 = 0. \tag{3.3}$$

This plane and the quartic surface meet at the ellipse E_0:

$$\frac{49}{4}(x + \frac{2}{7})^2 + \frac{21}{4}y^2 = 1, \tag{3.4}$$

on the above plane. On the interior of the convex domain $W(N)$, the function $\psi(x, y) = \sqrt{z - (x^2 + y^2)}$ is continuously differentiable. On the closed domain E bounded by the above ellipse, the function $\psi(x, y) = \sqrt{z - (x^2 + y^2)}$ is represented by

$$\psi(x, y) = \frac{4}{3}[1 - \frac{9}{16}(x - \frac{2}{3})^2 - \frac{9}{16}y^2]^{1/2}, \tag{3.5}$$

and its graph lies on an ellipsoid. For $0 < q < 1$, we consider the set

$$\Delta_q(N) = \{x + iy : (x, y) \in \mathbf{R}^2, (\frac{d\psi}{dx})^2 + (\frac{d\psi}{dy})^2(x, y) = \frac{q^2}{1 - q^2}\}. \tag{3.6}$$

Then the boudary of $W_q(N)$ satisfies

$$\partial W_q(N) \subseteq \{q\,(x + iy) + \sqrt{1 - q^2}w\psi(x, y) : x + iy \in \Delta_q(N), w \in \mathbf{C}, |w| = 1\}. \tag{3.7}$$

(cf. [4], Theorem 2). The intersection of $\Delta_q(N)$ with the elliptical disc E:

$$\frac{49}{4}(x + \frac{2}{7})^2 + \frac{21}{4}y^2 \leq 1, \tag{3.8}$$

is non-empty if and only if $1/2 \le q \le 13/14$. In the case $q = 1/2$ or $q = 13/14$, this set is a singleton on the ellipse $(49/4)(x + \frac{2}{7})^2 + (21/4)y^2 = 1$. In the case $1/2 < q < 13/14$, the intersection coincides with the intersection of the circle

$$\{x + iy : (x, y) \in \mathbf{R}^2, (x - \frac{2}{3})^2 + y^2 = (\frac{4q}{3})^2\}$$

and the elliptical disc $(49/4)(x+\frac{2}{7})^2+(21/4)y^2 \le 1$. The function ψ on $W(N)$ attains its maximum $27/22$ at $(x, y) = (3/11, 0)$. Corresponding to this fact, the range $W_0(N)$ is the circular disc $|z| \le 27/22$. The boundary of $W_1(N)$ lies on the dual curve of the algebraic curve

$$16 \det(I_3 + x\Re(N) + y\Im(N)) = 2x^3 + 2xy^2 - 9x^2 - 9y^2 + 16 = 0.$$

Hence the boundary of $W_1(N) = W(N)$ lies on the sextic curve in Theorem 1.1 (1).

4. Proof of Theorem 1.1

We shall prove Theorem 1.1 for $0 < q < 1$, that is (3), (4) of Theorem 1.1. By (2.5), every point $x + iy$ of $\partial W_q(N)$ is expreesed as

$$x = q\,X_0 + \sqrt{1-q^2}\,U, \quad y = q\,Y_0 + \sqrt{1-q^2}\,V, \tag{4.1}$$

for some boundary point (X_0, Y_0, U, V) of $\Gamma(N)$. If $0 < q \le 1/2$ or $13/14 \le q < 1$, then the above (X_0, Y_0, U, V) is chosen so that

$$L_0(X, Y, U, V) = L(X, Y, X^2 + Y^2 + U^2 + V^2) = 0, \tag{4.2}$$

and

$$L_1(X, Y : x, y, q) = L_0(X, Y, \frac{1}{\sqrt{1-q^2}}(x - qX), \frac{1}{\sqrt{1-q^2}}(y - qY), \tag{4.3}$$

satisfies

$$\frac{\partial L_1}{\partial X}(X_0, Y_0 : x, y, q) = \frac{\partial L_1}{\partial Y}(X_0, Y_0 : x, y, q) = 0. \tag{4.4}$$

By the relation (3.7), the boundary of $W_q(N)$ is also viewed as the envelop of the 1-parameter family of circles. In the case $1/2 < q < 13/14$, the set $\Delta_q(N) \cap E$ is a circular arc. But the whole $\Delta_q(N)$ is not contained in E. One part of the boundary of $W_q(N)$ satisfies the above equastions. Another part of $\partial W_q(N)$ lies on the outer envelope of the 1-parameter family of the circles:

$$\{x+iy : (x,y) \in \mathbf{R}^2, (X,Y) \in \mathbf{R}^2, (x-qX)^2+(y-qY)^2$$
$$= (1-q^2)\frac{4+4X-3X^2-3Y^2}{3},$$
$$(X-\frac{2}{3})^2+Y^2 = (\frac{4q}{3})^2, \frac{49}{4}(X+\frac{2}{7})^2+\frac{21}{4}Y^2 \le 1\}.$$

and hence it lies on the circle

$$(x-\frac{2q}{3})^2+y^2 = (\frac{4}{3})^2, \tag{4.5}$$

The main computations to obtain the boundary of the equation of $W_q(N)$ ($0 < q < 1$) consist in the eliminations of X_0, Y_0 from the equations (4.3), (4.4). We perform the successive eliminations of Y_0 and X_0. The equation of the envelope of the family of circles is obtained as a simple factor of the resultant of $K_0(x,y : X,q)$ and $\partial K_0(x,y : X,q)/\partial X$ with respect to X, where $K_0(x,y : X,q)$ is a simple factor of the resultant of $L_1(X,Y : x,y)$ and $\partial L_1(X,Y : x,y,q)/\partial Y$ with respect to Y. By direct long computations by a computer, we obtain the equation of the boundary of $W_q(N)$

$$K(x,y:q) = 203233536\,(x^2+y^2)^5\,\{(11q^2+16)^2x^4+2(297q^4-256q^2+256)x^2\,y^2+(27q^2-16)^2y^4\}$$
$$+98537472\,q\,x\,(x^2+y^2)^4\,\{(-605q^4+616q^2+2176)x^4+(-2046q^4+664q^2+4352)x^2y^2$$
$$+(-1377q^4+48q^2+2176)y^4\}+23328\,(x^2+y^2)^4\,\{(3147089q^6-5573106q^4+6132288q^2-4350464)x^4$$
$$+(8544690q^6-31872868q^4+32253440q^2-8700928)x^2\,y^2+(8109153q^6-30362994q^4+26121152q^2$$
$$-4350464)y^4\}+15552q\,x\,(x^2+y^2)^3\,\{(-4146109q^6+13793770q^4+5518656q^2-28651008)x^4$$
$$+(-12162138q^6+25620692q^4+27657600q^2-57302016)x^2\,y^2+(-6619293q^6+9033450q^4$$
$$+22138944q^2-28651008)y^4\}+81(x^2+y^2)^2\,\{(523461745q^8-2124877888q^6+6290634240q^4$$
$$-7032029184q^2+429981696)x^6+(1782657363q^8-9976494272q^6+24892826112q^4$$
$$-21265956864q^2+1289945088)x^4\,y^2+(2777208147q^8-14892417216q^6+30795194880q^4$$
$$-21435826176q^2+1289945088)x^2\,y^4+(1494943857q^8-6994663488q^6+12193003008q^4$$
$$-7201898496q^2+429981696)y^6\}+108q\,x\,(x^2+y^2)^2\{(-232777441q^8+851960576q^6$$
$$+1001346560q^4-3116924928q^2+1289945088)x^4+(-797967810q^8+1586611712q^6$$
$$+2819058688q^4-6506348544q^2+2579890176)x^2\,y^2+(-433553121q^8+631431936q^6+1816598016q^4$$
$$-3389423616q^2+1289945088)y^4\}+36q^2\,(x^2+y^2)\{(432453157q^8-2256714240q^6+8021457408q^4$$

$$
\begin{aligned}
&-9000419328q^2 + 3869835264)x^6 + (1502724999q^8 - 10675428864q^6 + 27841488384q^4 \\
&-28747726848q^2 + 11609505792)x^4\, y^2 + (2399499423q^8 - 14470295040q^6 + 31425928704q^4 \\
&-30409261056q^2 + 11609505792)x^2\, y^4 + (1271555901q^8 - 6051580416q^6 + 11605897728q^4 \\
&-10661953536q^2 + 3869835264)y^6\} + 192q^5\, x\{(-43100059q^6 + +187633440q^4 - 41534208q^2 \\
&+81616896)x^6 + (-179125119q^6 + 365573088q^4 - 17701632q^2 + 205037568)x^4\, y^2 + (-196049709q^6 \\
&+170176608q^4 + 89199360q^2 + 165224448)x^2\, y^4 + (-60745545q^6 - 7763040q^4 + 65366784q^2 \\
&+41803776)y^6\} + 32q^6\{(116960873q^6 - 428634560q^4 + 1430279424q^2 - 579280896)x^6 \\
&+(386391087q^6 - 2381624640q^4 + 4586257152q^2 - 1737842688)x^4\, y^2 + (619346763q^6 - 3365520192q^4 \\
&+4835669760q^2 - 1737842688)x^2\, y^4 + (334781829q^6 - 1412669376q^4 + 1679692032q^2 - 579280896)y^6\} \\
&+384q^7\, x\{(-3419435q^6 + 9806144q^4 + 28189440q^2 - 19906560)x^4 + (-9847230q^6 \\
&+220032q^4 + 66332160q^2 - 39813120)x^2\, y^2 + (-4967739q^6 - 9058752q^4 + 37847808q^2 - 19906560)y^4\} \\
&+384q^8\{(1034883q^6 - 866560q^4 + 11135232q^2 - 7962624)x^4 + (4363650q^6 - 17044992q^4 \\
&+31117824q^2 - 15925248)x^2\, y^2 + (3766455q^6 - 14683392q^4 + 19097856q^2 - 7962624)y^4\} \\
&+1024q^{11}\, x\{(-212641q^4 + 1266112q^2 - 857088)x^2 + (-122283q^4 + 727488q^2 - 525312)y^2\} \\
&+768q^{12}\{(62927q^4 + 37696q^2 - 73728)x^2 + (111639q^4 - 181440q^2 + 73728)y^2\} \\
&+3072q^{15}\,(2839q^2 - 2432)x + 1024q^{16}\,(289q^2 - 256).
\end{aligned}
\tag{4.6}
$$

In the case $0 < q \leq 1/2$ or $13/14 \leq q < 1$, we substitute (4.5) into (4.6). By this way, we determinte the chaging points $(x_1, \pm y_1)$ of the circular arc of $\partial W_q(N)$ and another part of $\partial W_q(N)$. It satisfies

$$9q^2\, x_1^2 + (-12q^3 + 30q)\, x_1 + (4q^4 - 8q^2 + 13) = 0,$$

and $x_1 \geq (2q/3) - 4/3$, hence

$$x_1 = \frac{-5 + 2q^2 + 2\sqrt{3}\sqrt{1-q^2})}{q}.$$

Thus we proved Theorem 1.1.

We present two graphics. Figure 1 is produced by plotting the contour of some function. Figure 2 is produced by using the formula (2.2).

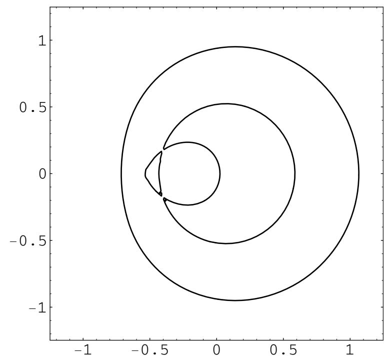

Figure 1: the graphic of the curve $K(x, y : 13/14) = 0$.

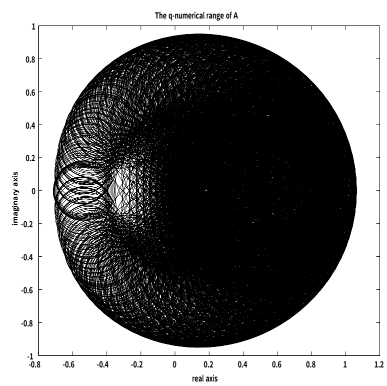

Figure 2: the numerical aproximation of the range $W_{13/14}(N)$ of N given by (1.5) by using C. K. Li's computer program ([11]).

References

[1] Y.-H. Au-Yeung and N. K. Tsing, An extension of the Hausdorff-Toeplitz theorem on the numerical range, Proc. Amer. Math. Soc., 89(1983), 215-218.

[2] R. Benedetti and J. J. Rister, Real algebraic and semi-algebraic sets, Hermann, Paris, 1990.

[3] P. Binding, Hermitian forms and the fibration of spheres, Proc. Amer. Math. Soc., 94(1985), 581-584.

[4] M. T. Chien and H. Nakazato, Davis-Wielandt shell and q-numerical range, Linear Algebra Appl., 340(2002), 15-31.

[5] M. T. Chien and H. Nakazato, The q-numerical range of a reducible matrix via a normal operator, Linear Algebra Appl., 419(2006), 440-465.

[6] M. T. Chien and H. Nakazato, The boundary of the q-numerical range of a reducible matrix, Linear and Multilinear Algebra, 55(2007), 275-292.

[7] M. T. Chien, H. Nakazato and P. Psarrakos, The q-numerical range and the Davis-Wielandt shell of reducible 3x3 matrices, Linear and Multilinear Algebra, 54(2006), 79-112.

[8] C. K. Li, q-numerical ranges of normal and convex matrices. Linear and Multilinear Algebra, 43(1998), 377-384.

[9] C.K. Li, P.P. Mehta abd L. Rodman, A generalized numerical range: the range of a constrained sesquilinear form, Linear and Multilinear Algebra, 37(1994), 25-49.

[10] C. K. Li and H. Nakazato, Some results on the q-numerical range, Linear and Multilinear Algebra, 43(1998), 385-409.

[11] C. K. Li, the q-numerical range of a general matrix, Matlab program presented in his home page.

[12] O. Toeplitz, Das algebraische Analogon zu einer Satze von Fejée, Math. Zeit., 2(1918), 187-197.

[13] N. K. Tsing, The constrained bilinear form and C-numerical range, Linear Algebra Appl., 56(1984), 195-206.

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