

Symplectic connections induced by the Chern connection

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Abstract. Let (M, ω) be a symplectic manifold and F be a Finsler structure on M . In the present paper we define a lift of the symplectic two-form ω on the manifold $TM \setminus 0$, and find the conditions that the Chern connection of the Finsler structure F preserves this lift of ω . In this situation if M admits a nowhere zero vector field then we have a non-empty family of Fedosov structures on M .

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1 Introduction

Connections are important objects in differential geometry. In 1917 a connection introduced on a manifold embedded into R^n by Levi-Civita. One year later H. Weyl introduced general symmetric linear connections in the tangent bundle. In 1922 E. Cartan studied non-symmetric linear connection which was applied in general relativity as a drastic tool ([6]).

An interesting field about the connections in differential geometry is the relation between connections and the other structures on a manifold. One of these structures is the symplectic structure on the manifolds. A symplectic connection is a symmetric connection which preserves the symplectic form. Many mathematicians worked on the symplectic connections. They found many interesting results about the relations of symplectic connections and the other structures on the manifolds and supermanifolds (see [2], [3], [5], [6], [7] and [8].).

Let (M, ω) be a symplectic manifold. A natural question is: "Which connections preserve ω ?"

Conversely, let ∇ be a connection on a manifold M , we can ask ourselves: "Which nondegenerate closed two-forms ω exist such that ∇ preserves them?"

I. Gelfand, V. Retakh and M. Shubin have obtained some interesting results about the first question in [6]. In this paper we try to answer to the second question in a special case.

Let (M, F) be a Finsler manifold also at the same time let (M, ω) be a symplectic

manifold. Suppose that $\tilde{\nabla}$ is the Chern connection arising from the Finsler structure F . We define a canonical lift of ω on the manifold $TM \setminus 0$ to find the conditions for ω , such that $\tilde{\nabla}$ preserves the canonical lift of ω . In this case if M admits a nowhere zero vector field then we have a non-empty family of symplectic connections on M .

2 Preliminaries and notations

In this section we give some important definitions and theorems which we need in the following. In the first, we review some of general definitions and theorems of symplectic geometry.

Definition 2.1. A *symplectic form* (or a *symplectic structure*) on a manifold M is a nondegenerate, closed two-form ω on M . A *symplectic manifold* (M, ω) is a manifold M together with a symplectic form ω on M (For more details see [1]).

Definition 2.2. Assume (M, ω) is a symplectic manifold. Let ∇ be a connection (covariant derivative) on M . We say that ∇ *preserves* ω if $\nabla\omega = 0$ or

$$Z(\omega(X, Y)) = \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y),$$

for any vector fields X, Y, Z (For more details see [6]).

Definition 2.3. If ∇ is symmetric and preserve the given symplectic form ω then we say that ∇ is a *symplectic connection*.

Definition 2.4. *Fedosov manifold* is a symplectic manifold with a given symplectic connection. We show a Fedosov manifold by (M, ω, ∇) , where ∇ is the symplectic connection such that preserves ω .

Theorem 2.5. (Darboux) Suppose ω is a nondegenerate two-form on a $2n$ -manifold M . Then $d\omega = 0$ if and only if there is a chart (U, ϕ) at each $m \in M$ such that $\phi(m) = 0$, and with $\phi(u) = (x^1(u), \dots, x^{2n}(u))$ we have

$$(2.1) \quad \omega|_U = \sum_{i=1}^n dx^i \wedge dx^{n+i}.$$

(For more details see [1].)

Now we give some fundamental concepts of Finsler geometry.

Definition 2.6. A Minkowski norm on R^n is a nonnegative function $\mathcal{F} : R^n \rightarrow [0, \infty)$ which has the following properties:

(i) \mathcal{F} is C^∞ on $R^n \setminus \{0\}$.

(ii) $\mathcal{F}(\lambda y) = \lambda \mathcal{F}(y)$ for all $\lambda > 0$ and $y \in R^n$.

(iii) The $n \times n$ matrix (g_{ij}) , where $g_{ij}(y) := [\frac{1}{2}\mathcal{F}^2]_{y^i y^j}(y)$, is positive-definite at all $y \neq 0$.

Definition 2.7. Let M be an n -dimensional smooth manifold and TM the tangent bundle of M . A function $F : TM \rightarrow [0, \infty)$ is called a Finsler metric if it has the following properties:

- (i) F is \mathcal{C}^∞ on the slit tangent bundle $TM \setminus 0$.
- (ii) For each $x \in M$, $\mathcal{F}_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$.

If the Minkowski norm satisfies $\mathcal{F}(-y) = \mathcal{F}(y)$, then one has the absolutely homogeneity $F(\lambda y) = |\lambda|F(y)$, for any $\lambda \in \mathbb{R}$. Every absolutely homogeneous Minkowski norm is a norm in the sense of functional analysis. Every Riemannian manifold (M, g) by defining

$$F(x, y) := \sqrt{g_x(y, y)} \quad x \in M, y \in T_x M$$

is a Finsler manifold.

Also we use of the following relations:

$$\begin{aligned} g_{ij} &:= \left(\frac{1}{2}F^2\right)_{y^i y^j} = FF_{y^i y^j} + F_{y^i} F_{y^j}, \\ A_{ijk} &:= \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} (F^2)_{y^i y^j y^k}, \\ \frac{\delta y^i}{F} &:= \frac{1}{F} (dy^i + N_j^i dx^j), \end{aligned}$$

where the N_j^i are collectively known as the nonlinear connection.

Theorem 2.8. (Chern) Let (M, F) be a Finsler manifold. The pulled-back bundle π^*TM admits a unique linear connection, called the Chern connection. Its connection forms (θ_j^i) are characterized by the structural equations:

$$* \text{ Torsion freeness: } d(dx^i) - dx^j \wedge \theta_j^i = -dx^j \wedge \theta_j^i = 0,$$

$$* \text{ Almost } g\text{-compatibility: } dg_{ij} - g_{kj} \theta_i^k - g_{ik} \theta_j^k = 2A_{ijs} \frac{\delta y^s}{F}.$$

(For more details see [4].)

Also we use of the following notations.

Let (x^i) and (\hat{x}^i) , $i = 1, \dots, n$ be local coordinates, we use of the following notations:

- (1) $\partial_i := \frac{\partial}{\partial x^i}$ and $\hat{\partial}_i := \frac{\partial}{\partial \hat{x}^i}$.
- (2) $\omega_{ij} := \omega(\partial_i, \partial_j)$.
- (3) $\nabla_i := \nabla_{\partial_i}$.
- (4) (*Christoffel symbols* Γ_{ij}^k), $\nabla_i \partial_j = \Gamma_{ij}^k \partial_k$ for symplectic connections.
- (5) (*Christoffel symbols* $\tilde{\Gamma}_{ij}^k$), $\tilde{\nabla}_i \tilde{\partial}_j = \tilde{\Gamma}_{ij}^k \tilde{\partial}_k$ for the Chern connection $\tilde{\nabla}$.

3 Symplectic connections induced by the Chern connection.

For Christoffel symbols of a symplectic connection we have the following theorem.

Theorem 3.1. *Let (M, ω, ∇) be a Fedosov manifold. In Darboux coordinate $(x^i)_{i=1}^{2n}$ for any $k = 1, \dots, 2n$ we have:*

$$\begin{aligned}\Gamma_{kj}^{i+n} &= \Gamma_{ki}^{j+n} & 1 \leq i \leq n, 1 \leq j \leq n, \\ \Gamma_{kj}^{i+n} &= -\Gamma_{ki}^{j-n} & 1 \leq i \leq n, n+1 \leq j \leq 2n, \\ \Gamma_{kj}^{i-n} &= -\Gamma_{ki}^{j+n} & n+1 \leq i \leq 2n, 1 \leq j \leq n, \\ \Gamma_{kj}^{i-n} &= \Gamma_{ki}^{j-n} & n+1 \leq i \leq 2n, n+1 \leq j \leq 2n.\end{aligned}$$

Proof. It suffices to use the equation $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$ and use of it in the following equation:

$$0 = \partial_k \omega(\partial_i, \partial_j) = \omega(\nabla_k \partial_i, \partial_j) + \omega(\partial_i, \nabla_k \partial_j).$$

□

Now we define a canonical lift of forms on the manifold $TM \setminus 0$.

Definition 3.2. Suppose that ω is a k -form on a manifold M of dimension n . In this case we define a k -form $\tilde{\omega}$ on the manifold $TM \setminus 0$, associated to pulled-back bundle π^*TM , in the following way and call it the *canonical lift of ω on π^*TM* :

$$\begin{aligned}\tilde{\omega}_{(x,y)} &: \overbrace{(\pi^*TM)_{(x,y)} \times \cdots \times (\pi^*TM)_{(x,y)}}^{k\text{-times}} \rightarrow R \\ \tilde{\omega}_{(x,y)}(z_1, \dots, z_n) &= \omega_x(z_1, \dots, z_n).\end{aligned}$$

For any $x \in M, y \in T_x M \setminus 0$ and $z_1, \dots, z_n \in T_x M$.

Definition 3.3. Let $X \in \mathcal{X}(M)$ be a vector field on M . We define a section \tilde{X} of π^*TM in the following way and call it the *canonical lift of X* :

$$\tilde{X}_{(x,y)} = X_x.$$

For any $x \in M$ and $y \in T_x M \setminus 0$. We use of the notation ∂_i with sense $\tilde{\partial}_i$.

Definition 3.4. Let (M, ω) be a symplectic manifold with a Finsler structure F on it. In this case we call the triple (M, ω, F) a *Finslerian symplectic manifold*.

Definition 3.5. Suppose that (M, ω, F) is a Finslerian symplectic manifold and $\tilde{\nabla}$ is the Chern connection of (M, F) . We say that $\tilde{\nabla}$ preserves ω if $\tilde{\nabla}$ preserves $\tilde{\omega}$, the canonical lift of ω , in the other words if for any $X, Y, Z \in \mathcal{X}(M)$ we have

$$\tilde{Z}(\tilde{\omega}(\tilde{X}, \tilde{Y})) = \tilde{\omega}(\tilde{\nabla}_{\tilde{Z}} \tilde{X}, \tilde{Y}) + \tilde{\omega}(\tilde{X}, \tilde{\nabla}_{\tilde{Z}} \tilde{Y}).$$

Where $\tilde{X}, \tilde{Y}, \tilde{Z}$ are the canonical lift of X, Y, Z and $\tilde{Z}(\tilde{\omega}(\tilde{X}, \tilde{Y})) := Z(\omega(X, Y))$. Suppose that $\tilde{\omega}_{ij} := \tilde{\omega}(\tilde{\partial}_i, \tilde{\partial}_j)$ then for $X = \partial_i, Y = \partial_j, Z = \partial_k$ we have

$$\tilde{\partial}_k \tilde{\omega}_{ij} = \tilde{\omega}_{il} \tilde{\Gamma}_{kj}^l - \tilde{\omega}_{jl} \tilde{\Gamma}_{ki}^l.$$

Theorem 3.6. Let (M, ω, F) be a Finslerian symplectic manifold and $\tilde{\nabla}$ be the Chern connection of (M, F) such that $\tilde{\nabla}$ preserves ω . Suppose that M admits a vector field $W \in \mathcal{X}(M)$ such that for any $x \in M$ we have $W_x \neq 0_{T_x M}$. Then there is a nonempty family of symplectic connections on M such that preserves ω , in the other words there is a nonempty family of Fedosov structures on M .

Proof. By using Christoffel symbols $\tilde{\Gamma}_{ij}^k$ of the Chern connection $\tilde{\nabla}$ on $TM \setminus 0$ and vector field W we define new Christoffel symbols Γ_{ij}^k on M in the following way:

$$\Gamma_{ij}^k(x) := \tilde{\Gamma}_{ij}^k(x, W_x).$$

Now we have a symmetric linear connection ∇ , generated by Christoffel symbols Γ_{ij}^k , on M such that preserves ω because for ∂_i, ∂_j and ∂_k we have:

$$\begin{aligned} \{\omega(\nabla_{\partial_k} \partial_i, \partial_j) + \omega(\partial_i, \nabla_{\partial_k} \partial_j)\}_x &= \{\omega(\Gamma_{ki}^l \partial_l, \partial_j) + \omega(\partial_i, \Gamma_{kj}^l \partial_l)\}_x \\ &= \{\tilde{\omega}(\tilde{\Gamma}_{ki}^l \tilde{\partial}_l, \tilde{\partial}_j) + \tilde{\omega}(\tilde{\partial}_i, \tilde{\Gamma}_{kj}^l \tilde{\partial}_l)\}_{(x, W_x)} \\ &= \{\tilde{\partial}_k \tilde{\omega}(\tilde{\partial}_i, \tilde{\partial}_j)\}_{(x, W_x)} \\ &= \{\partial_k \omega(\partial_i, \partial_j)\}_x. \end{aligned}$$

So ∇ is a symplectic connection on M . □

Corollary 3.7. In Theorem 3.6, if (M, F) is of Berwald type then the induced symplectic connection is unique.

Proof. By using the definition of Berwald spaces, Christoffel symbols of the Chern connection are functions only of variable $x \in M$ so the induced symplectic connection is not associated to vector field W . Therefore the induced symplectic connection is unique. □

Theorem 3.8. Let (M, ω, F) be a Finslerian symplectic manifold. Suppose that F is locally Minkowskian. Then the Chern connection preserves ω if and only if

$$\begin{aligned} \partial_k \omega_{ij} &= 0, \\ \partial_h \hat{x}^l \cdot \hat{\partial}_k \hat{\partial}_i x^h \cdot \omega_{lj} + \partial_h \hat{x}^l \cdot \hat{\partial}_k \hat{\partial}_j x^h \cdot \omega_{il} &= \hat{\partial}_k \omega_{ij}. \end{aligned}$$

Where $(x^i)_{i=1}^{2n}$ is the natural coordinates for the locally Minkowskian manifold M and $(\hat{x}^i)_{i=1}^{2n}$ is an arbitrary local coordinate on M .

Proof. In any Finsler manifold M we have

$$(3.1) \quad \hat{\Gamma}_{qr}^p = \partial_i \hat{x}^p \cdot \hat{\partial}_q \hat{\partial}_r x^i + \partial_i \hat{x}^p \cdot \tilde{\Gamma}_{jk}^i \cdot \hat{\partial}_q x^j \hat{\partial}_r x^k.$$

(See [4] p.43) In locally Minkowskian manifolds in natural coordinates we have $\tilde{\Gamma}_{jk}^i = 0$, so by using 3.5 and equation 3.1 we obtain the result. □

Now let us use the Randers metrics to obtain the Fedosov structure. Let $F(x, y) = \alpha(x, y) + \beta(x, y)$ be a Randers metric on a manifold M such that $d\beta$ be nondegenerate. Therefore $(M, d\beta)$ is a symplectic manifold. Now we calculate the conditions that the Chern connection preserves $d\beta$. Suppose that $\beta(x, y) = b_i(x)y^i$ or in other words $\beta(x) = b_i(x)dx^i$ and let $\omega = d\beta$, so we have

$$\omega = d(b_i dx^i) = db_i \wedge dx^i.$$

Therefore

$$\omega_{pq} = (db_i \wedge dx^i)(\partial_p, \partial_q) = \partial_p b_q - \partial_q b_p.$$

The Chern connection arising from $F = \alpha + \beta$, preserves ω if and only if

$$(\tilde{\Gamma}_{ki}^l \partial_l b_j - \tilde{\Gamma}_{kj}^l \partial_l b_i) + (\tilde{\Gamma}_{kj}^l \partial_i b_l - \tilde{\Gamma}_{ki}^l \partial_j b_l) + (\partial_k \partial_j b_i - \partial_k \partial_i b_j) = 0$$

and also in Darboux coordinate we have

$$(\tilde{\Gamma}_{ki}^l \partial_l b_j - \tilde{\Gamma}_{kj}^l \partial_l b_i) + (\tilde{\Gamma}_{kj}^l \partial_i b_l - \tilde{\Gamma}_{ki}^l \partial_j b_l) = 0.$$

4 Results from the curvature tensor

Let (M, ω, F) be a Finslerian symplectic manifold such that the Chern connection preserves ω and M admits a vector field $W \in \mathcal{X}(M)$ such that for any $x \in M$ we have $W_x \neq 0_{T_x M}$. Suppose that ∇ is the induced symplectic connection on M by the Chern connection and vector field W . We have the following relations for the curvature tensor of ∇ :

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

$$R(\partial_j, \partial_k) \partial_i = R_{ijk}^l \partial_l.$$

Therefore by using the Christoffel symbols we have:

$$\begin{aligned} R_{ijk}^l(x) &= \partial_j \Gamma_{ki}^l(x) - \partial_k \Gamma_{ij}^l(x) + \Gamma_{ki}^m(x) \Gamma_{jm}^l(x) - \Gamma_{ij}^m(x) \Gamma_{km}^l(x) \\ &= \partial_j \tilde{\Gamma}_{ki}^l(x, W(x)) - \partial_k \tilde{\Gamma}_{ij}^l(x, W(x)) \\ &+ \tilde{\Gamma}_{ki}^m(x, W(x)) \tilde{\Gamma}_{jm}^l(x, W(x)) - \tilde{\Gamma}_{ij}^m(x, W(x)) \tilde{\Gamma}_{km}^l(x, W(x)) \\ &= (D_j \tilde{\Gamma}_{ki}^l)(x, W(x)) + (D_p \tilde{\Gamma}_{ki}^l)(x, W(x)) \cdot \partial_j W^p(x) \\ &- (D_k \tilde{\Gamma}_{ij}^l)(x, W(x)) - (D_p \tilde{\Gamma}_{ij}^l)(x, W(x)) \cdot \partial_k W^p(x) \\ &+ \tilde{\Gamma}_{ki}^m(x, W(x)) \tilde{\Gamma}_{jm}^l(x, W(x)) \\ (4.1) \quad &- \tilde{\Gamma}_{ij}^m(x, W(x)) \tilde{\Gamma}_{km}^l(x, W(x)). \end{aligned}$$

Where in local coordinate (x^i) we have

$$W = W^p \partial_p,$$

and also D_p is derivative associated to component p .

Theorem 4.1. For any symplectic connection we have:

$$(4.2) \quad R_{ijkl} = R_{jikl}.$$

Proof. See [6]. □

Theorem 4.2. Let (M, ω, F) be a Finslerian symplectic manifold. If the Chern connection $\tilde{\nabla}$ preserves ω then for any nowhere zero $W \in \mathcal{X}(M)$ we have the following two conditions:

(1)

$$\begin{aligned} & \{ D_k \tilde{\Gamma}_{lj}^n + D_p \tilde{\Gamma}_{lj}^n \cdot \partial_k W^p - D_l \tilde{\Gamma}_{jk}^n - D_p \tilde{\Gamma}_{jk}^n \cdot \partial_l W^p + \tilde{\Gamma}_{lj}^p \cdot \tilde{\Gamma}_{kp}^n - \tilde{\Gamma}_{jk}^p \cdot \tilde{\Gamma}_{lp}^n \} \\ & + \{ D_l \tilde{\Gamma}_{jk}^n + D_p \tilde{\Gamma}_{jk}^n \cdot \partial_l W^p - D_j \tilde{\Gamma}_{kl}^n - D_p \tilde{\Gamma}_{kl}^n \cdot \partial_j W^p + \tilde{\Gamma}_{jk}^p \cdot \tilde{\Gamma}_{lp}^n - \tilde{\Gamma}_{kl}^p \cdot \tilde{\Gamma}_{jp}^n \} \\ & + \{ D_j \tilde{\Gamma}_{kl}^n + D_p \tilde{\Gamma}_{kl}^n \cdot \partial_j W^p - D_k \tilde{\Gamma}_{lj}^n - D_p \tilde{\Gamma}_{lj}^n \cdot \partial_k W^p + \tilde{\Gamma}_{kl}^p \cdot \tilde{\Gamma}_{jp}^n - \tilde{\Gamma}_{lj}^p \cdot \tilde{\Gamma}_{kp}^n \} \omega_{in} = 0, \end{aligned}$$

(2)

$$\begin{aligned} & \{ D_k \tilde{\Gamma}_{lj}^n + D_p \tilde{\Gamma}_{lj}^n \cdot \partial_k W^p - D_l \tilde{\Gamma}_{jk}^n - D_p \tilde{\Gamma}_{jk}^n \cdot \partial_l W^p + \tilde{\Gamma}_{lj}^p \cdot \tilde{\Gamma}_{kp}^n - \tilde{\Gamma}_{jk}^p \cdot \tilde{\Gamma}_{lp}^n \} \omega_{in} \\ & - \{ D_k \tilde{\Gamma}_{li}^n + D_p \tilde{\Gamma}_{li}^n \cdot \partial_k W^p - D_l \tilde{\Gamma}_{ik}^n - D_p \tilde{\Gamma}_{ik}^n \cdot \partial_l W^p + \tilde{\Gamma}_{li}^p \cdot \tilde{\Gamma}_{kp}^n - \tilde{\Gamma}_{ik}^p \cdot \tilde{\Gamma}_{lp}^n \} \omega_{jn} = 0. \end{aligned}$$

Proof. For (1) it suffices to use equation 4.1 and the first Bianchi identity for symmetric connections. We can obtain (2) by using equations 4.1 and 4.2. □

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