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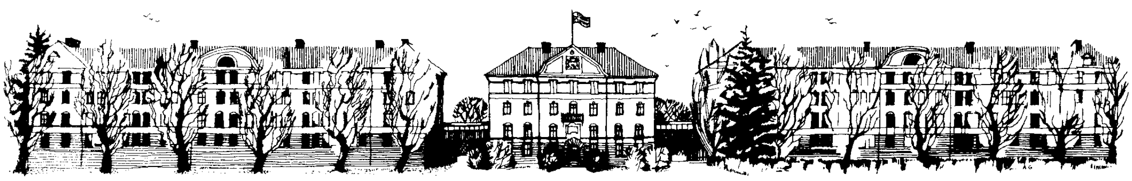

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**Abstract**

We give a short and elementary proof for the classification of the real vector product algebras.

A real *vector product algebra* is a Euclidean space $V = (V, \langle\,\rangle)$ together with an anti-symmetric bilinear map

$$V \times V \to V,\ (u, v) \mapsto uv$$

having the property that the set $\{u, v, uv\}$ is orthonormal whenever $\{u, v\}$ is. A morphism between vector product algebras $V$ and $W$ is a linear map $\varphi : V \to W$ such that $\varphi(uv) = \varphi(u)\varphi(v)$ and $\langle\varphi(u), \varphi(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$. Two vector product algebras $V$ and $W$ are isomorphic if there exists a bijective morphism $\varphi : V \to W$.

Note that in our definition, vector product algebras are *not* required to be finite-dimensional. This, however, follows from the theory. In the classification theorem, our Theorem 3, it is established that real vector product algebras exist in dimension 0, 1, 3 and 7 only, and form one isomorphism class in each dimension.

Given a vector product algebra $V$, the space $\mathbb{R} \times V$ with multiplication given by

$$(\alpha, v) \cdot (\beta, w) = (\alpha\beta - \langle v, w\rangle, \alpha v + \beta w + uv)$$

is a composition algebra. Conversely, every composition algebra arises in this way (see for example [2]).

Vector product algebras were first considered, and classified, by Eckmann [1] using topological methods. A treatment based on the correspondence with composition algebras is found in [3], as well as an alternative approach relying on results from algebraic topology. Rost [4] has considered vector product algebras over general fields $k$ with $\operatorname{char} k \neq 2$, and by elementary means proven that the equation $d(d-1)(d-3)(d-7) = 0$ must hold in the ground field, if $d$ is the dimension of a vector product algebra.

However, our method differs those of the above mentioned. We believe that the notion of multiplicatively independent sets will be helpful for the reader to understand the structure of vector product algebras. In what follows, $V$ and $W$ will always denote real vector product algebras.

From the definition, it is clear that $\langle yx, x\rangle = 0$ for all $x, y \in V$. Substituting $x + z$ for $x$ we get

$$\begin{aligned}0 = \langle y(x+z), x+z\rangle = \langle yx, x\rangle + \langle yx, z\rangle + \langle yz, x\rangle + \langle yz, z\rangle = \\ = \langle yx, z\rangle + \langle yz, x\rangle = \langle x, yz\rangle - \langle xy, z\rangle\end{aligned}$$

which yields the important identity

$$\langle xy, z\rangle = \langle x, yz\rangle \quad \text{for all } x, y, z \in V. \tag{1}$$

The following lemma provides means to control the multiplication in $V$.

**Lemma 1** *Suppose that $u, v \in V$ are orthogonal and $\|u\| = 1$. Then the following identities hold.*

1. $u(uv) = -v$
2. $v(uv) = u$

*If in addition $w \in V$ is orthogonal to $u$, $v$ and $uv$, then*

3. $u(vw) = -(uv)w = (vu)w$

*In particular,* $\operatorname{span}\{u, v, uv\}$ *is a vector product subalgebra of $V$.*

*Proof:* Note that, as $u$ is unit vector, the mapping $u^\perp \to u^\perp$, $x \mapsto ux$ is an isometry. For all $x \in V$ we therefore have

$$\langle x, u(uv)\rangle = \langle xu, uv\rangle = -\langle ux, uv\rangle = \langle ux, u(-v)\rangle = \langle x, -v\rangle.$$

Hence $u(uv) = -v$. The identity 2 is just a reformulation of 1, using that the multiplication is anti-commutative.

By 1, we also have $u(uw) = -w$. Moreover,

$$\begin{aligned}-\|u+w\|^2 v = (u+w)\left((u+w)v\right) &= (u+w)(uv+wv) = \\ &= u(uv) + u(wv) + w(uv) + w(wv) = \\ &= -\|u\|^2 v + u(wv) + w(uv) - \|w\|^2 v = \\ &= -\|u+w\|^2 v - u(vw) - (uv)w.\end{aligned}$$

Hence $u(vw) = -(uv)w$. □

Our tool of investigation will be so called *multiplicatively independent sets*. We say that a subset $E \subset V$ is multiplicatively independent if for any $e \in E$, we have $\|e\| = 1$ and $e$ is orthogonal to the subalgebra $\langle E_e\rangle \subset V$ generated by $E_e = E \setminus \{e\}$.

Let $E \subset V$ be a multiplicatively independent set and $e \in E$. In view of Lemma 1, $\langle E\rangle$ may be written as $\langle E\rangle = \langle E_e\rangle + \langle e\rangle + \langle E_e\rangle e$. As for any $x, y \in E_e$, $\langle x, ye\rangle = \langle xy, e\rangle = 0$ and $\langle xe, e\rangle = \langle x, e\cdot e\rangle = 0$ we indeed have

$$\langle E\rangle = \langle E_e\rangle \oplus \langle e\rangle \oplus \langle E_e\rangle e \tag{2}$$

the different summands being pairwise orthogonal.

**Lemma 2** *If $E \subset V$ is a multiplicatively independent set and $f \in V$ is orthogonal to $\langle E \rangle$, then $E \cup \{f\}$ is also multiplicatively independent.*

*Proof:* We need to show that any $e \in E$ is orthogonal to $\langle E_e \cup \{f\} \rangle$. But $\langle E_e \cup \{f\} \rangle = \langle E_e \rangle \oplus \langle f \rangle \oplus \langle E_e \rangle f$. Since $e \perp \langle E_e \rangle$, $e \perp f$ and $e \perp \langle E \rangle f \supset \langle E_e \rangle f$, our assertion holds true. □

Certainly, $\langle \emptyset \rangle = \{0\}$, that is $\dim \langle \emptyset \rangle = 0$. From (2) it follows that $\dim \langle E \rangle = 2 \dim \langle E_e \rangle + 1$ for any finite multiplicatively independent set $E$ and $e \in E$. Therefore, if $n$ is the number of elements in $E$, then $\dim \langle E \rangle = 2^n - 1$. Lemma 2 implies that if $V$ is finite-dimensional, then $V = \langle E \rangle$ for some multiplicatively independent set $E \subset V$. Such a set we call a *multiplicative basis* for $V$.

**Theorem 3** *Every vector product algebra is isomorphic to one of the following:*

- *The trivial algebra $\{0\}$.*
- *The 1-dimensional algebra $\mathbb{R}$ with zero multiplication.*
- *The 3-dimensional algebra $\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}$ with multiplication given by the upper left part of Table 1.*
- *The 7-dimensional algebra determined by Table 1.*

| $\cdot$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $-e_3$ | $0$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_2$ | $-e_1$ | $0$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $0$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $0$ | $-e_3$ | $e_2$ |
| $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $0$ | $-e_1$ |
| $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | $0$ |

Table 1: 7-dimensional vector product algebra

*Proof:* Suppose that $E = \{u, v, w, z\} \subset V$ is a multiplicatively independent set. Evaluating $u\,(v(wz))$ we get

$$u(v(wz)) = u((wv)z) = ((wv)u)z = -((vw)u)z$$

but also

$$u(v(wz)) = (vu)(wz) = (w(vu))z = ((vw)u)z.$$

So $u(v(wz)) = 0$, which contradicts the multiplicative independence of $E$. Hence any multiplicatively independent set contains at most 3 elements. This rules out the existence of infinite-dimensional vector product algebras, since in such algebras we would be able to find multiplicatively independent sets of any finite cardinality. Moreover, it tells that every vector product algebra has dimension 0, 1, 3 or 7.

For any multiplicatively independent set $E \subset V$, $e \in E$ and orthonormal $x, y \in \langle E_e \rangle$, the set $\{y, e, xe\}$ is also multiplicatively independent. Lemma 1 yields the following identities.

$$y(xe) = (xy)e \tag{3}$$

$$x(xe) = -e \tag{4}$$

$$e(xe) = -e(ex) = x \tag{5}$$

$$(xe)(ye) = (y(xe))\,e = ((xy)e)\,e = -xy \tag{6}$$

Now suppose that $E$ and $F$ are multiplicative bases for $V$ and $W$ respectively. Further suppose that $e \in E$, $f \in F$ and that $\varphi : \langle E_e \rangle \to \langle F_f \rangle$ is an isomorphism (hereby it follows that $E$ and $F$ have the same number of elements). We define $\tilde{\varphi} : V \to W$ by $\tilde{\varphi}(x + \lambda e + ye) = \varphi(x) + \lambda f + \varphi(y)f$, for $x, y \in \langle E \rangle$, $\lambda \in \mathbb{R}$. It is straightforward to prove that $\tilde{\varphi}$ is a morphism of vector product algebras. Since $\tilde{\varphi}$ is bijective, $V$ and $W$ are isomorphic. By induction, we deduce that any two vector product algebras having multiplicative bases which contain the same number of elements (equivalently, having the same dimension) must be isomorphic.

On the other hand, anyone may convince himself, by a straightforward calculation, that $V = \mathbb{R}^7$ with multiplication given by Table 1 satisfies the axioms for a vector product algebra. Then it is also clear that span($\emptyset$), span$\{e_1\}$ and span$\{e_1, e_2, e_3\}$ are vector product subalgebras of $V$ of the types listed in the proposition. □

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