Some remarks on the Riemann zeta distribution

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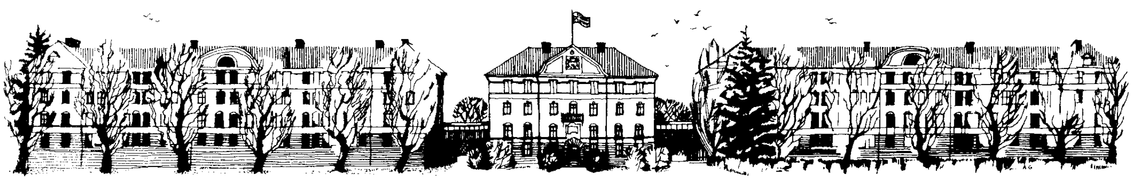

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Abstract

The Riemann zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, where $z=\sigma+it\in\mathbb{C}$, plays an important role in connection with the distribution of primes. A probabilistically pleasant fact is that the zeta function, as a function of t, properly normalized, is a characteristic function, the distribution of which is compound Poisson. This property is exploited and some facts from analytic number theory are (re)established.

1 Introduction

The Riemann zeta function is

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z},\quad z=\sigma+it\in\mathbb{C}.$$

The function is convergent for $\sigma>1$ and absolutely convergent for $\sigma\geq 1+\delta$ for any $\delta>0$, and, hence, a regular analytic function for $\sigma>1$. The function plays an important role in connection with the distribution of primes, and is also famous for the *Riemann hypothesis*, which is *not* the topic of the present paper.

Inspired by a recent article by Lin and Hu [8], the aim of this note is to re-derive some facts from analytic number theory involving the Riemann zeta function and its derivatives by probabilistic means.

The main point is that, for $\sigma>1$ fixed, one can view the normalized zeta function $\varphi_\sigma(t)=\frac{\zeta(\sigma+it)}{\zeta(\sigma)}$ as the characteristic function of, as it turns out, a compound Poisson distribution. This allows us to use results for moments of sums of a random number of random variables to (re)prove relations between i.a. the von Mangoldt function, the Möbius function, and the Riemann zeta function and its derivatives. We also re-derive (and extend somewhat) an identity due to Selberg. An appendix with some terminology from analytic number theory closes the paper.

2 The normalized zeta function is a characteristic function

Set $z=\sigma+it$ with $\sigma>1$. We normalize the Riemann zeta function into

$$\varphi_\sigma(t)=\frac{\zeta(\sigma+it)}{\zeta(\sigma)}=\sum_{n=1}^{\infty}\frac{1}{\zeta(\sigma)\cdot n^{\sigma+it}}=\sum_{n=1}^{\infty}\frac{1}{\zeta(\sigma)\cdot n^{\sigma}}\cdot n^{-it}\,,\tag{2.1}$$

in order for the value at zero to equal one.

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Let U $(= U_\sigma)$ be a random variable with probability mass function

$$P(U = n) = \frac{1}{\zeta(\sigma)\cdot n^\sigma}, \quad n = 1, 2, \dots,$$

and set $Y = -\log U$ (that is, $Y_\sigma = -\log U_\sigma$). Then

$$\psi_Y(t) = E\exp\{tY\} = E\exp\{-t\log U\} = E\,U^{-t} = \sum_{n=1}^{\infty} \frac{1}{\zeta(\sigma)\cdot n^\sigma}\cdot n^{-t} = \frac{\zeta(\sigma+t)}{\zeta(\sigma)}, \quad (2.2)$$

so that, for $\sigma > 1$,

$$\varphi_Y(t) = \psi_Y(it) = \varphi_\sigma(t). \quad (2.3)$$

We have thus shown that $\varphi_\sigma(t)$ is a characteristic function and exhibited the corresponding random variable Y:

$$P(Y = -\log n) = \frac{1}{\zeta(\sigma)\cdot n^\sigma}, \quad n = 1, 2, \dots. \quad (2.4)$$

Since moments are obtained via derivatives at 0 of moment generating functions or characteristic functions, it follows that

$$E\,Y^k = \frac{\zeta^{(k)}(\sigma)}{\zeta(\sigma)} \quad \text{for} \quad k = 1, 2, \dots, \quad (2.5)$$

in particular,

$$E\,Y = \frac{\zeta'(\sigma)}{\zeta(\sigma)} \quad \text{and} \quad \operatorname{Var} Y = \frac{\zeta''(\sigma)}{\zeta(\sigma)} - \Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^2. \quad (2.6)$$

In addition, since variances are non-negative, (2.6) tells us that

$$\frac{d^2}{d\sigma^2}\log\zeta(\sigma) = \frac{d}{d\sigma}\,\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{\zeta''(\sigma)\zeta(\sigma) - (\zeta'(\sigma))^2}{\zeta(\sigma)^2} \geq 0, \quad (2.7)$$

that is, $\zeta(\sigma)$ is logconvex (for $\sigma > 1$). This is of course(?) no news (although we have not found any reference).

Remark 2.1 The inequality in (2.7) is strict, since U is non-degenerate. □

3 The von Mangoldt function

According to a famous formula due to Euler (see e.g. [5], Theorem 280, or [10], p. 1), an alternative expression for the Riemann zeta function is

$$\zeta(\sigma) = \prod_p \frac{1}{1 - p^{-\sigma}}, \quad \text{where the product runs over all primes,} \quad (3.1)$$

which suggests that the distribution of primes plays an important role in the study of $\zeta(z)$. An important role in this context is played by the von Mangoldt function:

Definition 3.1 *The* von Mangoldt function Λ *is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and some integer } k, \\ 0, & \text{otherwise.} \end{cases}$$

□

Let $\sigma > 1$. In order to express the Riemann zeta function with the aid of the von Mangoldt function we have, following [1], p. 239,

$$\log \zeta(\sigma) = -\sum_p \log(1 - p^{-\sigma}) = \sum_p \sum_{k=1}^{\infty} \frac{p^{-\sigma k}}{k}.$$

Now, since

$$p^{-\sigma k} = \begin{cases} n^{-\sigma}, & \text{when} \quad n = p^k, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \frac{1}{k} = \begin{cases} \frac{\log p}{\log n} = \frac{\Lambda(n)}{\log n}, & \text{when} \quad n = p^k, \\ 0, & \text{otherwise}, \end{cases}$$

the representation

$$\log \zeta(\sigma) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n \cdot n^{\sigma}} \quad \text{or, equivalently,} \quad \zeta(\sigma) = \exp\Big\{ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \cdot n^{-\sigma} \Big\}, \tag{3.2}$$

emerges, and via analytic continuation,

$$\zeta(\sigma + it) = \exp\Big\{ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \cdot n^{-(\sigma+it)} \Big\} \quad \text{for} \quad \sigma > 1. \tag{3.3}$$

4 The Riemann zeta distribution

Inserting the representation (3.2) into the expression for the moment generating function ψ_Y in (2.2) we obtain

$$\begin{aligned} \psi_Y(t) &= \exp\{\log \zeta(\sigma + t) - \log \zeta(\sigma)\} = \exp\Big\{ \log \zeta(\sigma) \Big(\frac{\log \zeta(\sigma + t)}{\log \zeta(\sigma)} - 1 \Big) \Big\} \\ &= \exp\Big\{ \log \zeta(\sigma) \Big(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log \zeta(\sigma) \cdot \log n \cdot n^{\sigma+t}} - 1 \Big) \Big\} \\ &= \exp\Big\{ \log \zeta(\sigma) \Big(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log \zeta(\sigma) \cdot \log n \cdot n^{\sigma}} \exp\{-t \log n\} - 1 \Big) \Big\}. \end{aligned}$$

The sum in the exponent can be identified as the moment generating function of $-\log V$, where

$$P(V = n) = \frac{\Lambda(n)}{\log \zeta(\sigma) \cdot \log n \cdot n^{\sigma}}, \quad n = 2, 3, \ldots . \tag{4.1}$$

Since the probability generating function of the $\text{Po}(\lambda)$-distribution equals $\exp\{\lambda(t-1)\}$ we may reformulate the expression for ψ_Y as

$$\psi_Y(t) = g_N\big(\lambda(\psi_X(t) - 1)\big), \tag{4.2}$$

where $\lambda = \log \zeta(\sigma)$, $N \in \text{Po}(\lambda)$ and $X \stackrel{d}{=} -\log V$, which, in turn, leads to the representation

$$Y \stackrel{d}{=} X_1 + X_2 + \cdots + X_N, \tag{4.3}$$

where X_1, X_2, $\ldots$ are i.i.d. random variables distributed as X, and independent of N. This means that Y has a *compound Poisson distribution*, and, hence, is *infinitely divisible* or, equivalently, that $\varphi_\sigma(t)$ is an infinitely divisible characteristic function.

This is no news ([7], p. 35, [3], pp. 76-77, [2], Theorem 1.2.25, Lin and Hu [8], Theorem 2); the point here being that infinite divisibility is established via the compound Poisson distribution. (In [8], Theorem 3, the authors identify the distribution as an infinite sum of weighted Poisson random variables, and in Theorem 4 as an infinite sum of independent geometric random variables.)

In addition,

- the moments of X are

$$E\,X^k = (-1)^k \sum_{n=2}^{\infty} \frac{\Lambda(n)(\log n)^{k-1}}{\log\zeta(\sigma)\cdot n^{\sigma}}, \quad k = 1, 2, \ldots, \tag{4.4}$$

  in view of (4.1) and the fact that $X \stackrel{d}{=} -\log V$,

- an application of the rules for derivation of Dirichlet series in (A.1) to the Riemann zeta function yields

$$\zeta^{(k)}(\sigma) = (-1)^k \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^{\sigma}}, \tag{4.5}$$

- by joining (A.1) and (3.2) (cf. [1], p. 236), it follows that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)(\log n)^{k-1}}{n^{\sigma}} = (-1)^k \frac{d^k}{d\sigma^k}\log\zeta(\sigma), \quad \text{for any} \quad k = 1, 2, \ldots. \tag{4.6}$$

- A further inspection of (3.3) and (4.1) tells us that $\log\varphi_{\sigma}(t)$ is the Fourier transform of a positive, finite measure or, equivalently, that

$$\frac{\log\zeta(\sigma+it)}{\log\zeta(\sigma)}, \quad -\infty < t < \infty, \tag{4.7}$$

  is a characteristic function (namely, of the random variable X). This statement has connections to [6].

In the following section we re-derive relation (4.6) for $k = 1$ and $k = 2$ via the compound Poisson distribution and the expressions for the moments of X and Y in (4.4) and (2.5), respectively, that is, without the detour via Dirichlet differentiation of (3.2). In a final section we also consider moments of order three and four, leaving higher order details to the reader(s). In between those sections we present an alternative proof of the Selberg identity and a sketch on extensions to more general Dirichlet series.

5 The von Mangoldt function, $E\,Y$ and $\operatorname{Var} Y$

Let $X, X_1, X_2, \ldots$ and N be given as above. Although we already know $E\,Y$ and $\operatorname{Var} Y$ from (2.6) it is rewarding to re-derive them from the representation (4.3) and the well known relations (see e.g. [4], Theorem 2.15.1):

$$E\,Y = E\,N\cdot E\,X \quad \text{and} \quad \operatorname{Var} Y = E\,N\cdot\operatorname{Var} X + (E\,X)^2\cdot\operatorname{Var} N. \tag{5.1}$$

Since $E\,X = -\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log\zeta(\sigma)\cdot n^{\sigma}}$ by (4.4) with $k = 1$, and $E\,N = \log\zeta(\sigma)$, we obtain, departing from (2.6),

$$\frac{d}{d\sigma}\log\zeta(\sigma) = \frac{\zeta'(\sigma)}{\zeta(\sigma)} = E\,Y = \log\zeta(\sigma)\cdot\Big(-\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log\zeta(\sigma)\cdot n^{\sigma}}\Big) = -\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}. \tag{5.2}$$

In order to compute the variance we observe that, in this particular case, the random index, N, is a Poisson random variable, which means that mean and variance coincide, turning the variance formula in (5.1) into

$$\operatorname{Var} Y = E\,N \cdot \big(\operatorname{Var} X + (E\,X)^2\big) = E\,N \cdot E\,X^2, \tag{5.3}$$

so that, recalling (2.6) and (4.4) with $k = 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{\Lambda(n)\log n}{n^{\sigma}} &= \log\zeta(\sigma)\cdot E\,X^2 = \log\zeta(\sigma)\cdot\frac{\operatorname{Var} Y}{E\,N} = \log\zeta(\sigma)\cdot\frac{\operatorname{Var} Y}{\log\zeta(\sigma)}\\ &= \operatorname{Var} Y = \frac{\zeta''(\sigma)}{\zeta(\sigma)} - \Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^2 = \frac{d^2}{d\sigma^2}\log\zeta(\sigma).\end{aligned} \tag{5.4}$$

Remark 5.1 If, instead, we would have used Dirichlet differentiation (Theorem A.3), we would have used the fact that $-\sum_{n=2}^{\infty}\frac{\Lambda(n)\log n}{n^{\sigma}}$ is the Dirichlet series representation of the derivative of the function corresponding to the Dirichlet series

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}, \qquad \text{namely} \qquad -\frac{\zeta'(\sigma)}{\zeta(\sigma)},$$

so that, by term-wise differentiation of the latter and (4.5) (i.e., Theorem A.3 once more), we would obtain

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)\log n}{n^{\sigma}} = \frac{\zeta''(\sigma)}{\zeta(\sigma)} - \Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^2, \tag{5.5}$$

from which the expression for $\operatorname{Var} Y$ would follow as before. □

6 The Selberg identity

This section is devoted to an identity due to Selberg, but first a definition and two facts.

Definition 6.1 *The* Möbius function $\mu(n)$ *([1], p. 24, [5], p. 254, [10], p. 3) is*

$$\mu(n) = \begin{cases} 1, & \textit{for} \quad n = 1,\\ (-1)^k, & \textit{if } n \textit{ is a product of } k \textit{ distinct primes},\\ 0, & \textit{otherwise}. \end{cases}$$

□

For the proof we also need the following relations which can be obtained from Theorem A.2 – as for the first one, cf. also [1], Example 1, p. 228, [5], Theorem 287, or [10], p. 3.

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{\sigma}} = \frac{1}{\zeta(\sigma)}, \quad \sigma > 1, \tag{6.1}$$

$$\Big(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\Big)^2 = \sum_{n=2}^{\infty}\frac{(\Lambda*\Lambda)(n)}{n^{\sigma}}. \tag{6.2}$$

Consider $E\,Y$ as a warm up to the Selberg identity, which, in turn, is established with the aid of second moments. For $\sigma > 1$ we obtain, via (5.2) and (6.1),

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma}} = -E\,Y = \sum_{n=1}^{\infty}\frac{\log n}{\zeta(\sigma)\cdot n^{\sigma}} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^{\sigma}}\sum_{n=1}^{\infty}\frac{\log n}{n^{\sigma}} = \sum_{n=2}^{\infty}\frac{\big(\mu*\log\big)(n)}{n^{\sigma}},$$

which, due to uniqueness, Theorem A.1, tells us that

$$\Lambda(n) = \big(\mu * \log\big)(n). \tag{6.3}$$

For the traditional derivation, see e.g [1], Theorem 2.11, or [5], Theorem 295.

Here is now the Selberg identity as given in [1], Theorem 2.27, p. 46; cf. [9], formula (2.6) or [5], Theorem 433, for the original variant.

Theorem 6.1 $\Lambda(n)\log n + (\Lambda * \Lambda)(n) = (\mu * \log^2)(n).$

Proof. Let $\sigma > 1$. From (5.2) and (5.4) we know that

$$\begin{aligned} E\,Y^2 &= \operatorname{Var} Y + (E\,Y)^2 = \sum_{n=2}^{\infty} \frac{\Lambda(n)\log n}{n^\sigma} + \Big(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma}\Big)^2 \\ &= \sum_{n=2}^{\infty} \frac{\Lambda(n)\log n}{n^\sigma} + \sum_{n=2}^{\infty} \frac{(\Lambda * \Lambda)(n)}{n^\sigma}, \end{aligned} \tag{6.4}$$

the last inequality being a consequence of (6.2). On the other hand, cf. (2.4) and (6.1),

$$E\,Y^2 = \sum_{n=1}^{\infty} \frac{(\log n)^2}{\zeta(\sigma)\cdot n^\sigma} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\sigma} \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^\sigma} = \sum_{n=2}^{\infty} \frac{\big(\mu * \log^2\big)(n)}{n^\sigma}. \tag{6.5}$$

Comparing the two expressions for $E\,Y^2$ we have thus shown that

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)\log n + (\Lambda * \Lambda)(n)}{n^\sigma} = \sum_{n=2}^{\infty} \frac{\big(\mu * \log^2\big)(n)}{n^\sigma}. \tag{6.6}$$

Since this is true for any $\sigma > 1$, Theorem A.1 (uniqueness) tells us that the numerators are term-wise equal. □

As a bonus we observe that, by joining (5.2), (5.4), (6.2), and the fact that $E\,Y^2 = \operatorname{Var} Y + (E\,Y)^2$ with (6.4), we obtain (for $\sigma > 1$)

$$\sum_{n=2}^{\infty} \frac{(\Lambda * \Lambda)(n)}{n^\sigma} = \Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^2, \tag{6.7}$$

$$\sum_{n=2}^{\infty} \frac{\big(\mu * \log^2\big)(n)}{n^\sigma} = \frac{\zeta''(\sigma)}{\zeta(\sigma)}. \tag{6.8}$$

Remark 6.1 By reviewing the situation "backwards" we observe that summing the identity normalized by n^σ over all n yields the (well-known) relation $E\,Y^2 = \operatorname{Var} Y + (E\,Y)^2$.

Remark 6.2 In their paper Lin and Hu, [8], p. 823, *exploit* the Selberg identity to compute the left-hand side of (6.7) in order to *compute* $\operatorname{Var} Y$, whereas we have used the decomposition of the variance in order to prove the Selberg idenity. □

7 An extension to general Dirichlet series

Recalling Definition A.1 it is tempting to extend the results to general Dirichlet series. For infinite divisibility we refer to [8], Theorem 2, the proof of which differs slightly from ours as presented above.

The representation of Dirichlet series via the von Mangoldt function is guaranteed if the arithmetic function is completely multiplicative ([1], Example 2, p. 239). For infinite divisibilty of the corresponding distribution or characteristic function this is, however, not necessary ([8], Remark 1).

Definition 7.1 *The arithmetical function* $f = \{f(n),\, n \geq 1\} \not\equiv 0$ *is* multiplicative *if and*

$$f(m \cdot n) = f(m) \cdot f(n) \quad \textit{whenever } (m,n) = 1 \quad (\textit{i.e., } m \textit{ and } n \textit{ are relative prime.})$$

The sequence is completely multiplicative *if*

$$f(m \cdot n) = f(m) \cdot f(n) \quad \textit{for all} \quad m, n. \qquad \square$$

Suupose that $\eta(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z}$ is a Dirichlet series with completely multiplicative non-negative coefficients. By replacing ζ in the derivations above with η the following conclusions are (essentially) immediate.

Set $z = \sigma + it$, and let $\sigma > \sigma_0$, where σ_0 is the abscissa of convergence of the Dirichlet series, and put

$$\widehat{\varphi}_\sigma(t) = \frac{\eta(\sigma + it)}{\eta(\sigma)} = \sum_{n=1}^{\infty} \frac{a(n)}{\eta(\sigma) \cdot n^{\sigma+it}} = \sum_{n=1}^{\infty} \frac{a(n)}{\eta(\sigma) \cdot n^{\sigma}} \cdot n^{-it}, \quad \sigma > \sigma_0. \tag{7.1}$$

Then $\widehat{\varphi}_\sigma(t)$ is the characteristic function of $\widehat{Y} = -\log \widehat{U}$, where

$$P(\widehat{U} = n) = \frac{a(n)}{\eta(\sigma) \cdot n^{\sigma}}, \quad n \geq 1.$$

Moreover,

- $E(\widehat{Y})^k = \frac{\eta^{(k)}(\sigma)}{\eta(\sigma)}$ for $k = 1, 2, \ldots$,
- $E\,\widehat{Y} = -\sum_{n=2}^{\infty} \frac{a(n) \cdot \Lambda(n)}{n^{\sigma}} = \frac{\eta'(\sigma)}{\eta(\sigma)} = \frac{d}{d\sigma} \log \eta(\sigma)$,
- $\operatorname{Var} \widehat{Y} = \sum_{n=2}^{\infty} \frac{a(n) \cdot \Lambda(n) \cdot \log n}{n^{\sigma}} = \frac{\eta''(\sigma)}{\eta(\sigma)} - \left(\frac{\eta'(\sigma)}{\eta(\sigma)}\right)^2 = \frac{d^2}{d\sigma^2} \log \eta(\sigma)$,
- $\widehat{\varphi}_\sigma(t)$ is logconvex and infinitely divisible,
- $\frac{\log \eta(\sigma+it)}{\log \eta(\sigma)}$ is a characteristic function.

Finally, an appeal to Theorem A.3 tells us that a representation via the von Mangoldt function analogous to that in Section 4 holds true, that is,

$$\widehat{\varphi}_\sigma(t) = g_N\big(\widehat{\lambda}(\varphi_{\widehat{X}}(t) - 1)\big),$$

where $\widehat{\lambda} = \log \eta(\sigma)$, $N \in \operatorname{Po}(\widehat{\lambda})$, and

$$P(\widehat{X} = -\log n) = \frac{a(n) \cdot \Lambda(n)}{\log \eta(\sigma) \cdot \log n \cdot n^{\sigma}}, \quad n = 2, 3, \ldots$$

Once again we have arrived at a(n infinitely divisible) compound Poisson distribution.

Following is an extension of (6.3) and the Selberg identity.

Theorem 7.1 *Let, as before,* $\eta(\sigma) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}}$, *where* $\{a(n),\, n \geq 1\}$ *is completely multiplicative and non-negative, and set, for* $n \geq 1$,

$$\widehat{\Lambda}(n) = \Lambda(n) \cdot a(n), \quad \widehat{\mu}(n) = \mu(n) \cdot a(n), \quad \widehat{\log}(n) = \log n \cdot a(n), \quad \widehat{(\log)^2}(n) = (\log n)^2 \cdot a(n).$$

Further, suppose that the abscissa of convergence σ_a *of* η *is finite. Then*

$$\widehat{\Lambda}(n) = \big(\widehat{\mu} * \widehat{\log}\big)(n), \tag{7.2}$$

$$\widehat{\Lambda}(n) \log n + (\widehat{\Lambda} * \widehat{\Lambda})(n) = (\widehat{\mu} * \widehat{(\log)^2})(n). \tag{7.3}$$

Proof. Let $\sigma > \sigma_a$. The proof follows the pattern of the proofs from Section 6. However, we first note that the Dirichlet series involving the sequences $\{\widehat{\mu}(n)\}$, $\{\widehat{\log}(n)\}$, and $\{\widehat{(\log)^2}(n)\}$ are convergent, since $|\mu(n)| \leq 1$ for all n, and $\log n = o(n^\delta)$ for any $\delta > 0$ as $n \to \infty$, and that the analog of (6.1) – see [1], Example 3, p. 229 – is

$$\frac{1}{\eta(\sigma)} = \sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n^\sigma}.$$

In order to prove (7.2) we now follow the derivation of (6.3) with the random variable $\widehat{Y}$ playing the role of Y there:

$$\sum_{n=2}^{\infty} \frac{\widehat{\Lambda}(n)}{n^\sigma} = -E\,\widehat{Y} = \sum_{n=1}^{\infty} \frac{a(n)\cdot \log n}{\eta(\sigma)\cdot n^\sigma} = \sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n^\sigma} \sum_{n=1}^{\infty} \frac{\widehat{\log}(n)}{n^\sigma} = \sum_{n=2}^{\infty} \frac{\big(\widehat{\mu} * \widehat{\log}\big)(n)}{n^\sigma},$$

which, in view of Theorem A.1 establishes the claim.

As for (7.3) we copy the arguments that produced (6.4) and (6.5), respectively:

$$E\,\widehat{Y}^2 = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{\widehat{\Lambda}(n)\log n}{n^\sigma} + \sum_{n=2}^{\infty} \frac{(\widehat{\Lambda} * \widehat{\Lambda})(n)}{n^\sigma}, \\ \displaystyle\frac{1}{\eta(\sigma)} \sum_{n=1}^{\infty} \frac{\widehat{(\log)^2}(n)}{n^\sigma} = \sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n^\sigma} \sum_{n=1}^{\infty} \frac{\widehat{(\log)^2}(n)}{n^\sigma} = \sum_{n=2}^{\infty} \frac{\big(\widehat{\mu} * \widehat{(\log)^2}\big)(n)}{n^\sigma}. \end{cases}$$

Equationg the extreme members and an appeal to Theorem A.1 finish the proof. □

The analogs of (6.7) and (6.8) are

$$\sum_{n=2}^{\infty} \frac{(\widehat{\Lambda} * \widehat{\Lambda})(n)}{n^\sigma} = \Big(\frac{\eta'(\sigma)}{\eta(\sigma)}\Big)^2 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{\big(\widehat{\mu} * \widehat{(\log)^2}\big)(n)}{n^\sigma} = \frac{\eta''(\sigma)}{\eta(\sigma)}.$$

8 Higher order moments

It is, of course, possible to obtain relations between the von Mangoldt function and moments of X analogous to those obtained for mean and variance in Section 5. In this section we provide analogs for moments of order three and four.

From (4.1) we know that

$$E\,X^3 = -\sum_{n=2}^{\infty} \frac{\Lambda(n)(\log n)^2}{\log\zeta(\sigma)\cdot n^\sigma} \quad \text{and} \quad E\,X^4 = \sum_{n=2}^{\infty} \frac{\Lambda(n)(\log n)^3}{\log\zeta(\sigma)\cdot n^\sigma}. \tag{8.1}$$

By conditioning in (4.3) – $E\,Y^k = E(E(Y^k \mid N))$ – or by differentiation of the moment generating function (4.2) – recall that $\lambda = \log\zeta(\sigma)$ – we obtain

$$\begin{aligned} E\,Y &= \psi'(0) = \log\zeta(\sigma)E\,X, \\ E\,Y^2 &= \psi''(0) = (\log\zeta(\sigma)E\,X)^2 + \log\zeta(\sigma)E\,X^2, \\ E\,Y^3 &= \psi'''(0) = (\log\zeta(\sigma)E\,X)^3 + 3(\log\zeta(\sigma))^2 E\,X E\,X^2 + \log\zeta(\sigma)E\,X^3, \\ E\,Y^4 &= \psi^{(4)}(0) = (\log\zeta(\sigma)E\,X)^4 + 6(\log\zeta(\sigma))^3(E\,X)^2E\,X^2 + 3(\log\zeta(\sigma)E\,X^2)^2 \\ &\qquad +4(\log\zeta(\sigma))^2E\,X E\,X^3 + \log\zeta(\sigma)E\,X^4. \end{aligned}$$

We now join these facts with (2.5) for $k = 1, 2, 3, 4$ and (4.4) with $k = 1, 2$.

As for EY^3 this produces

$$\begin{aligned}\frac{\zeta'''(\sigma)}{\zeta(\sigma)} &= (\log\zeta(\sigma))^3\Big(\frac{1}{\log\zeta(\sigma)}\cdot\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^3 \\ &\quad +3(\log\zeta(\sigma))^2\frac{1}{\log\zeta(\sigma)}\cdot\frac{\zeta'(\sigma)}{\zeta(\sigma)}\frac{1}{\log\zeta(\sigma)}\Big(\frac{\zeta''(\sigma)}{\zeta(\sigma)}-\Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^2\Big) \\ &\quad -\log\zeta(\sigma)\sum_{n=2}^{\infty}\frac{\Lambda(n)(\log n)^2}{\log\zeta(\sigma)\cdot n^{\sigma}},\end{aligned}$$

which after reshuffling tells us that

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)(\log n)^2}{n^{\sigma}} = (-1)^3\Big\{\frac{\zeta'''(\sigma)}{\zeta(\sigma)} - 2\Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^3 - 3\frac{\zeta'(\sigma)\zeta''(\sigma)}{(\zeta(\sigma))^2}\Big\}. \tag{8.2}$$

For EY^4 a completely analogous procedure leads to the relation

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\Lambda(n)(\log n)^3}{n^{\sigma}} &= (-1)^4\Big\{\frac{\zeta^{(4)}(\sigma)}{\zeta(\sigma)} - 6\Big(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)^4 - 3\Big(\frac{\zeta''(\sigma)}{\zeta(\sigma)}\Big)^2 \\ &\quad +12\frac{(\zeta'(\sigma))^2\zeta''(\sigma)}{(\zeta(\sigma))^3} - 4\frac{\zeta'(\sigma)\zeta'''(\sigma)}{(\zeta(\sigma))^2}\Big\}.\end{aligned} \tag{8.3}$$

The final verification that the expressions within braces in the right-hand sides of (8.2) and (8.3) coincide with the third and fourth derivatives of $\log\zeta(\sigma)$, respectively, is an exercise in differentiation, which we omit.

A Appendix – Some facts about Dirichlet series

Among the important tools in analytic number theory are Dirichlet series:

Definition A.1 *A* Dirichlet series with coefficients $\{f(n)\}$ *is a series if the form*

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^z}, \quad z\in\mathbb{C},$$

where $f(n)$ is an arithmetical function*, that is a real- or complex-valued function defined on the positive integers.* □

The Riemann zeta function is an example of a Dirichlet series (put $f(n)=1$ for all n).

Theorem A.1 Uniqueness; cf. [1], Theorem 11.5 *If*

$$A(z)=\sum_{n=1}^{\infty}\frac{a(n)}{n^z} \quad \textit{for} \quad \sigma>\sigma_0 \qquad \textit{and} \qquad B(z)=\sum_{n=1}^{\infty}\frac{b(n)}{n^z} \quad \textit{for} \quad \sigma>\sigma_0,$$

are such that $A(z_k)=B(z_k)$ for $z_k\nearrow\infty$ as $k\to\infty$, then $a(n)=b(n)$ for all n. □

Remark A.1 The probabilistic analog is the uniqueness theorem for transforms. □

Definition A.2 *The* Dirichlet product or convolution *h of the arithmetical functions f and g is*

$$h(n)=(f*g)(n)=\sum_{d|n}f(d)g\Big(\frac{n}{d}\Big).$$

(The notation $d|n$ means that the sum runs over the divisors of n). □

The following theorem, cf. [1], Theorem 11.5, tells us that the coefficients of the product of two Dirichlet series are obtained by convolving the respective coefficients, the probabilistic analog of which is that the transform of the sum of two independent random variables is obtained by multiplying the individual transforms.

Theorem A.2 *The product (convolution) of the two Dirichlet series*

$$A(z)=\sum_{n=1}^{\infty}\frac{a(n)}{n^z}\quad \text{for}\quad \sigma>a\qquad\text{and}\qquad B(z)=\sum_{n=1}^{\infty}\frac{b(n)}{n^z}\quad\text{for}\quad\sigma>b,$$

is

$$A(z)B(z)=\sum_{n=1}^{\infty}\frac{(a*b)(n)}{n^z}\quad\text{for}\quad\sigma>\max\{a,b\}.\qquad\square$$

Definition A.3 *The* derivative f' of an arithmetical function f *is*

$$f'(n)=f(n)\log n.\qquad\square$$

Our final tool ([1], Theorem 11.12), connects Dirichlet functions and their derivatives:

Theorem A.3 *The sum function*

$$F(z)=\sum_{n=1}^{\infty}\frac{f(n)}{n^z}$$

of a Dirichlet series is analytic in its half-plane of convergence, and its derivative $F'(z)$ is represented in the same half-plane by the Dirichlet series

$$F'(z)=\sum_{n=1}^{\infty}\frac{f(n)\log n}{n^z},$$

which is obtained by differentiating term by term. □

By repeated application of this result it follows, as a corollary (cf. [1], p. 236), that

$$F^{(k)}(z)=(-1)^k\sum_{n=1}^{\infty}\frac{f(n)(\log n)^k}{n^z}.\qquad\text{(A.1)}$$

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