Quaternionic Determinants

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1 Introduction

The classical matrix groups are of fundamental importance in many parts of geometry and algebra. Some of them, like $\mathrm{Sp}(n)$, are most conceptually defined as groups of quaternionic matrices. But, the quaternions not being commutative, we must reconsider some aspects of linear algebra. In particular, it is not clear how to define the determinant of a quaternionic matrix. Over the years, many people have given different definitions. In this article I will try to discuss some of these. I would like to thank Jon Berrick, P. M. Cohn, Soo Teck Lee and the referee for help with improving this paper.

Let us first briefly recall some basic facts about quaternions. The quaternions were discovered on October 16 1843 by Sir William Rowan Hamilton. (For more on the history, I recommend [19, 31, 47, 48].) They form a noncommutative, associative algebra over $\mathbb{R}$:

$$\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\},$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

We can also express $z \in \mathbb{H}$ in the form $z = x + jy$, where $x, y \in \mathbb{C}$, but then we have to remember that $yj = j\overline{y}$ for $y \in \mathbb{C}$. Notice that $\mathbb{H}$ is not an algebra over $\mathbb{C}$, since the center of $\mathbb{H}$ is only $\mathbb{R}$. Conjugation is defined by

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$\overline{a + ib + jc + kd} = a - ib - jc - kd$ and satisfies $\overline{uv} = \overline{v}\,\overline{u}$. We will call the quaternions of the form $ib + jc + kd$ with $b, c, d \in \mathbb{R}$ the pure quaternions.

For any ring R, we let R^* denote the set of units in R, i.e., the invertible elements of R. If R is a skewfield, then $R^* = R - \{0\}$. Let $M(n, R)$ be the ring of $n \times n$ matrices with entries in R. We will denote the set of invertible $n \times n$ matrices over R by $GL(n, R)$. (Some readers might worry about our definition of invertible in $M(n, \mathbb{H})$: Is there a distinction between left and right inverses? We will see later that there is no such problem. See also [15, 32].)

2 Cayley

The most simple-minded approach when trying to define the determinant of a quaternionic matrix would be to use the usual formula. But then the question is: Which usual formula? For a 2×2 determinant we could use $a_{11}a_{22} - a_{12}a_{21}$ (expanding along the first row) or $a_{11}a_{22} - a_{21}a_{12}$ (expanding along the first column) or some other ordering of the factors in the usual formula. To a modern mathematician, this lack of a canonical definition is an indication that this is *not* the correct approach. But we might still ask ourselves: What exactly would happen if we tried one of these formulas?

In 1845, just two years after Hamilton's discovery of the quaternions, Arthur Cayley ([10, 35]) did precisely this. He chose to expand both the original matrix and all the minors along the first column (or vertical row as he called it). If we denote the Cayley determinant by Cdet, we get

$$\operatorname{Cdet}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1$$

and

$$\operatorname{Cdet}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

Is this a good definition? Cayley himself points out that if two rows are the same in a 2×2 matrix, then

$$\operatorname{Cdet}\begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ab = 0,$$

while if two columns are the same in a 2×2 matrix, then

$$\operatorname{Cdet}\begin{pmatrix} a & a \\ b & b \end{pmatrix} = ab - ba,$$

which in general is nonzero. For some reason, this didn't seem to bother Cayley much, and he happily proceeded to write a couple more pages about his new function. But it should bother us.

Let us try to clarify the situation by first deciding on which properties we want the determinant to satisfy. Based on our experience with complex matrices, we will call $d\colon M(n, \mathbb{H}) \to \mathbb{H}$ a determinant if it satisfies the following three axioms.

Axiom 1 $d(A) = 0$ if and only if A is singular.

Axiom 2 $d(AB) = d(A)d(B)$ for all $A, B \in M(n, \mathbb{H})$.

Axiom 3 If A' is obtained from A by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $d(A') = d(A)$.

Let me make some comments about these axioms. It can be shown ([7]) that if d is not constantly equal to 0 or 1, then Axiom 2 implies that $d(A) = 0$ for all singular matrices. Thus we only need to define the determinant of invertible matrices.

Notice that in Axiom 3 there is a distinction between left and right scalar multiplication. Consider the mapping $T(v) = cv$. Then

$$T(dv) = c(dv)$$

is in general different from

$$dT(v) = d(cv),$$

while

$$T(vd) = c(vd) = cvd = T(v)d.$$

We see that we must write the coefficients of a linear transformation on the opposite side of what we use for the vector space structure. I will identify vectors with columns and identify linear transformations with matrices on the left, but consider all vector spaces to be right vector spaces.

Axiom 3 can be expressed in terms of matrix multiplication. Let e_{ij} be the matrix with a 1 in the (i, j) entry and 0 otherwise. Define

$$B_{ij}(b) = I_n + be_{ij}, \qquad \text{for } i \neq j.$$

Multiplying a matrix A by $B_{ij}(b)$ on the left adds the j-th row multiplied by b on the left to the i-th row, while multiplying A by $B_{ij}(b)$ on the right adds the i-th column multiplied by b on the right to the j-th column. So Axiom 3 can be restated (using Axiom 2) as saying that $d(B_{ij}(b)) = 1$.

It is easy to see that

$$B_{ij}(b)^{-1} = B_{ij}(-b),$$

so it follows that products of $B_{ij}(b)$'s generate a subgroup of $GL(n, \mathbb{H})$, which we will denote by $SL(n, \mathbb{H})$. Notice that when K is a field, we define $SL(n, K)$ to be the set of matrices with determinant equal to 1. But since we don't have a determinant yet, we must define $SL(n, \mathbb{H})$ in some other way, and then hope that once we have our determinant, it will have $SL(n, \mathbb{H})$ as its kernel. That Axiom 3 can be restated as saying that matrices in $SL(n, \mathbb{H})$ have determinant equal to 1 is therefore promising.

An obvious question is now whether such determinants exist. Let me first state a simple obstruction.

Theorem 1. *Assume that d is a determinant, i.e., d satisfies our three axioms. Then the image $d\,(M(n, \mathbb{H}))$ is a commutative subset of $\mathbb{H}$.*

This theorem essentially says that when trying to define a quaternionic determinant, we must keep it complex-valued. This rules out Cayley's definition, since Cdet is onto $\mathbb{H}$.

The proof of Theorem 1 depends on the next two lemmas. We first observe that the definition of $B_{ij}(b)$ only involves two indices. We can therefore often assume without loss of generality that $n = 2$. A simple calculation proves the following lemma.

Lemma 2. *Let $a \neq 0$ and d be a determinant. Then*

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}-1 \\ 0 & 1 \end{pmatrix}$$

and

$$d\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 1$$

The next lemma is crucial.

Lemma 3. *Every $A \in GL(n, \mathbb{H})$ can be written in the form*

$$A = D(x)B,$$

where

$$D(x) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & x \end{pmatrix}$$

and $B \in SL(n, \mathbb{H})$.

Proof. Since A is invertible, there must be at least one nonzero element in the first row, say $a_{1j} \neq 0$. By adding the j-th column multiplied by $a_{1j}^{-1}(1 - a_{11})$ on the right to the first column, we get a matrix with $a_{11} = 1$. We can then make all the other entries in the first row equal to zero, and proceed by induction. □

The observant reader may now be wondering about the uniqueness of the $A = D(x)B$ decomposition. But it is more urgent to prove Theorem 1.

Proof of Theorem 1. Define $f\colon \mathbb{H} \to \mathbb{H}$ by

$$f(x) = d\left(D(x)\right).$$

It follows from Lemma 3 that $f(\mathbb{H}) = d\left(M(n, \mathbb{H})\right)$. For simplicity of notation we will assume that $n = 2$. We have

$$d\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = d\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\right) = f(x)$$

by Axiom 2 and Lemma 2. But then

$$f(x)f(y) = d\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\right) = d\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = d\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = f(y)f(x).$$

and we see that $f(\mathbb{H}) = d(M(n, \mathbb{H}))$ is commutative. □

It is now time to ask how Cayley's definition fits into this. It clearly cannot satisfy all the three axioms. In fact, it doesn't satisfy any of them! Consider the matrix

$$M = \begin{pmatrix} k & j \\ i & 1 \end{pmatrix}.$$

It is easy to prove that if

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then $x = y = 0$, so M is invertible. But

$$M^t\begin{pmatrix} -1 \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so M^t is singular. But $\operatorname{Cdet} M = 0$ and $\operatorname{Cdet} M^t = 2k$, so we see that Axiom 1 fails. This also shows that the transpose is not a very useful concept in quaternionic linear algebra. The reason is that it is neither an automorphism nor an antiautomorphism! (But notice that Hermitian involution, $M^* = \overline{M}^t$, *is* an antiautomorphism, i.e., $(MN)^* = N^*M^*$.) For similar reasons, the concept of rank is also more complicated. The right column rank is the same as the left row rank, but they might be distinct from the the left column rank, which is equal to the right row rank ([12]). Noting that

$$\mathrm{Cdet}\left(\begin{pmatrix}1 & i\\ j & k\end{pmatrix}\begin{pmatrix}k & j\\ i & 1\end{pmatrix}\right) = 2 - 2k$$

while

$$\mathrm{Cdet}\begin{pmatrix}1 & i\\ j & k\end{pmatrix}\mathrm{Cdet}\begin{pmatrix}k & j\\ i & 1\end{pmatrix} = 0,$$

we see that Axiom 2 also fails.

As for Axiom 3, we have

$$\mathrm{Cdet}\begin{pmatrix} ab & b\\ a & 1\end{pmatrix} = 0,$$

but after subtracting the second row multiplied by b on the left from the second row, we get

$$A' = \begin{pmatrix}ab - ba & 0\\ 0 & 1\end{pmatrix},$$

and $\mathrm{Cdet}(A') = ab - ba$, which in general is nonzero.

This clearly shows that Cdet is not the way to go. A more promising lead is before us, in Lemma 3. It will be followed up later.

Let me finish this section with a remark about Theorem 1. It is inspired by a related theorem proved by the physicist and mathematician Freeman J. Dyson in 1972 ([21]). He used a different third axiom:

Axiom $3'$ Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. If for some row index r we have

$$a_{ij} = b_{ij} = c_{ij}, \quad i \neq r, \qquad \text{and} \qquad a_{ri} + b_{ri} = c_{ri},$$

then $d(A) + d(B) = d(C)$.

In other words, d should be additive in the rows. He then proved that if d satisfies Axioms 1, 2 and $3'$, then the image of d is commutative. It is easy to see that Axioms 1, 2 and $3'$ imply Axiom 3. We just have to prove that $d(B_{ij}(b)) = 1$. Let B' be the matrix obtained by replacing the i-th entry along the diagonal in $B_{ij}(b)$ by a 0. Then B' is singaular, and it follows from Axiom $3'$ that $d(B_{ij}(b) = 1$.

It follows that his definition of determinant is more restrictive that ours. But it is in fact too restrictive. Determinants satisfying his three axioms simply don't exist over the quaternions! Why? It follows from Axiom 2 that $d(I_n) = 1$. Define

$$D(x) = \begin{pmatrix}1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & x\end{pmatrix}.$$

Since $I_n + D(-1) = 2D(0)$ is singular, it follows from Axioms 1 and 3′ that $d(D(-1)) = -1$. Since $-1 = iji^{-1}j^{-1}$, we get $D(-1) = D(i)D(j)D(i)^{-1}D(j)^{-1}$, so $D(-1)$ is a commutator in $GL(n, \mathbb{H})$. But Axiom 2 and Theorem 1 then implies that $d(D(-1)) = 1$, which is a contradiction.

3 Study

Concerning quaternionic determinants, nothing much happened during the 75 years after Cayley. In the second (posthumous) edition of W. R. Hamilton's book *Elements of Quaternions* ([24]) from 1889, the editor added an appendix, which was just a restatement of Cayley's paper. And a paper by J. M. Peirce ([38]) from 1899 is just a laborious elaboration on the Cayley determinant. But in 1920 a very interesting paper by Eduard Study appeared ([44]). (For more details, see also [16, 23, 46].) His idea was to transform a quaternionic matrix into a complex $2n \times 2n$ matrix and then take the determinant.

I will start by discussing some important homomorphisms between quaternionic, complex and real matrices. Recall that any complex $n \times n$ matrix can be written uniquely as $N = C + iD$, where C, D are real $n \times n$ matrices. We can then define an injective algebra homomorphism $\phi\colon M(n, \mathbb{C}) \to M(2n, \mathbb{R})$ by

$$\phi(C + iD) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}.$$

Set

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Let R_i be right-multiplication by i on $\mathbb{C}^n$. The corresponding matrix is iI, and $J = \phi(iI) = \phi(R_i)$ (I will sometimes identify a linear transformation and its standard matrix). This gives a complex structure on $\mathbb{R}^{2n}$, and we know that $P \in M(2n, \mathbb{R})$ corresponds to a complex linear transformation if and only if P commutes with the complex structure. Hence

$$\phi(M(n, \mathbb{C})) = \{\, P \in M(2n, \mathbb{R}) \mid JP = PJ \,\}.$$

In a similar way, any quaternionic $n \times n$ matrix can be expressed uniquely in the form $M = A + jB$, where A, B are complex $n \times n$ matrices. (We write j on the left since we work with right vector spaces.) We can therefore define $\psi\colon M(n, \mathbb{H}) \to M(2n, \mathbb{C})$ by

$$\psi(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}.$$

It is straightforward to show that this map is an injective algebra homomorphism. (This implies in particular that there is no distinction between left- and right-inverses in $GL(n, \mathbb{H})$.)

Let R_j be right-multiplication by j on $\mathbb{H}^n$. Notice that any $\mathbb{H}$-linear transformation commutes with R_j, but that R_j is *not* $\mathbb{H}$-linear . Thus there is no matrix associated to R_j, and it doesn't make sense to talk about $\psi(R_j)$, but we can still consider the corresponding map of $\mathbb{C}^{2n}$ given by $\widetilde{R_j}(x,y) = (-\overline{y}, \overline{x})$. We see that $\widetilde{R_j}$ corresponds to first multiplying by J and then conjugating. This gives a quaternionic structure on $\mathbb{C}^{2n}$, and we know that $N \in M(2n, \mathbb{C})$ corresponds to a quaternionic linear transformation if and only N commutes with the quaternionic structure. Since $N\overline{Jv} = \overline{\overline{N}Jv}$, we have $N\overline{Jv} = \overline{JNv}$ if and only if $\overline{N}J = JN$, so

$$\psi(M(n, \mathbb{H})) = \{ N \in M(2n, \mathbb{C}) \mid JN = \overline{N}J \}. \tag{1}$$

Notice that this is simply a generalization of the formula $jz = \overline{z}j$ for $z \in \mathbb{C}$.

It follows immediately from (1) that $\det_{\mathbb{C}} \psi(M) \in \mathbb{R}$, but we will soon see that in fact we have $\det_{\mathbb{C}} \psi(M) \geq 0$. (I will sometimes write $\det_{\mathbb{R}}$ or $\det_{\mathbb{C}}$ to stress that I'm taking the determinant of a real or complex matrix.)

By applying the homomorphism $\phi_1 \colon \mathbb{C} \cong M(1, \mathbb{C}) \to M(2, \mathbb{R})$ to each element of $M \in M(n, \mathbb{C})$ we get a map $\widetilde{\phi} \colon M(n, \mathbb{C}) \to M(2n, \mathbb{R})$. ($\phi(N)$ consists of 4 n-blocks, while $\widetilde{\phi}(N)$ consists of n^2 2-blocks.) The important thing here is that the 2-blocks in $\widetilde{\phi(N)}$ are easier to manage than the the n-blocks in $\phi(N)$. Since $\mathbb{C}$ is commutative and ϕ_1 is a homomorphism, the 2-blocks in $\widetilde{\phi}(N)$ commute. This allows us to use the following folklore theorem. (It has been rediscovered numerous times, but to the best of my knowledge it is originally due to M. H. Ingraham ([26]).)

Theorem 4. *If $A = (A_{ij})$ is a square block matrix, where the A_{ij} are mutually commutative $m \times m$ matrices, and B is the $m \times m$ matrix obtained by taking the determinant of A with the A_{ij} as elements, then* $\det A = \det B$.

For example, if A_{11}, A_{12}, A_{21}, A_{22} are mutually commutative, then

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det \left(A_{11}A_{22} - A_{12}A_{21} \right).$$

In other words, you evaluate by "taking the determinant twice".

By shuffling some rows and columns, we see that $\det_{\mathbb{R}} \phi(N) = \det_{\mathbb{R}} \widetilde{\phi}(N)$, and we can now apply Theorem 4 to get ([6])

$$\begin{aligned} \det_{\mathbb{R}} \phi(N) &= \det_{\mathbb{R}} \widetilde{\phi}(N) = \det_{\mathbb{R}} (\phi_1(\det_{\mathbb{C}} N)) \\ &= \det_{\mathbb{R}} \begin{pmatrix} \operatorname{Re} \det_{\mathbb{C}} N & -\operatorname{Im} \det_{\mathbb{C}} N \\ \operatorname{Im} \det_{\mathbb{C}} N & \operatorname{Re} \det_{\mathbb{C}} N \end{pmatrix} = |\det_{\mathbb{C}} N|^2, \end{aligned} \tag{2}$$

for $N \in M(n, \mathbb{C})$. This discussion leads to the following important theorem.

Theorem 5. *For any complex matrix N, we have*

$$\det_{\mathbb{R}} \phi(N) = |\det_{\mathbb{C}} N|^2 \geq 0. \tag{3}$$

For any quaternionic matrix M, we have

$$\det_{\mathbb{C}} \psi(M) = \sqrt{\det_{\mathbb{R}} \phi\big(\psi(M)\big)} \geq 0. \tag{4}$$

Proof. The first part follows from (2). It follows from (1) that $\det_{\mathbb{C}} \psi(M) \in \mathbb{R}$ and since $\det \phi(GL(n, \mathbb{H}))$ is a connected subset of $\mathbb{R}$, we get that Sdet $M \geq 0$ for quaternionic matrices. We then deduce (4) from (2). $\square$

We are now finally ready to define the Study determinant Sdet by

$$\operatorname{Sdet} M = \det_{\mathbb{C}} \psi(M).$$

The obvious question is now which axioms the Study determinant satisfies. The Study determinant satisfies Axiom 2 since ψ is a homomorphism. Let us show that Axiom 1 holds. (Notice that the proof of this statement is wrong in both editions of the otherwise excellent book by Morton L. Curtis ([16]).) We know that if Sdet $M = \det_{\mathbb{C}} \psi(M) \neq 0$, then $\psi(M)$ is invertible in $M(2n, \mathbb{C})$, but we need to know that the inverse actually lies in $\psi(M(n, \mathbb{H}))$. But by conjugating and inverting the formula $J\psi(M) = \overline{\psi(M)}J$, we see that $J\psi(M)^{-1} = \overline{\psi(M)^{-1}}J$. But then it follows from (1) that $\psi(M)^{-1}$ lies in $\psi(M(n, \mathbb{H}))$.

To show that Axiom 3 holds, it suffices to prove that Sdet $B_{ij}(b) = 1$. If $b = b_1 + jb_2$, then

$$\psi(B_{ij}(b)) = \begin{pmatrix} I_n + b_1 e_{ij} & -\overline{b_2} e_{ij} \\ b_2 e_{ij} & I_n + \overline{b_1} e_{ij} \end{pmatrix}.$$

But since $e_{ij} e_{ij} = 0$, we can apply Theorem 4 to get $\det(\psi(B_{ij}(b)) = \det(I_n) = 1$

Thus the Study determinant satisfies all our axioms, and it is used frequently in differential geometry and Lie theory ([23]). But bear in mind that it is a *quadratic* function of the entries, not multilinear in the rows and the columns like the usual determinant.

Let me finish this section with a couple of additional comments. The Study determinant was defined above by identifying $\mathbb{H}$ with $\mathbb{C}^2$. What would happen if we instead identified $\mathbb{H}$ with $\mathbb{R}^4$? After all, the center of $\mathbb{H}$ is $\mathbb{R}$, not $\mathbb{C}$, so the quaternions form an $\mathbb{R}$-algebra. We can write $M \in M(n, \mathbb{H})$ uniquely as

$M = A_0 + iA_1 + jA_2 + kA_3$ where A_0, A_1, A_2, A_3 are real $n \times n$ matrices and apply the homomorphism $\mu\colon M(n, \mathbb{H}) \to M(4n, \mathbb{R})$ given by

$$\mu(A_0 + iA_1 + jA_2 + kA_3) = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix}.$$

Notice that

$$\phi\psi(A_0+iA_1+jA_2+kA_3) = \begin{pmatrix} A_0 & -A_2 & -A_1 & A_3 \\ A_2 & A_0 & A_3 & A_1 \\ A_1 & -A_3 & A_0 & -A_2 \\ -A_3 & -A_1 & A_2 & A_0 \end{pmatrix} \neq \mu(A_0+iA_1+jA_2+kA_3),$$

but it is easy to see that by shuffling some rows, columns and signs, we get (see also ([4, 30])

$$\det{}_{\mathbb{R}}\mu(M) = \det{}_{\mathbb{R}}\phi(\psi(M)) = \mathrm{Sdet}(M)^2.$$

Finally, we also note that in general

$$\psi(M^t) = \psi(A^t + jB^t) = \begin{pmatrix} A^t & -\overline{B}^t \\ B^t & \overline{A}^t \end{pmatrix} \neq \begin{pmatrix} A^t & B^t \\ -\overline{B}^t & \overline{A}^t \end{pmatrix} = \psi(M)^t,$$

while

$$\psi(M^*) = \psi(\overline{A}^t + \overline{jB}^t) = \psi(\overline{A}^t - \overline{B}^t j) = \psi(\overline{A}^t - jB^t) = \begin{pmatrix} \overline{A}^t & \overline{B}^t \\ -B^t & A^t \end{pmatrix} = \psi(M)^*.$$

Hence $\mathrm{Sdet}\, M^* = \overline{\mathrm{Sdet}\, M} = \mathrm{Sdet}\, M$; but in general $\mathrm{Sdet}\, M^t \neq \mathrm{Sdet}\, M$, since as we saw earlier, M can be invertible while M^t is singular.

4 Dieudonné

Study was not the only one studying quaternionic determinants in his time. In the next 10 years, A. Heyting, E. H. Moore, Ø. Ore and A. R. Richardson all wrote about this topic ([25, 34, 36, 42, 43]). The paper by Øystein Ore ([36]) is important because it introduces the concept of the ring of fractions for a noncommutative ring. But from the point of view of determinants, the most interesting are the papers by A. R. Richardson ([42, 43]) (this is the Richardson in the Littlewood-Richardson rule, where Littlewood is not the one in Hardy-Littlewood). His main contribution was to make it apparent that commutators play a key role. His papers are filled with formulas involving commutators.

Let us go back to studying $SL(n, \mathbb{H})$ and take a closer look at Lemma 3. It is easy to see that $SL(n, \mathbb{H})$ is a normal subgroup of $GL(n, \mathbb{H})$, and it can be shown ([1, 15, 17, 40]) that $SL(n, \mathbb{H})$ is the commutator subgroup of $GL(n, \mathbb{H})$.

Lemma 6. $SL(n, \mathbb{H}) = [GL(n, \mathbb{H}), GL(n, \mathbb{H})]$.

Let me mention in passing that for any field k the commutator of $GL(n, k)$ is $SL(n, k)$, except when $n = 2$ and k is $\mathbb{Z}_2$ or $\mathbb{Z}_3$ ([15]).

The main reason why Lemma 3 is so crucial is that it shows that we only need to define our determinant on the matrices $D(x)$. But you may be impatient for me to get back to the issue of uniqueness. Since $SL(n, \mathbb{H})$ is normal in $GL(n, \mathbb{H})$, the question becomes: For which $x \in \mathbb{H}$ does $D(x)$ lie in $SL(n, \mathbb{H})$? The answer is given by the following lemma.

Lemma 7.

$$D(x) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & x \end{pmatrix}$$

is a commutator in $GL(n, \mathbb{H})$ *(i.e., it lies in* $SL(n, \mathbb{H})$*) if and only if* x *is a commutator in* $\mathbb{H}^*$.

Proof. One direction is trivial:

$$\begin{pmatrix} 1 & 0 \\ 0 & aba^{-1}b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

The other direction, however, is not so easy. It is essentially equivalent to showing that the Dieudonné determinant is well defined, so it is an easy consequence of results in [1, 17, 40], and I refer the reader to those excellent sources for the details. □

It follows that in the decomposition $A = D(x)B$, neither x nor B is unique, but that the coset $x[\mathbb{H}^*, \mathbb{H}^*] \in \mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$ *is* unique. This is exactly what Jean Dieudonné used in his 1943 paper ([17]). His goal was to show how the determinant could be expressed in terms of group theory. We would expect

$$\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \det\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix},$$

but then we probably need the determinant to take values in a commutative ring, and we get that by considering $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$. His main theorem states that for any skewfield K, there's an isomorphism

$$GL(n, K)/[GL(n, K), GL(n, K)] \to K^*/[K^*, K^*].$$

For $K = \mathbb{H}$, this is immediate from Lemmas 3 and 7. We therefore define the Dieudonné determinant by

$$\det A = \det(D(x)B) = x[\mathbb{H}^*, \mathbb{H}^*]$$

Thanks to Lemma 7, we see that this is well defined, and that the kernel is precisely $SL(n, \mathbb{H})$, i.e., our definition of $SL(n, \mathbb{H})$ agrees with the usual one, once we have the determinant.

If we now extend to $M(n, \mathbb{H})$, we get a determinant that takes values in $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*] \cup \{0\}$. But what does this set look like? We need the following lemma.

Lemma 8. $[\mathbb{H}^*, \mathbb{H}^*]$ *is isomorphic to the set of quaternions of length one.*

Proof. It is clear that every commutator has length one. The set of quaternions of length one can be identified with S^3, and $\psi(S^3) = SU(2)$. But every element of $SU(2)$ is conjugate to a diagonal element, so it follows that every element in S^3 is conjugate to an element of S^1, the unit circle of $\mathbb{C} \subset \mathbb{H}$. (This also follows from the Noether-Skolem Theorem.) So given $z \in S^3$, we can write $z = xyx^{-1}$ with $y \in S^1$.

We can identify the pure quaternions with $\mathbb{R}^3$, and for $p, q \in \mathbb{R}^3$ we have $p^{-1} = \overline{p}/|p|^2 = -p/|p|^2$ and

$$pq = -\langle p, q\rangle + p \times q,$$

where $\langle\,,\rangle$ is the usual inner product on $\mathbb{R}^3$ and $\times$ is the vector product in $\mathbb{R}^3$. From this we can easily deduce that every quaternion can be written as the product of two pure quaternions.

Since y is complex, we can find $w \in \mathbb{C}$ with $y = w^2$ and it follows from the above that we can write $w = pq$ where $p, q \in R^3$. Since $|w| = |y| = 1$, we can also assume that $|p| = |q| = 1$, so $p^{-1} = -p$ and $q^{-1} = -q$. But then

$$\begin{aligned} z = xpqpqx^{-1} &= xpq(-p)(-q)x^{-1} = xpqp^{-1}q^{-1}x^{-1} \\ &= (xpx^{-1})(xqx^{-1})(xpx^{-1})^{-1}(xqx^{-1})^{-1}. \end{aligned}$$ □

For other proofs, see [9, 17, 50]. It follows that $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$ is isomorphic to the positive real numbers. Define

$$\omega\colon \mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*] \to \mathbb{R}^+ \qquad \text{by} \qquad \omega(x[\mathbb{H}^*, \mathbb{H}^*]) = |x|,$$

and define the normalized Dieudonné determinant by

$$\mathrm{Ddet}(M) = \omega(\det(M)).$$

Dieudonné showed ([17]) that any determinant function satisfying our three axioms will be of the form

$$d(M) = \mathrm{Ddet}^r(M) \tag{5}$$

for some $r \in \mathbb{R}$. In particular, we can easily check the following theorem.

Theorem 9.

$$\mathrm{Sdet}\, M = \det_{\mathbb{C}}(\psi(M)) = \mathrm{Ddet}^2(M) \tag{6}$$

$$\det_{\mathbb{R}}\mu(M) = \det_{\mathbb{R}}\phi(\psi(M)) = \mathrm{Ddet}^4(M). \tag{7}$$

Let me also point out that it follows from (6) that the Study determinant corresponds to the reduced norm ([15]).

Equation (5) has been generalized by L. E. Zagorin ([52]). If ν is a homomorphism of $\mathbb{H}$ into $M(s, \mathbb{C})$, and $\overline{\nu}$ is the corresponding homomorphism of $M(n, \mathbb{H})$ into $M(ns, \mathbb{C})$, then $\det_{\mathbb{C}} \overline{\nu}(M) = \mathrm{Ddet}^s(M)$.

In addition to our three axioms, the Dieudonné determinant satisfies several other properties ([1, 17, 40]). Interchanging rows i and j corresponds to left multiplying by the matrix $P_{ij} = B_{ij}(1)B_{ji}(-1)B_{ij}(1)$. But since $-1 \in [\mathbb{H}^*, \mathbb{H}^*]$, we get $\det P_{ij} = 1[\mathbb{H}^*, \mathbb{H}^*]$, so interchanging two rows doesn't change the determinant.

When $n = 2$,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a & b \\ 0 & d - ca^{-1}b \end{pmatrix} = (ad - aca^{-1}b)[\mathbb{H}^*, \mathbb{H}^*], \qquad \text{if } a \neq 0$$

$$\det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = cb[\mathbb{H}^*, \mathbb{H}^*] = -bc[\mathbb{H}^*, \mathbb{H}^*].$$

We can also show that multiplying a row on the left by m or multiplying a column on the right by m multiplies the determinant by $m[\mathbb{H}^*, \mathbb{H}^*]$. (This last product can be either on the left or on the right, since $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$ is commutative.) But since

$$\det \begin{pmatrix} 1 & a \\ b & ab \end{pmatrix} = (ab - ba)[\mathbb{H}^*, \mathbb{H}^*],$$

while

$$b[\mathbb{H}^*, \mathbb{H}^*] \det \begin{pmatrix} 1 & a \\ 1 & a \end{pmatrix} = 0,$$

we see that we cannot factor out a right multiple of a row.

In addition, it doesn't behave well with respect to addition. Consider the determinant as a function of the first row, keeping the other rows fixed. Denote this function by $m(v)$. Define addition in $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$ by setting

$$a[\mathbb{H}^*, \mathbb{H}^*] + b[\mathbb{H}^*, \mathbb{H}^*] = \{ak_1 + bk_2 \mid k_1, k_2 \in [\mathbb{H}^*, \mathbb{H}^*]\}.$$

It can then be shown ([1]) that

$$m(v_1 + v_2) \subset m(v_1) + m(v_2).$$

If we use Ddet instead of det and denote the corresponding function by $M(v)$, we get a sort of triangle inequality.

$$M(v_1 + v_2) \leq M(v_1) + M(v_2).$$

5 Moore

We started out by showing what was wrong with the Cayley determinant. But sometimes it does work. Granted that his formula doesn't make sense in general, does it still make sense for certain matrices? The answer is that if we restrict to Hermitian quaternionic matrices ($M^* = M$), then we get a useful function by specifying a certain ordering of the factors in the $n!$ terms in the sum. This was first studied by Eliakim Hastings Moore (for biographical information about Moore, see [37]), and I will denote it by Mdet.

Let σ be a permutation of n. Write it as a product of disjoint cycles. Permute each cycle cyclically, until the smallest number in the cycle is in front. Then sort the cycles in decreasing order according to the first number of each cycle. In other words, write

$$\sigma = (n_{11} \cdots n_{1l_1})(n_{21} \cdots n_{2l_2}) \cdots (n_{r1} \cdots n_{rl_r}),$$

where for each i, we have $n_{i1} > n_{ij}$ for all $j > 1$, and $n_{11} > n_{21} > \cdots > n_{r1}$. Then we define

$$\operatorname{Mdet} M = \sum_{\sigma \in S_n} |\sigma| m_{n_{11}n_{12}} \cdots m_{n_{1l_1}n_{11}} m_{n_{21}n_{22}} \cdots m_{n_{rl_r}n_{r1}}.$$

If H is Hermitian, then Mdet H is a real number. I will not go into details, but refer to the work of Moore, Jacobson, Dyson, Mehta, Chen, van Praag and Piccinni. ([5, 11, 12, 20, 21, 27, 28, 32, 33, 34, 39, 49, 50]). But I would again like to make some comments.

In general, it is difficult to talk about eigenvalues of a quaternionic matrix ([13, 29]). Since we work with right vector spaces, we must consider right eigenvalues. If

$$Mx = x\lambda,$$

then for $q \neq 0$, we get

$$M(xq) = x\lambda q = (xq)q^{-1}\lambda q.$$

Hence all the conjugates of λ are also eigenvalues.

Let us study the conjugacy classes more closely. For $q \in \mathbb{H}$, we define $\rho(q)$ by $\rho(q)(x) = qxq^{-1}$. Since $\rho(q)$ leaves the real axis invariant and is orthogonal, we can restrict to $\mathbb{R}^3$. It is easy to see ([18]) that if we write $q = q_0 + q'$ with $q_0 \in \mathbb{R}$ and $q' \in \mathbb{R}^3$, then $\rho(q)$ represents the rotation of $\mathbb{R}^3$ with axis q' and angle $2\arctan(|q'|/q_0)$. From this we get that if x is real, then the conjugacy class of x is just $\{x\}$, while for $x \in S^3 - \{\pm 1\}$, we get a copy of S^2 containing x and orthogonal to the real axis. Suppose that $\lambda = \lambda_0 + \lambda'$ with $\lambda_0 \in \mathbb{R}$ and $\lambda' \in \mathbb{R}^3$. Then $q\lambda q^{-1} = \lambda_0 + q\lambda' q^{-1}$, and the conjugacy class of λ' intersects the i-axis at $\pm|\lambda'|i$. It follows that the conjugacy class of a non-real eigenvalue contains exactly two complex numbers and that they are conjugate.

If p is complex and $v = u + jw$, then $Mv = vp$ if and only if $\psi(M)(u\,w)^t = (u\,w)^t p$, and it can be proved by induction ([29]) that the eigenvalues of $\psi(M)$ occur in conjugate pairs. It follows that the eigenvalues of $\psi(M)$ are precisely the $2n$ numbers $\lambda_1, \ldots, \lambda_n$ and $\overline{\lambda_1}, \ldots, \overline{\lambda_n}$, while the eigenvalues of M are the elements of the conjugacy classes of $\lambda_1, \ldots, \lambda_n$, where we can replace λ_i by $\overline{\lambda_i}$.

It is now easy to show ([29]) that M is symplectically similar to a triangular matrix with diagonal elements d_i, where d_i equals λ_i or $\overline{\lambda_i}$. For more about normal forms of quaternionic matrices see [27, 29, 41, 45, 51]

If we restrict to a Hermitian matrix, H, then it turns out that all its eigenvalues are real (and there are therefore precisely n of them, since each conjugacy class only contains one element) and that the matrix can be symplectically diagonalized. That is, we can find $P \in GL(n, \mathbb{H})$ such that

$$PH\overline{P}^t = D,$$

where $\overline{P}^t = P^{-1}$ and D is diagonal and real.

We can now prove the following theorem that relates the Moore determinant to the other determinants.

Theorem 10. *Let H be a Hermitian quaternionic matrix. Then*

$$|\operatorname{Mdet} H| = \operatorname{Ddet} H \qquad \textit{and} \qquad \operatorname{Mdet} H[\mathbb{H}^*, \mathbb{H}^*] = \det H. \tag{8}$$

For any quaternionic matrix M, we have

$$\operatorname{Sdet} M = \operatorname{Mdet}(MM^*). \tag{9}$$

Proof. It can be shown that for a Hermitian matrix, H, the Moore determinant is equal to the product of the eigenvalues, so $\operatorname{Mdet} H$ is real valued. But the normalized Dieudonné determinant of a diagonal matrix is the norm of the product of the diagonal elements, so (8) follows. To prove (9), we just have to observe that the eigenvalues of AA^* are positive, and use the product rule and (6). □

Finally, if H is Hermitian, then

$$(J\psi(H))^t = -\psi(H)^t J = -J\overline{\psi(H)}^t = -J\psi(H),$$

so $J\psi(H)$ is skewsymmetric, and we can take its Pfaffian ([14]). But since

$$\mathrm{pf}(-J\psi(H))^2 = \det_{\mathbb{C}}(-J\psi(H)) = \mathrm{Ddet}^2 H = \mathrm{Mdet}^2 H,$$

we get

$$\mathrm{Mdet}(H) = \mathrm{pf}(-J\psi(H)).$$

For other applications of the Pfaffian, see [2, 3].

6 $SP(n)$

I would like to finish with a simple application of these ideas. As mentioned in the introduction, the group $SP(n)$ can be defined as the group preserving the norm on $\mathbb{H}^n$. But the usual description of this group is by considering its image under ψ in $M(\mathbb{C}, 2n)$. It is easy to see that all such matrices have determinant ± 1. There are different ways of proving that in fact the determinant is equal to 1, but this also follows from the results above, since all matrices in $\psi(GL(\mathbb{H}, n))$ have positive determinant.

In conclusion, I would also like to mention the recent work of Gelfand ([22]). Unfortunately, it is beyond the scope of this article to report on his work.

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