Estimating Fatigue Damage from Stress Power Spectral Density Functions
Revision A

By Tom Irvine
Email: tom@vibrationdata.com

August 5, 2014

Introduction

The rainflow method is a method for counting fatigue stress-reversal cycles from a time history, as shown in Reference 1. The rainflow method allows the application of the Palmgren-Miner rule in order to assess the fatigue life of a structure subject to complex loading.

Fatigue counting can also be performed in the frequency domain using semi-empirical methods. A comprehensive review of such methods for estimating fatigue damage from variable amplitude loading is presented. The dependence of fatigue damage accumulation on power spectral density (PSD) is investigated for random processes relevant to real structures such as in offshore or aerospace applications.

Beginning with the Rayleigh (or narrow band) approximation, attempts at improved approximations or corrections to the Rayleigh approximation are examined by comparison to rainflow analysis of time histories simulated from PSD functions representative of simple theoretical and real world applications. Methods investigated include corrections by Wirsching and Light, Lutes et al., Ortiz and Chen, the Dirlik formula, and the Single-Moment method, among other more recent proposed methods.

An average cumulative damage index is proposed based on these methods.

Variables

$D(\tau)$	is the accumulated damage function due to stresses or strains occurring up to the time τ
D_{NB}	is the narrowband damage
A	is the fatigue strength coefficient
α	is the irregularity factor
ε	is the spectral width
β_k	is the generalized spectral bandwidth
ν_0^+	is the equivalent frequency (Hz) based on rate of positive slope zero crossings

ν_p	is the rate of peaks
m	is the fatigue strength exponent from the material S-N curve
F	is the frequency (Hz)
W_S	is the one-sided, wideband stress power spectral density (stress^2/Hz)
M_j	is the *j* th moment of the one-sided spectral density
S	is the stress cycle range (peak-to-valley)
N(S)	is the histogram function for stress cycles
τ	is the exposure time (sec)
σ_S	is the response stress overall RMS level
λ	is the narrowband correction factor
Γ	is the gamma function

The fatigue strength coefficient *A* for the purpose of this paper is taken as the stress level on the S-N curve where N=1/2 cycles, assuming no reduction in slope due to strain hardening in the low-cycle range.

An alternate method is to calculate the *A* value using fracture mechanics for the case of a preexisting crack.

Note that

$$M_j = \int_0^\infty f^{\,j}\, W_S(f)\, df \tag{1}$$

The index *j* may be a non-integer.

The rate of zero up-crossings can be estimated as

$$\nu_0^+ = \sqrt{M_2/M_0} \tag{2}$$

The rate of peaks is

$$\nu_p = \sqrt{M_4/M_2} \tag{3}$$

The irregularity factor is

$$\alpha_i = \frac{M_i}{\sqrt{M_0\, M_{2i}}} \tag{4}$$

A special case is

$$\alpha_2 = \frac{M_2}{\sqrt{M_0 \, M_4}} = \frac{\nu_0^+}{\nu_p} \tag{5}$$

The spectral width is

$$\varepsilon = \sqrt{1 - \alpha_2{}^2} \tag{6}$$

Narrowband Stress

The narrowband damage D_{NB} from Reference 2 is

$$D_{NB} = \frac{\nu_0^+ \tau}{A} \left(\sqrt{2}\,\sigma_S\right)^m \Gamma\left(\frac{1}{2} m + 1\right) \tag{7}$$

Note that failure is assumed to occur when D $\geq$ 1.0. But some references use thresholds which are conservatively lower.

Wideband Stress Methods

Correction Factor

The wideband damage can be estimated from the narrowband damage as

$$D = \lambda \, D_{NB} \tag{8}$$

The λ value is a generic scale factor. Specific types are given

Wirsching and Lite

Wirsching and Lite developed the following correction factor by simulating processes having a variety of spectral shapes, in Reference 2.

$$\lambda_W(\varepsilon, m) = a(m) + [1 - a(m)](1 - \varepsilon)^{b(m)} \tag{9}$$

$$a(m) = 0.0926 - 0.033 \text{ m} \tag{10}$$

$$b(m) = 1.587 \text{ m} - 2.323 \tag{11}$$

Note that Wirsching and Lite used the S-N slope m values of 3, 4, 5 and 6 for the simulations.

Larsen and Lutes

Larsen and Lutes have given the following empirical correction factor, referred to in the literature as the Single-Moment method, in References 3 and 4.

$$\lambda_L = \frac{\left(M_{2/m}\right)^{m/2}}{\sigma_S^m \ \nu_0^+} \tag{12}$$

Note that the $\left(M_{2/m}\right)^{m/2}$ term is evaluated using j = 2/m in equation (1)

The direct Larsen and Lutes cumulative damage is

$$D = \frac{\tau}{A}\left(\sqrt{2}\right)^m \left(M_{2/m}\right)^{m/2} \Gamma\left(\frac{1}{2}m+1\right) \tag{13}$$

Equation (13) shows the single-moment dependency.

Oritz and Chen

Ortiz and Chen developed the following correction factor by applying the generalized spectral bandwidth to the Rayleigh distribution, from Reference 2.

$$\lambda_k = \frac{\beta_k^m}{\alpha_2} \tag{14}$$

The generalized spectral bandwidth is

$$\beta_k = \sqrt{\frac{M_2 M_k}{M_0 M_{k+2}}} \quad , \quad k = 2.0/m \tag{15}$$

Benasciutti & Tovo

The first Benasciutti & Tovo correction factor from Reference 5 is

$$\lambda_{BT} = \left[b + (1-b)\,\alpha_2^{m-1} \right] \alpha_2 \tag{16}$$

where

$$b = \frac{(\alpha_1 - \alpha_2)\left[1.112\left(1 + \alpha_1\alpha_2 - (\alpha_1 + \alpha_2)\right)\exp\left(2.11\alpha_2\right) + \left(\alpha_1 - \alpha_2\right)\right]}{\alpha_2} \tag{17}$$

$\alpha_{0.75}$ Method

The $\alpha_{0.75}$ correction factor from Reference 5 proposed by Benasciutti & Tovo is

$$\lambda_{\alpha} = \alpha_{0.75}^{2} \tag{18}$$

Dirlik

The Dirlik method is shown in Appendix A, from Reference 5. It approximates the cycle-amplitude distribution by using a combination of one exponential and two Rayleigh probability densities.

Zhao-Baker

Zhao and Baker combined theoretical assumptions and simulation results to give an expression for the cycle distribution as a linear combination of the Weibull and Rayleigh probability density function. There method is shown in Appendix B, from Reference 5.

<u>Rainflow Cycle Count in the Time Domain</u>

A time history with a normal distribution can be synthesized to match an applied force or base excitation power spectral density. The excitation can then be applied to a structural model. The structure's response can then be calculated via a modal transient analysis. The resulting stress can then be calculated as a post-processing step from the strain response. The rainflow cycles can then be calculated using the method in Reference 1. The cumulative damage is then calculated via the Palmgren-Miner formula, per Reference 6.

Palmgren-Miner Index

Palmgren-Miner's cumulative damage index D is given by

$$D = \sum_{i=1} \frac{n_i}{N_i} \tag{19}$$

where

n_i is the number of stress cycles accumulated during the vibration testing at a given level stress level represented by index i

N_i is the number of cycles to produce a fatigue failure at the stress level limit for the corresponding index

The damage index D for a single-segment S/N curve can be expressed as

$$D = \frac{1}{A} \sum_{i=1} S_i^{\,m} \tag{20}$$

Analysis Approach

The input and stress response levels for a sample dynamic system are given in Appendix C.

The resulting fatigue cycles were calculated using the time domain simulation with rainflow cycle counting, as well as the frequency domain methods previously given. The time domain results are considered as the reference by which the frequency domain methods are evaluated, although the time domain results remain a simulation.

Assumptions

1. Single-segment S-N curve
2. Neglect endurance limit
3. All stresses below yield
4. Fatigue cycles are fully reversed with zero mean stress
5. Stationary stress time histories

Materials

Three material cases were considered as shown in Table 1. These materials have nearly the same specific stiffness values such that a common stress PSD can be applied for the three materials.

Table 1. Material Properties, For Reference Only			
Material	M	A	Reference
Butt-Welded Steel	3.5	1.26e+11 ksi^3.5	2
Stainless Steel[1]	6.54	1.32E+18 ksi^6.54	7
Aluminum 6061-T6	9.25	9.77e+17 ksi^9.5	6

Note that offshore steel structures with welded joints typically have a fatigue exponent of m ≅ 3 per Reference 8.

Example 1

A single-mode stress response PSD is shown in Figure C-3. The fatigue analysis results are shown in Table 2.

Table 2. Cumulative Damage Results D, Single-mode Stress Response PSD			
Method	Material		
	Butt-Welded Steel	Stainless Steel	Aluminum
Rainflow Cycle Count	0.00373	0.00559	0.00119
Narrowband	0.00394	0.00620	0.00139
Wirsching Light	0.00339	0.00447	0.000867
$\alpha_{0.75}$	0.00382	0.00602	0.00135
Ortiz Chen	0.00402	0.00631	0.00141
Lutes Larsen	0.00379	0.00591	0.00132
Benasciutti Tovo	0.00369	0.00546	0.00118
Zhao Baker	0.00372	0.00591	0.00139
Dirlik	0.00377	0.00565	0.00113
Average	0.00380	0.00588	0.00130

The average is obtained by using the six methods in Table 2 starting with $\alpha_{0.75}$ and ending with Dirlik, in Tables 2 through 4.

[1] Martensitic precipitation/age-hardening stainless steel, unnotched PH13-8Mo (H1000) hand forging

Example 2

A bimodal stress response PSD is shown in Figure C-4.

The resulting fatigue cycles are shown in Table 3.

Table 3. Cumulative Damage Results D, Bimodal Stress Response PSD			
Method	Material		
	Butt-Welded Steel	Stainless Steel	Aluminum
Rainflow Cycle Count	0.0212	0.0818	0.0400
Narrowband	0.0244	0.0961	0.0488
Wirsching Light	0.0203	0.0684	0.0303
$\alpha_{0.75}$	0.0209	0.0825	0.0419
Ortiz Chen	0.0213	0.0814	0.0409
Lutes Larsen	0.0203	0.0767	0.0384
Benasciutti Tovo	0.0197	0.0672	0.0320
Zhao Baker	0.0215	0.0831	0.0420
Dirlik	0.0199	0.0717	0.0324
Average	0.0206	0.0771	0.0379

Example 3

A tri-modal stress response PSD is shown in Figure C-5.

The resulting fatigue cycles are shown in Table 4.

Table 4. Cumulative Damage Results D, Tri-modal Stress Response PSD			
Method	Material		
	Butt-Welded Steel	Stainless Steel	Aluminum
Rainflow Cycle Count	0.0211	0.0774	0.0393
Narrowband	0.0252	0.0929	0.0480
Wirsching Light	0.0209	0.0661	0.0298
$\alpha_{0.75}$	0.0207	0.0760	0.0393
Ortiz Chen	0.0210	0.0737	0.0375
Lutes Larsen	0.0199	0.0688	0.0348
Benasciutti Tovo	0.0196	0.0620	0.0302
Zhao Baker	0.0219	0.0790	0.0407
Dirlik	0.0196	0.0654	0.0301
Average	0.0204	0.0708	0.0354

Summary

The damage results from the three examples are given in Table 5.

Table 5. Absolute Error Ratios for Frequency Domain Relative to Time Domain Damage

Case	$\alpha_{0.75}$	Ortiz Chen	Lutes Larsen	Benasciutti Tovo	Zhao Baker	Dirlik	Average
1	0.024	0.078	0.016	0.011	0.003	0.011	0.019
2	0.077	0.129	0.057	0.023	0.057	0.011	0.052
3	0.134	0.185	0.109	0.008	0.168	0.050	0.092
4	0.014	0.005	0.042	0.071	0.014	0.061	0.028
5	0.009	0.005	0.062	0.178	0.016	0.123	0.057
6	0.048	0.023	0.040	0.200	0.050	0.190	0.052
7	0.019	0.005	0.057	0.071	0.038	0.0196	0.033
8	0.018	0.048	0.111	0.199	0.021	0.0654	0.085
9	0.000	0.046	0.115	0.232	0.036	0.0301	0.099
Mean	0.038	0.058	0.068	0.110	0.045	0.062	0.057
Max	0.134	0.185	0.115	0.232	0.168	0.190	0.099

The error values are absolute values.

The six cases represent stress PSD and material combinations.

Cases 1 through 3 are taken from Table 2, for butt-welded steel, stainless steel, and aluminum, respectively. Cases 4 through 6 are taken from Table 3. Cases 7 through 9 are likewise taken from Table 3.

The average in the last column is not the average error for a given case. Rather it is the error of the average frequency domain damage relative to the time domain rainflow cycle damage.

The results in Table 5 show that the $\alpha_{0.75}$ method gives the smallest mean damage error.

The Lutes-Larsen method gives the least maximum error.

The average of the six methods gives a maximum error than any of the individual methods.

Table 6. Error Ratios for Frequency Domain Relative to Time Domain Damage						
Case	$\alpha_{0.75}$	Ortiz Chen	Lutes Larsen	Benasciutti Tovo	Zhao Baker	Dirlik
1	0.024	0.078	0.016	-0.011	-0.003	0.011
2	0.077	0.129	0.057	-0.023	0.057	0.011
3	0.134	0.185	0.109	-0.008	0.168	-0.050
4	-0.014	0.005	-0.042	-0.071	0.014	-0.061
5	0.009	-0.005	-0.062	-0.178	0.016	-0.123
6	0.048	0.023	-0.040	-0.200	0.050	-0.190
7	-0.019	-0.005	-0.057	-0.071	0.038	-0.071
8	-0.018	-0.048	-0.111	-0.199	0.021	-0.155
9	0.000	-0.046	-0.115	-0.232	0.036	-0.234
Mean	0.027	0.035	-0.027	-0.110	0.044	-0.096

The results in Table 6 show that the $\alpha_{0.75}$ and Lutes Larsen methods give the smallest mean damage error. Zhao Baker is the most conservative.

Conclusions

The several frequency domain methods tend to provide reasonably good estimates of the cumulative fatigue damage with some limitations. The Narrowband method should only be used with an applied correction factor. In addition, the Wirsching-Light method should only be used for cases with fatigue exponents in the range of 3 to 6, inclusive.

An average cumulative damage index based on the six methods in Table 5 is recommended for calculating an estimate with enhanced reliability. Note that the computation time for implementing this averaging method is negligibly low.

An effort was made to derive a weighted average method, but the simple average was determined to be the best metric for the six cases.

The quest for greater precision in the damage calculation is justified but somewhat academic. Uncertainties in S-N curves, stress concentration factors, mean stress, load sequence, thermal and corrosion environments, natural frequencies, damping and other variables must also be considered.

References

1. ASTM E 1049-85 (2005) Rainflow Counting Method, 1987.

2. P. Wirsching, T. Paez, K. Ortiz, Random Vibrations Theory and Practice, Dover, New York, 2006.

3. C. Larsen & L. Lutes, Predicting the Fatigue Life of Offshore Structures by the Single-Moment Spectral Method, Probabilistic Engineering Mechanics, 6(2), 1991.

4. L. Lutes & C. Larsen, Improved Spectral Method for Variable Amplitude Fatigue Prediction, ASCE Journal of Structural Engineering, 116(4), 1990.

5. M. Mrsnik, J. Slavic and M. Boltezar, Frequency-domain Methods for a Vibration-fatigue-life Estimation -Application to Real Data. International Journal of Fatigue, Vol. 47, p. 8-17, 2013.

6. T. Irvine, Miner's Cumulative Damage via Rainflow Cycle Counting, Rev G, Vibrationdata, 2014.

7. MIL-HDBK-5J, Metallic Materials and Elements for Aerospace Vehicle Structures, U.S. Department of Defense, 2003.

8. Commentary on the Guide for the Fatigue Assessment of Offshore Structures, American Bureau of Shipping, New York, April 2003.

9. T. Irvine, The Longitudinal Vibration Response of a Rod to an Applied Force, Rev D, Vibrationdata, 2014.

APPENDIX A

Dirlik Method

The Dirlik histogram formula N(S) for stress cycles ranges is

$$N(S) = \nu_p \cdot \tau \cdot p(S) \tag{A-1}$$

The function p(S) is

$$p(S) = \frac{\frac{D_1}{Q}\exp\left(-\frac{Z}{Q}\right) + \frac{D_2 Z}{R^2}\exp\left(\frac{-Z^2}{2R^2}\right) + D_3 Z \exp\left(\frac{-Z^2}{2}\right)}{2\sqrt{m_0}} \tag{A-2}$$

The coefficients and variables are

$$D_1 = \frac{2\left(x_m - \gamma^2\right)}{1+\gamma^2} \tag{A-3}$$

$$D_2 = \frac{1-\gamma - D_1 + D_1^2}{1-R} \tag{A-4}$$

$$D_3 = 1 - D_1 - D_2 \tag{A-5}$$

$$Z = \frac{S}{2\sqrt{m_0}} \tag{A-6}$$

$$Q = \frac{1.25\left(\gamma - D_3 - D_2 R\right)}{D_1} \tag{A-7}$$

$$R = \frac{\gamma - x_m - D_1^2}{1-\gamma - D_1 + D_1^2} \tag{A-8}$$

$$\gamma = \frac{m_2}{\sqrt{m_0\, m_4}} \tag{A-9}$$

$$x_m = \frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \tag{A-10}$$

The Dirlik cumulative damage is

$$D = \left(\frac{1}{2^m\ A}\right)\int_0^\infty S^m\ N(S)\, dS \tag{A-11}$$

APPENDIX B

Zhao & Baker Method

The probability density function $p(Z)$ for stress cycles amplitude is

$$p(Z) = w\, \alpha\beta\, Z^{\beta-1} \exp\left(-\alpha Z^{\beta}\right) + (1-w) Z \exp\left(-Z^2/2\right) \tag{B-1}$$

where

$$Z = S/\sigma_S \tag{B-2}$$

The weighting factor w is

$$w = \frac{1-\alpha_2}{1-\sqrt{\frac{2}{\pi}}\;\Gamma\left(1+\frac{1}{\beta}\right)\alpha^{-1/\beta}} \tag{B-3}$$

The Weibull parameters are

$$\alpha = 8 - 7\alpha_2 \tag{B-4}$$

$$\beta = \begin{cases} 1.1 \text{ for } \alpha_2 < 0.9 \\ 1.1 + 9\left(\alpha_2 - 0.9\right) \text{ for } \alpha_2 \geq 0.9 \end{cases} \tag{B-5}$$

The Zhao & Baker cumulative damage is

$$D = \left(\frac{\nu_p\, \tau}{A}\right) {\sigma_S}^m \int_0^{\infty} Z^m\, p(Z)\, dZ\ , \quad dZ = dS/\sigma_S \tag{B-6}$$

APPENDIX C

Rod, Applied Force and Response

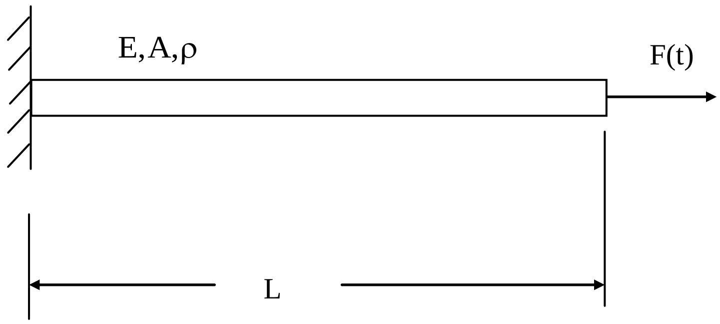

Figure C-1.

Consider a thin, fixed-free rod subjected to an applied force at the free end.

Table C-1. Sample Rod Properties, Aluminum			
Length	L	=	84 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch^2
Area Moment of Inertia	I	=	0.0491 inch^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Mass Density	ρ	=	0.1 lbm/in^3
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	$\begin{cases} 0.05 & \text{for mode 1} \\ 0.02 & \text{for mode 2} \end{cases}$

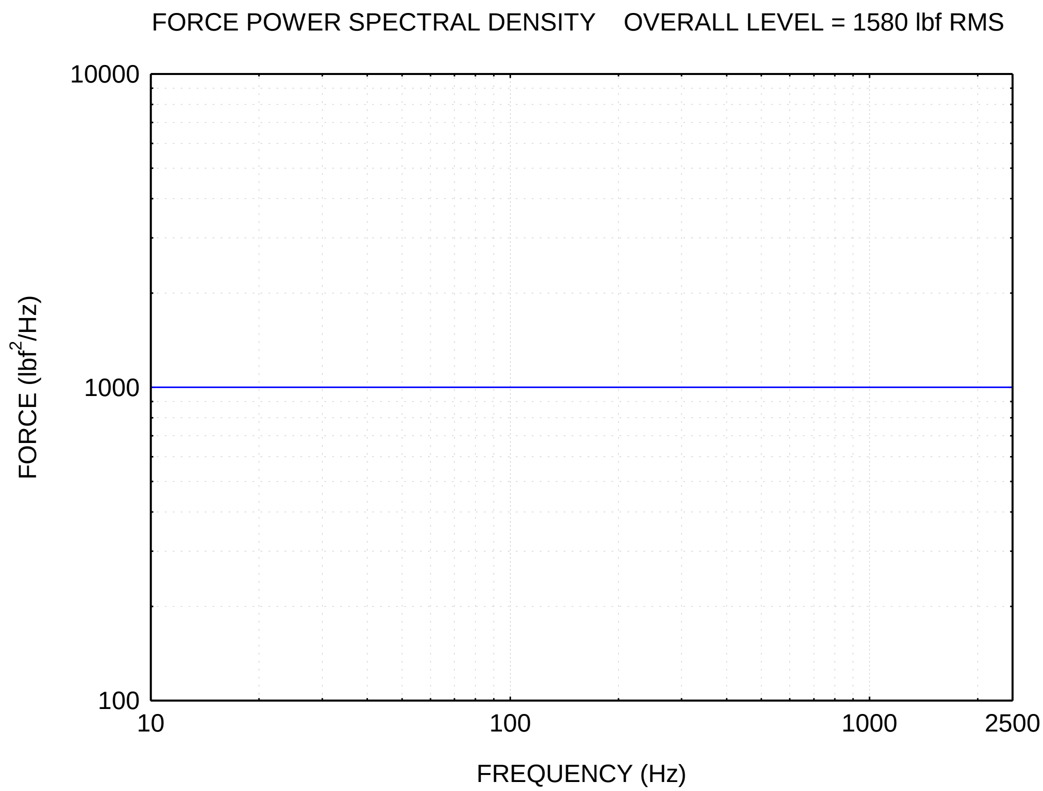

Figure C-2.

Force PSD, 1580 lbf RMS, 600 seconds	
Frequency (Hz)	Force (lbf^2/Hz)
10	1000
2500	1000

The level in Table C-1 was increase by a factor of 12 dB for the case of stainless steel in order to induce a non-trivial cumulative damage index.

The one and two-mode stress responses for the nominal force input are shown in Figures C-2 and C-3, respectively. The stress level are calculated the rod's fixed boundary per Reference 9.

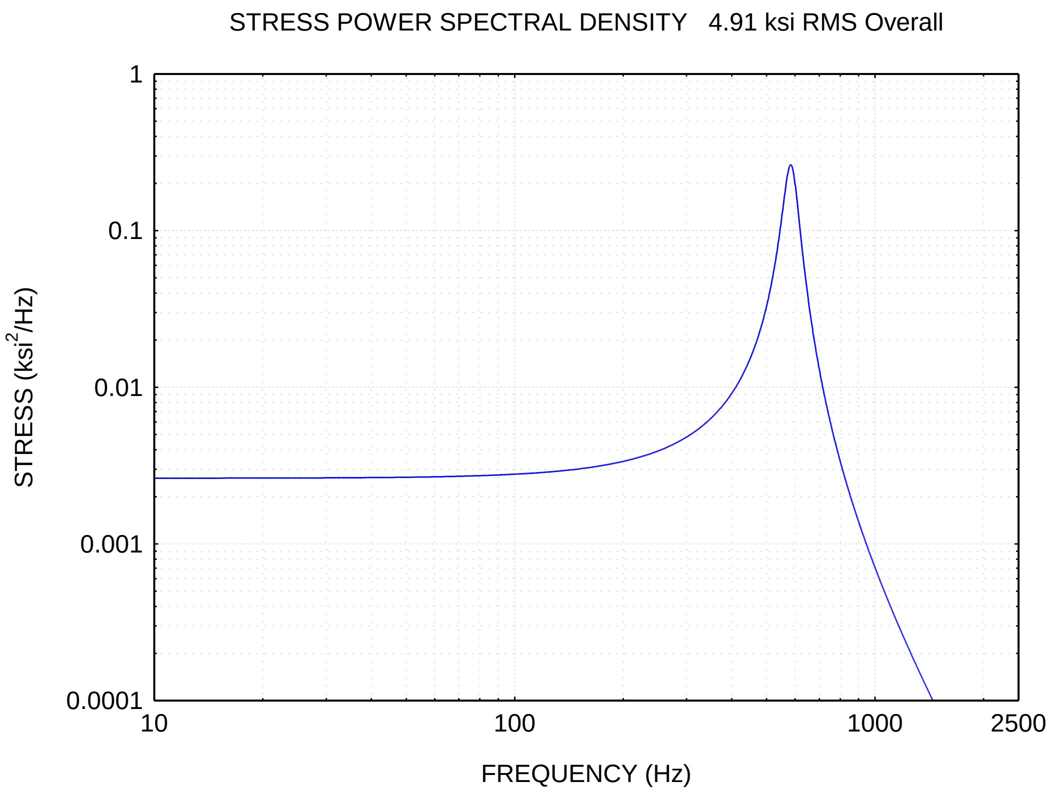
STRESS POWER SPECTRAL DENSITY 4.91 ksi RMS Overall
STRESS (ksi²/Hz)
1
0.1
0.01
0.001
0.0001
10
100
1000
2500
FREQUENCY (Hz)

Figure C-3.

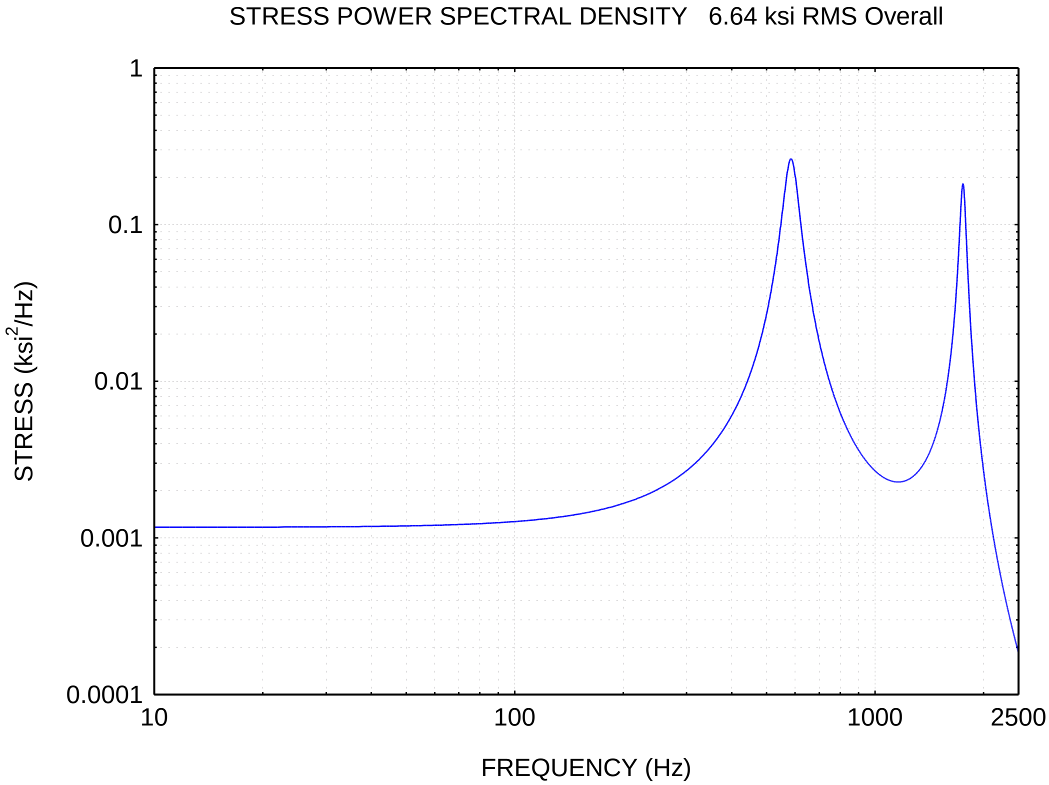
STRESS POWER SPECTRAL DENSITY   6.64 ksi RMS Overall
1
0.1
0.01
0.001
0.0001
STRESS (ksi^2/Hz)
10
100
1000
2500
FREQUENCY (Hz)

Figure C-4.

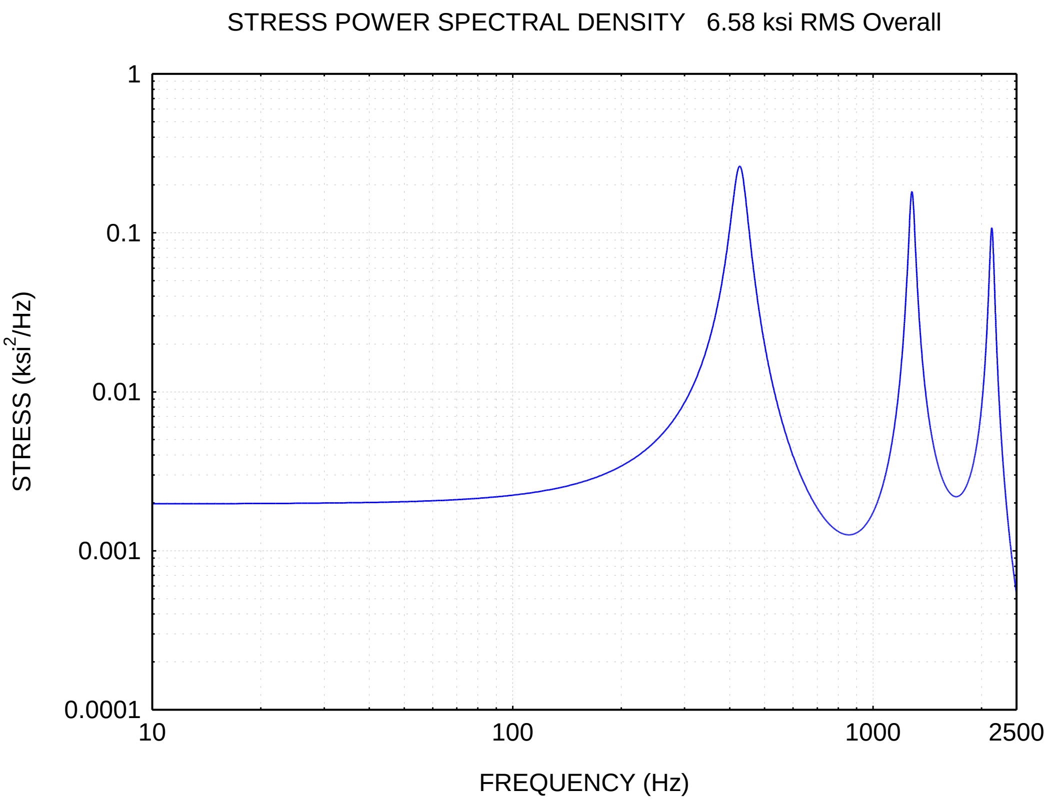

Figure C-5.

The length of the rod was increased to 115 inches for the tri-modal case in Figure C-5. The damping for the third mode was set at 0.0156.