Homological Perturbation Theory
Hochschild Homology and Formal Groups

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§1 Introduction and Notation

As Don Schack mentioned in his plenary talk [**SDS**], Gerstenhaber [**MG**] observed that if A is an associative algebra, then the Hochschild cohomology of A is a graded commutative algebra with an additional structure, viz., that of a Lie algebra. The two structures satisfy a compatibility condition (graded Poission structure). We agree to call such structures *G–algebras* (Gerstenhaber–algebras). Schack pointed out that if A is a smooth K-algebra, and we give the exterior algebra $\Lambda^*_{H^0(A,A)}(H^1(A,A))$ the Nijenhuis-Schouten bracket, making it into a G-algebra, we have the classic theorem of Hochschild, Kostant, and Rosenberg (§5): There is an isomorphism of G-algebras

$$\Lambda^*_{H^0(A,A)}(H^1(A,A)) \to H^*(A,A).$$

If A is a *filtered* algebra over a commutative ring with unit R, i.e., there is a sequence of R-submodules $F_{n-1}(A) \subset F_n(A)$, we let

$$E^0(A) = \oplus_n F_n(A)/N_{n-1}(A)$$

be the *associated graded* R-module ($E^0_n(A) = F_n(A)/F_{n-1}(A)$). A is *multiplicatively filtered* if $F_p(A)F_q(A) \subset F_{p+q}(A)$. In this case, $E^0(A)$ is a graded algebra.

Under certain conditions, if A is multiplicatively filtered, there is a spectral sequence (4.1), (5.3)

$$E_2 = H^*(E^0(A), E^0(A)) \implies H^*(A,A).$$

In case $E^0(A)$ is smooth, Brylinski has identified the E_2 term using the theorem of Hochschild, Kostant, and Rosenberg and has given a natural interpretation of the differential d_2 in terms of a G-algebra structure on $E^0(A)$ (5.3.1) (the grading is slightly different there so that one sees d_1). We will review some related results concerning the commutative case in §5. We will also look at the May spectral sequence (4.1.3) for multiplicatively filtered algebras where $E^0(A)$ is not necessarily commutative.

The main concern in this paper will not be with multiplicative structure in cohomology or in the associated chain complexes, but with the construction of *resolutions* over A and

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its enveloping algebra A^e (3.1). It should be pointed out that for algebras over a field of characteristic zero, a theory of *multiplicative* resolutions is worked out in [**SS1**], [**SS2**]. The constructions we discuss involve what has come to be known as "homological perturbation theory" [**VG1**], [**RB**], [**LS**], [**JH1**], [**JH2**], [**GS**], [**GL**], [**GLS1**], [**GLS2**], [**HK**], [**HT**], [**LL3**].

We define the notion of a "model" for the May spectral sequence in (4.1) using notions from homological perturbation theory. We believe that this concept is an important computational technique in homological algebra and should prove useful in the context of other spectral sequences as well. Indeed, we may say that the work of Gugenheim and May in [**GM**] provides models for the Eilenberg-Moore spectral sequence and has provided inspiration for the concept of spectral sequence model in this paper.

Although multiplicative results can be obtained through the use of the theory presented here, it involves the use of some of the more subtle results in [**GL**], [**GLS1**], [**GLS1**], [**HK**] and this paper should provide an starting point for such a study.

Finally, it should be mentioned that there are some interesting and useful alternate viewpoints that may be found in the series of papers by J. Huebschmann and the paper of J. Huebschmann and T. Kadeishvili [**JH1**], [**JH2**], [**JH3**], [**JH4**], [**HK**].

In what follows, we will assume that we are working over a ground ring R which is commutitave with unit. Much of what we will say about modules and algebras over R can be done for *differential graded* modules and algebras. In this paper however, we will deal only with Ext and Tor over ordinary algebras (possibly graded, but with zero differential).

§2 Some basic homological perturbation theory

Some of the main ideas of **basic homological perturbation theory** are reviewed in this section.

One of the motivations for homological perturbation theory comes from the what we call the **Gugenheim principle** [**LL3**]: *if X is a resolution over an object A and the object P is a perturbation of A, then there is a perturbation X_P of X which is a resolution over P.* The statement is purposely vague with regard to which category the objects are in and what is meant by a resolution. For example, in this paper, A and P are algebras and X and X_P are resolutions in the sense of classical homological algebra. One might however think of A and P as fibrations and X and X_P as models for their loop/path spaces [**LS**], [**VG2**], [**GLS1**], [**GLS2**], [**HK**].

Thus, if P is an algebra over some ring R and there is a filtration on P and a resolution X over the associated graded algebra $E^0(P)$, we expect that there should be a resolution over P which is obtained by some well-defined (perturbation) of X. One of the first examples of this that one sees is in the theory of Lie algebras. The classic theorem of Poincaré-Birkhoff-Witt states that over a field, for example, if the universal enveloping algebra $P = \mathcal{U}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ is filtered by length of monomial generators, then $E^0(P)$ is the symmetric algebra, $A = Sym(\mathcal{G})$ on the underlying vector space of $\mathcal{G}$. It is well-known that this also holds if the underlying R-module of $\mathcal{G}$ is projective over R. We have a resolution of the ground ring R over A given by the Koszul complex $A \otimes E_R[\mathcal{G}]$, where $E_R[\mathcal{G}]$ is the exterior algebra on the underlying vector space of $\mathcal{G}$. Of course, $A =$

$Sym(\mathcal{G}) \cong R[x_1, \ldots, x_n]$, the polynomial algebra in n indeterminants $\{x_1, \ldots, x_n\}$, where $n = dim(\mathcal{G})$. Although it is not often presented this way, the differential in the Chevalley-Eilenberg complex [**ChE**] is a perturbation of this differential. When the ground ring is understood, we will also write the exterior algebra $E_R[M]$ on a module M over R as $\Lambda(M)$. Summarizing, we have

Theorem 2.1 [JK], [GH]. *If $A = Sym(V)$ is the symmetric algebra on a projective module V over R, then the complex $(A \otimes \Lambda(V), \partial)$ where*

$$\partial(v \otimes v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{n} (-1)^{i-1} v v_i \otimes v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_n$$

is a resolution of R over A. (As usual, a $\widehat{\ }$ indicates that the corresponding term is omitted). ▮

and

Theorem 2.2 [ChE]. *If $\mathcal{G}$ is a Lie algebra over R and $\mathcal{G}$ is projective as an $R-$module, then the complex $(\mathcal{U}(\mathcal{G}) \otimes \Lambda(\mathcal{G}), \partial)$ where*

$$\begin{aligned}\partial(g \otimes g_1 \wedge \cdots \wedge g_n) = &\sum_{i=1}^{n} (-1)^{i-1} g g_i \otimes g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge g_n \\ &+ \sum_{j<k} (-1)^{j+k} g \otimes [g_j, g_k] \wedge g_1 \cdots \wedge \widehat{g_j} \wedge \cdots \wedge \widehat{g_k} \wedge \cdots \wedge g_n\end{aligned}$$

is a resolution of R over $\mathcal{U}(\mathcal{G})$. ▮

If we think of the universal enveloping algebra as a perturbation of the symmetric algebra, then the perturbation vanishes if the Lie algebra is trivial, i.e., abelian. In a corresponding way, if the Lie algebra is abelian, i.e., all brackets $[g, h] = 0$, then the perturbation in the second resolution above vanishes and it becomes the first resolution. This actually fits into a hierarchy of perturbations involving *formal groups*, as we will see below (§9).

More abstractly, we may say that **homological perturbation theory** is a body of knowledge which deals with the transference of (differential) algebraic structure from one object to another preserving (chain homotopy) equivalence.

Quite often, the transference of structure is to be made from one differential object to another which is a **strong deformation retraction** of the first. This sort of transference is the subject of several papers such as [**KC**], [**VG1**], [**VG2**], [**GS**], [**LS**], [**GL**], [**GLS2**], [**GLS1**], [**RB**], [**VG1**], [**JH2**], [**JH3**], [**JH4**], [**HK**], [**BL**], [**SH**], to name a few.

A statement of the transference problem was formulated and investigated in [**BL**]. That statement is reviewed here. To begin, let (A, d) be a DG-module over R and let

$$\phi : A \to A, \qquad (2.3)$$

be a degree 1, R-linear map of graded R-modules. Suppose, in addition, that

(2.4) $$\phi^2 = 0$$

(2.5) $$\phi d \phi = \phi.$$

We call such a map ϕ a *splitting homotopy.* Note that a splitting homotopy give rise to a splitting of A (as a DG-object)

$$A = im(\pi) \oplus ker(\pi)$$

where $\pi = 1 - (d\phi + \phi d)$. This is due to the fact that π is a degree 0 map and $\pi^2 = \pi$ as easily follows from (2.3)-(2.4). All of this is in [**BL**].

We have the following transference problem [**BL** §**2**]:

(2.6) Transference Problem. *Given a splitting homotopy $\phi : (A, d) \to (A, d)$ and a new differential $d' : A \to A$, find a splitting homotopy $\phi' : (A, d') \to (A, d')$ such that $im(\pi) \cong_R im(\pi')$, (isomorphic as R-modules ignoring differentials), where $\pi = 1 - (d\phi - \phi d)$, and $\pi' = 1 - (d'\phi' + \phi' d')$.* ▮

The transference problem may also be stated in terms of what we have called SDR–data (Strong Deformation Retraction data) in several of the references above. In this paper, we will shorten this to simply an "SDR". Originally, such objects and maps were said to form a contraction [**EM1,** §**12**].

We recall that if M is a DG-module over R and A and ϕ are given as above, except that (2.4), and (2.5) are relaxed, then we say that

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi)$$

is an SDR (M is an SDR of A) if and only if ∇ and f are chain maps such that

$$f\nabla = 1_M, \quad \text{and} \quad \nabla f = 1_A - (d\phi + \phi d).$$

It was shown in [**LS**] that the additional conditions

$$\phi^2 = 0, \quad \phi\nabla = 0, \quad f\phi = 0$$

could be assumed (if necessary, by replacing ϕ by a composition involving d and ϕ [**LS**]). In this paper, whenever we consider an SDR, we will assume that all of these conditions hold.

The map f above is called the *projection*, and the map ∇ is called the *inclusion.*

Strong deformation retractions are fairly common. It is not difficult to see that if M and A are chain complexes, which are free as R-modules and $f : A \to M$ is an onto map which induces an isomorphism in homology, then f may be "completed" to an SDR $(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi)$. This situation is encountered in the Cartan Seminar '54 [**HC**] for resolutions which are compared with the bar construction [**SM**].

In terms of strong deformation retractions, the transference problem may be stated [**BL** §**2**]:

(2.7) Transference Problem (SDR version): *Given an SDR*

$$(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi)$$

and a change t in the differential such that $(d+t)^2 = 0$, find a new SDR

$$((M, d') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (A, d+t), \phi').$$

The perturbation t of the original differential is called the **initiator**. ▮

It is easy to see that the two versions of the transference problem are equivalent (set $M = im(\pi)$, where $\pi = 1 - (d\phi + \phi d)$, see [**BL,** §**2**] for more details). The point is that an SDR is completly determined by the corresponding splitting homotopy. This was noted in [**BL**] and first observed in [**VG2**] in the case that the differential on M is zero. Thus the inclusion, the projection, and the new differential are determined by the splitting homotopy. It should be pointed out however that there can be some surprises in this. For example, it can turn out that when both M and A are algebras, one of f, and ∇ are maps of DGA-algebras, but not the other [**GMu**], [**GL**], [**GLS2**].

There is a formal solution to the transference problem. This can be found in [**WS**], [**RB**] and [**VG1**]. One forms the sequence of sums (for a *geometric series* involving $t\phi$)

$$\phi_n = 1 - t\phi + \cdots + (-t\phi)^n + \ldots.$$

The formal solution is $\phi_\infty = lim_{n\to\infty}\phi_n$ and we also obtain formal solutions for the new (limit) inclusion and projection. We have

Formal Solution 2.8. [VG1], [RB]: *With the above hypotheses let*

$$\mathcal{S}_n = t - t\phi t + \cdots + (-t\phi)^n t + \ldots. \tag{2.8.1}$$

The formal solution to the transference problem is

$$\begin{aligned} f_\infty &= f - f\mathcal{S}_\infty\phi, & \nabla_\infty &= \nabla - \phi\mathcal{S}_\infty\nabla \\ \partial_\infty &= d_M + f\mathcal{S}_\infty\nabla, & \phi_\infty &= \phi - \phi\mathcal{S}_\infty\phi. \end{aligned} \tag{2.8.2}$$

Note that we have used the sign convention from [**BL**] here which differs from the convention given in [**VG1**]. This really makes no difference if one simply replaces t by $-t$ above. We will switch back to the original convention in §8.

There are straightforward conditions that may be given to insure the convergence of these series. For example, there may be decreasing filtrations of M and A so that f, ∇ and ϕ are filtration preserving and such that $t\phi$ lowers filtration. Such conditions were given and used in [**VG1**]. There are examples however where the series converge and where such filtrations are not obvious (§9). The following theorem given in [**BL**] is a criterion for the existance of a solution to the transference problem which does not require explicit examination of the formal series above.

Theorem 2.9 [BL §4]. *If $\nu : A \to A$ is a splitting homotopy with respect to d and t is a change in the differential on A (so that $(d+t)^2 = 0$), then if $\nu' : A \to A$ is another degree 1 map such that*

$$\begin{aligned}(1 + \nu t)\nu' &= \nu \\ \nu t \nu' &= \nu' t \nu,\end{aligned}$$

then

1. *ν' is a splitting homotopy with respect to the new differential $\xi = d + t$ and is a solution to the transference problem,*
2. *if $(M \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A, \nu)$ is the SDR corresponding to the original splitting homotopy ν, then we obtain a new SDR $((M, d') \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} (A, \xi), \nu')$ where*

$$\begin{aligned} d' &= d_M + \beta t \alpha - \beta \xi \nu' \xi \alpha \\ \alpha' &= \alpha - \nu' \xi \alpha \\ \beta' &= \beta - \beta \xi \nu' \quad \blacksquare \end{aligned}$$

Before closing this section, a few remarks about other transference problems may be in order. As we have already mentioned, one encounters strong deformation retractions quite often in situations involving "small resolutions", e.g. minimal resolutions of a module over an algebra. A classic example is that of the Koszul resolution $K = A \otimes E_R[u_1, \ldots, u_n]$ related to the ideal $I = (x_1, \ldots, x_n)$ in the polynomial ring $A = R[x_1, \ldots, x_n]$. Here A is an augmented algebra over R and we may form the *bar construction* or standard resolution $B(A)$ for R over A (§3), [**SM**], [**HC**], [**CE**]. Now K is also a resolution of R over A and so, by the comparison theorem, we have a chain homotopy equivalence $K \to B(A)$; however, it can be seen that a map exists which is the inclusion for an SDR $K \rightleftarrows B(A)$ [**PM, §7**]. Much more will be said about this kind of situation in (6.1). Here we want to mention that it sometimes happens that we need to compare two resolutions neither of which is the standard resolution. It becomes rarer to find strong deformation retractions in that case. We always have two homotopy inverse chain maps by the comparison theorem however. In fact, this situation can also arise even when the standard resolution is involved if the other resolution is "too big". There is a satisfactory theory for transference of a change in differential from one side to another in these cases as well [**HK**]. It is observed ([**HK**], [**BL, §6**]) that this can by done by using the mapping cylinder to reduce the problem to the "one sided" transference problem above.

Another variation involves the preservation of algebraic properties. For example, if one starts out with an SDR in which the differentials are (co)derivations and the chain homotopy equivalences are (co)algebra maps, then one might ask for the maps in a solution to the transference problem to be (co)derivations and (co)algebra maps. This is the sort of problem addressed in [**GLS1**], [**GLS2**], [**HK**], [**SH**].

Still another class of transference problems involve the situations encountered in [**GMu**], [**CW**], and discussed in [**BL2**].

In this paper only the first kind of transference problem will be considered.

§3 Standard Resolutions, Twisting Cochains, and Twisted Tensor Products

The bar construction (standard complex) is briefly reviewed in this section.

3.1 The two–sided bar construction.

The bar construction [**SM**], [**HC**] is a functor of differential graded augmented algebras

$$\epsilon : A \to R.$$

There is also a "two-sided" bar construction, as it is called in [**PM**], which is used to define the Hochschild homology of an algebra. It is presented in the case that the differential in A is zero in [**CE**] and more generally, one can find a definition in [**GM**]. We will briefly review both constructions here and indicate some of the relations between them. We will adopt the terminology "one sided bar construction" for the first and "two sided bar construction" for the second.

In each case, we make use of the R-module $\bar{A} = \mathit{coker}(\sigma)$, where $\sigma : R \to A$ is the unit for A. We define

$$\bar{B}_0(A) = R$$
$$\bar{B}_n(A) = \otimes^n \bar{A}.$$

The element of $B_0(A)$ corresponding to the identity element of R is denoted by $[\,]$ and the element $a_1 \otimes \cdots \otimes a_n$ of $B_n(A)$ is denoted by $[a_1|\ldots|a_n]$.

We begin with the two sided bar construction which is defined for any R-algebra A. It is useful to introduce the "enveloping algebra" $A^e = A \otimes A^{op}$, (tensor product algebra) where A^{op} is A with the opposite multiplication $\circ : A^{op} \otimes A^{op} \to A^{op}$ given by $a \circ b = ba$. We have a standard augmentation

$$\epsilon : A^e \to A$$
$$\epsilon(a \otimes b) = ab.$$

In a straightforward manner, we have a category isomorphism

$${}_{A^e}\mathcal{M} \cong {}_A\mathcal{M}_A$$

of the category of left-A^e-modules and the category of A-bimodules [**CE, §IX.3**].

The two–sided bar construction is defined so that we obtain a resolution of A over A^e. As an A^e-module, i.e., as an A-bimodule, the two sided bar construction is $B(A, A) = A^e \otimes_R \bar{B}(A) \cong A \otimes_R \bar{B}(A) \otimes_R A$.

The augmentation $\epsilon : A^e \to A$, gives an augmentation $\epsilon_{B(A,A)} : B(A, A) \to A$ by simply taking $\epsilon_{B(A,A)}$ to be ϵ on $B_0(A, A)$ and zero elsewhere. The complex $B(A, A)$ is defined in such a way that $\epsilon_{B(A,A)}$ is a map of DG-modules where A is given the zero differential. In fact, define

$$\sigma_A : A \to B(A,A)$$
$$\sigma_A(a) = [\,]a,$$

then the conplex $B(A,A)$ is defined so that we have an SDR

$$(A \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} B(A,A), s)$$

where

$$s : B(A,A) \to B(A,A)$$
$$s(a[a_1|\dots|a_n]a') = [a|a_1|\dots|a_n]a'$$

i.e., s is a *splitting homotopy*, although it is not always stated that way. Of course, this means that $B(A,A)$ and A are chain homotopy equivalent (where A has zero differential) and ϵ is a quasi-isomorphism, i.e., $B(A,A) \to A$ is a resolution. It is a remarkable fact that the condition

$$s\partial + \partial s = 1_{B(A,A)} - \sigma\epsilon$$

completely defines the A^e-linear map ∂ by induction. If the reader is not familiar with this well-known fact, he is encouraged to work out the first few terms (as done below). We will not consider the case that A itself is a differential algebra in the examples presented in this paper, but the theory presented will cover that case. A similar remark applies to the one-sided bar construction presented below.

In general, when A is an ordinary algebra, i.e. $d_A = 0$, (but A may be graded) we have [**SM**], [**HC**], [**CE**]

$$\begin{aligned}\partial(a[a_1|\dots a_n]a') = aa_1[a_2|\dots|a_n]&\\ \sum \pm[a_1|\dots|a_ia_{i+1}|\dots a_n]&\\ \pm [a_1|\dots a_{n-1}]a_n.&\end{aligned}$$

We are not concerned with the signs here; they follow from the recursive definition of ∂ above in the general case. The first few cases (for an ordinary ungraded algebra) are

$$\begin{aligned}\partial(a[a_1]a') &= a(a_1[\,] - [\,]a_1)a'\\ \partial(a[a_1|a_2]a') &= a(a_1[a_2] - [a_1a_2] + [a_1]a_2)a'\\ \partial(a[a_1|a_2|a_3]a') &= a(a_1[a_2|a_3] - [a_1a_2|a_3] + [a_1|a_2a_3] - [a_1|a_2]a_3)a'.\end{aligned}$$

3.2 The one–sided bar construction.

Now assume that A is an *augmented* (i.e., *supplemented* [**CE**]) algebra. This means that there is given an algebra map

$$\epsilon_R : A \to R$$

so that $\epsilon_R\sigma_R = 1_R$ (recall that σ_R is the unit of A). Form the R-module $B(A) = A\otimes_R\bar{B}(A)$ and define R-module maps $\epsilon_{B(A)}$ and σ (similar to the two-sided case)

$$\epsilon(a[a_1|\dots|a_n]) = \begin{cases} 0, & n = 0 \\ \epsilon_R(a), & n = 0 \end{cases}$$

$$\sigma(r) = r[\,].$$

The one-sided bar construction as a complex gives an SDR

$$(R \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} B(A), s)$$

where the splitting homotopy s is given by

$$s : B(A) \to B(A)$$
$$s(a[a_1|\dots|a_n]) = [a|a_1|\dots|a_n].$$

Again, it is true that the differential ∂ in $B(A)$ is completly determined recursively by the equation [**EM1,2**], [**HC**], [**PM**]

$$s\partial + \partial s = 1_{B(A)} - \sigma\epsilon.$$

Since ϵ is a chain homotopy equivalence, we have that $B(A)$ is a resolution of R over A.

Now note that R is an A-module via ϵ. In fact, we may use ϵ to make R into an A-bimodule in the obvious way as well. Thus we may consider R as a left-A^e-module and it is clear that, as differential graded modules,

$$B(A) \cong B(A, A) \otimes_A R.$$

Thus, the one-sided bar construction is seen to be a special case of the two-sided bar construction. It is pointed out in [**CE** §**X.6**] however that the two-sided bar construction can be recovered from the one-sided bar construction by a *straightforward tensor product* in some important cases, e.g., that of a group-ring $A = R(G)$ with the usual augmentation, and the universal enveloping algebra of a Lie algebra $\mathcal{U}(\mathcal{G})$ with the usual augmentation ([**CE** §**XIII.5**]). We will review these results here and examine the two-sided bar construction for any augmented algebra from the viewpoint of *twisted tensor products.* This process is well-known in topology and has some very useful applications in homological algebra as well since it gives us a good way to model the two-sided bar construction. This will be mentioned again in (§8).

To begin, we recall the theorem on the "inverse–process" from Cartan and Eilenberg [**CE** §**X.6.1**]

Theorem 3.2.1: *Given an augmented algebra $\epsilon : A \to R$. Suppose that there exists an algebra map $\varsigma : A \to A^e$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\epsilon_R} & R \\ {\scriptstyle\varsigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A^e & \xrightarrow{\epsilon_A} & A \end{array}$$

commutes. Letting $J = ker(\epsilon_A)$, and $I = ker(\epsilon_R)$, suppose also that

(3.2.2) $J = A^e_\varsigma$

(3.2.3) A^e_ς *is a projective right-A-module*

where A^e_ς is A^e as a left- A^e-module and is a right A-module via $\varsigma : A \to A^e$. Then if

$$X \to R \in {}_R\mathcal{M}$$

is a projective resolution, then

$$A^e_\varsigma \otimes_A X \to A \in {}_{A^e}\mathcal{M}$$

is a projective resolution. ∎

Informally, this theorem says that under the stated hypotheses, we can obtain the two-sided bar construction $B(A, A) = A \otimes_R \bar{B}(A) \otimes_R A$ from the one-sided bar construction $B(A) = A \otimes_R \bar{B}(A)$ by appropriately tensoring another A onto the right of $B(A)$. In any case, we can "twist" A onto the right of $B(A)$ to obtain $B(A, A)$ as we will see in the next section.

3.3 Twisting cochains and Twisted Tensor Products

If $\epsilon : C \to R$ is a differential graded coalgebra with counit ϵ, (including the case that C is an "ordinary" coalgebra, i.e., the differential in C is zero and/or C is concentrated in degree 0), and A is a differential graded algebra with unit $\sigma : R \to A$, (including the case that A is an ordinary algebra), then if the R-bimodule $M = A \otimes C$ has a differential d which makes it into a differential A-module, and a differential C-module, i.e., the following diagrams commute

$$\begin{array}{ccccccc} A \otimes M & \xrightarrow{\mu} & M & \quad & M & \xrightarrow{\rho} & M \otimes C \\ {\scriptstyle d_A \otimes 1_M + 1_A \otimes d_M}\downarrow & & \downarrow{\scriptstyle d_M} & & {\scriptstyle d_M}\uparrow & & \uparrow{\scriptstyle d_M \otimes 1_C + 1_M \otimes d_C} \\ A \otimes M & \xrightarrow{\mu} & M & & M & \xrightarrow{\rho} & M \otimes C \end{array}$$

where μ and ρ are the (obvious) module and comodule structures on M, then we simply say that M is an A-C-module/comodule and write $M \in {}_A\mathcal{M}^C$.

With the notation above, recall the convolution algebra structure on $[C, A]_R$ defined in the usual way: if $f, g : C \to A$, then $fg : C \to A$ is given by

$$\begin{array}{ccc} C & \xrightarrow{fg} & A \\ \triangle\downarrow & & \uparrow m \\ C \otimes C & \xrightarrow{f \otimes g} & A \otimes A, \end{array}$$

(m is the product in A and $\triangle$ is the coproduct in C).

It is also useful to define the composites j_A and i_C

$$j_A : A \otimes C \xrightarrow{1_A \otimes \epsilon} A \otimes R \cong A$$
$$i_C : C \cong R \otimes C \xrightarrow{\sigma \otimes 1_C} C \otimes A.$$

Finally, for a map $\tau : C \to A$ define $\widetilde{\tau}$ to be the composite

$$\begin{array}{ccc} M & \xrightarrow{\widetilde{\tau}} & M \\ {\scriptstyle 1_A \otimes \triangle}\downarrow & & \uparrow{\scriptstyle m \otimes 1_C} \\ A \otimes C \otimes C & \xrightarrow{1_A \otimes \tau \otimes 1_C} & A \otimes A \otimes C. \end{array}$$

Gugenheim's theorem on twisting cochains is

Theorem 3.3.1 [VG1]: *Suppose that $M = A \otimes_R C \in {}_A\mathcal{M}^C$. Then*

$$d_M = d^{\otimes} + \widetilde{\tau}$$
$$d_M^2 = D\tau + \tau^2 = 0$$

where τ is the composite

$$\tau = j_A d_M i_C.$$

Here $D\tau$ is the differential in $[C, A]_R$:

$$D\tau = d_A \tau + \tau d_C,$$

$\tau^2 = \tau\tau$ is the product of τ with itself in the convolution algebra structure on $[C, A]_R$ and $d^{\otimes} = d_A \otimes 1_C + 1_A \otimes d_C$ is the tensor product differential. ▮

By definition, an R-module map $\tau : C \to A$ is called a *twisting cochain* if $D\tau + \tau^2 = 0$. For a map $\tau : C \to A$, we put $d_\tau = d^{\otimes} + \widetilde{\tau}$. Gugenheim goes on to prove

Theorem 3.3.2 [VG1]: *Suppose that $\tau : C \to A$ is any R-module map, then the following are equivalent*

$$D\tau + \tau^2 = 0$$
$$d_\tau^2 = 0.$$

and if one of these equivalent conditions hold, then $(A \otimes C, d_\tau) \in {}_A\mathcal{M}^C$. ▮

Thus if τ is a twisting cochain, then we obtain a *perturbation* of the tensor product differential on $M = A \otimes_R C$ giving a new complex $M \in {}_A\mathcal{M}^C$. We call any complex arising in this way a **twisted tensor product complex** (see [**EB**]).

3.4 The Universal Twisting Cochain

We can obtain a differential $\bar{\partial}$ on $\bar{B}(A)$ for an augmented algebra A by taking

$$(\bar{B}(A), \bar{\partial}) \cong (R \otimes_A B(A), 1_R \otimes_A \partial).$$

It is well-known that $(\bar{B}(A), \bar{\partial})$ is a differential graded coalgebra with counit σ

$$\triangle[a_1|\dots|a_n] = \sum \pm[a_1|\dots|a_i] \otimes [a_i|\dots|a_n]$$
$$\sigma(r) = [\,]r.$$

For example, in the case of an ordinary algebra, we have

$$\triangle[\,] = [\,] \otimes [\,]$$
$$\triangle[a_1] = [\,] \otimes [a_1] + [a_1] \otimes [\,]$$
$$\triangle[a_1|a_2] = [\,] \otimes [a_1|a_2] + [a_1] \otimes [a_2] + [a_1|a_2] \otimes [\,]$$

etc.

It is not difficult to see that the differential ∂ on $B(A)$ gives this complex the structure of an A-$\bar{B}(A)$-module/comodule and so theorem (3.5) applies and we easily compute the corresponding twisting cochain π to be

(3.4.1) $$\pi[a_1|\dots|a_n] = \begin{cases} 0, & n > 0 \\ a_1 - \epsilon(a_1), & n = 0. \end{cases}$$

We thus recover $B(A)$ as a twisted tensor product complex as is well-known. For reasons that we won't go into here, we call π the *universal twisting cochain* [**GMu**], [**HMS**], [**GL**].

3.5 More Twisted Tensor Products

If $\tau : C \to A$ is a twisting cochain, and N is an A-module, then we may form the *associated* tensor product complex $M = N \otimes_R C$ where the differential is given by $d_\tau = d^{\otimes} + \widetilde{\tau}$ and $\widetilde{\tau}$ is the composite

$$\begin{array}{ccc} M & \xrightarrow{\widetilde{\tau}} & M \\ {\scriptstyle 1_N \otimes \triangle}\downarrow & & \uparrow{\scriptstyle \mu \otimes 1_C} \\ N \otimes C \otimes C & \xrightarrow{1_N \otimes \tau \otimes 1_C} & N \otimes A \otimes C \end{array}$$

where $\mu : N \otimes A \to N$ is the (right) A-module structure on A. Obviously, this gives the twisted tensor product of the previous section when N is taken to be A with the standard A-module structure.

Dually, we have the associated twisted tensor product complex (M, d_τ) where $M = D \otimes A$ for and C-comodule D:

$$\begin{array}{ccc} D \otimes A & \xrightarrow{d_\tau} & D \otimes A \\ {\scriptstyle \rho}\downarrow & & \uparrow{\scriptstyle 1_D \otimes m} \\ D \otimes C \otimes A & \xrightarrow{1_D \otimes \tau \otimes 1_A} & D \otimes A \otimes A \end{array}$$

where $\rho : D \to D \otimes C$ is the comodule structure. Obviously, this gives the twisted tensor product of the previous section when D is taken to be C with the standard C-comodule structure.

We may apply these remarks to the universal twisting cochain $\pi : \bar{B}(A) \to A$ and the comodule $B(A)$ over $\bar{B}(A)$ to obtain an associated twisted tensor product complex $B(A) \otimes_R A$ and we have the following

Theorem 3.5.1: *Let $\epsilon : A \to R$ be an augmented algebra and $\pi : \bar{B}(A) \to A$ be the universal twisting cochain. The associated twisted tensor product complex $B(A) \otimes_R A \cong A \otimes_R \bar{B}(A) \otimes_R A$ is just the two-sided bar construction $B(A, A)$.* ∎

The proof is an easy computation using the definitions. As a consequence, we may think of the differential ∂ in the two-sided bar construction as a perturbation of the tensor product $1_A \otimes \bar{\partial} \otimes 1_A$ where $\bar{\partial}$ is the differential in the coalgebra $\bar{B}(A)$. By simply tracing the composites involved, one can easily identify ξ in the following.

Theorem 3.5.2: *With the hypotheses above, the differential ∂ in the two-sided bar construction is a perturbation of the tensor product differential*

$$\partial = 1_A \otimes \bar{\partial} \otimes 1_A + \xi \quad \blacksquare \tag{3.5.3}$$

4. The May Spectral Sequence

We review an important concept in homological algebra in this section and discuss some interesting results involving it.

4.1. The Spectral Sequence.

When A is augmented, there is a rather general spectral sequence which was introduced by May in [**PM,** §**4**]. May's interest in that paper is with Tor and Ext over A and so he deals with the resolutions $R \otimes_A B(A, A) \otimes_A M$, for a left A-module M. The homology of this complex is, of course $Tor^A(R, M)$ [**CE,** §**X.2**] which is denoted by $H_*(A; M)$.

Suppose that A is a multiplicatively filtered algebra (as in §1) and the filtration satisfies

$$F_n A = 0, \quad \text{if } n \geq 0, \quad F_{-1}A = ker(\epsilon), \quad \cap_n F_n(A) = 0 \tag{4.1.1}$$

or

$$F_n A = 0, \quad \text{if } n < 0, \quad F_0 A = R, \quad \cup_n F_n A = A. \tag{4.1.2}$$

Suppose that M is a filtered A-module or has the filtration given by $F_n(M) = F_n(A)M$. Then there is a filtration of the bar construction resolution $B = R \otimes_A B(A, A) \otimes_A M$ such that in the resulting spectral sequence, one has

$$E^1 = E^0(B) \cong R \otimes_{E^0(A)} B(E^0(A), E^0(A)) \otimes_{E^0(A)} E^0(M).$$

Furthermore, May shows that the differential d^1 corresponds to the differential in the bar construction resolution involved in the isomorphism above and hence

Theorem 4.1.3. [PM, §4]: *Let $\epsilon : A \to R$ be an augmented algebra which is multiplicatively filtered and satisfies either (4.1.1) or (4.1.2). Suppose that M is a filtered left A-module. There is a filtration of $R \otimes_A B(A, A) \otimes_A M$ such that the associated spectral sequence satisfies*

$$E^2 = H_*(E^0(A); E^0(M)) \implies H_*(A; M). \quad \blacksquare$$

We should point out that the cases of interest in [**PM**] are such that the inverse process of Cartan and Eilenberg (3.2.1) is available, so that a two-sided version, i.e., a version for Hochschild homology follows from (4.1.3); however, a version of the May spectral sequence may be derived directly using the methods in [**PM**]. This was done independently for a certain class of algebras in [**JB**] (see §5.3 below).

Generally, if A is a multiplicatively filtered algebra and there is an associated filtration of the bar construction which gives rise to a spectral sequence and satisfies $E^0B = BE^0$, we will call the spectral sequence a **May spectral sequence** for A (with respect to the given filtration). In §8, for a given multiplicative filtration of A, we will define associated transference problems ((8.1.3) and (8.1.8)). We will call any solutions to these particular transference problems **models** of the corresponding May spectral sequence.

4.2. Certain Quadratic Algebras and the Complex of Priddy.

In [**SP**], Priddy looks at a class of (possibly graded) algebras over a *field* R which have a presentation in terms of generators $\{a_i\}$ and relations $\{r_j\}$ where the r_j are at most quadratic in the a_i (and at least linear, since he assumes that the relations are all of the form $\sum_i r_i a_i + \sum_{j,k} r_{j,k} a_j a_k$, so that Sridharan's algebras (5.2) below are not considered). If A is filtered by length of monomial (in the generators a_i), then $E^0(A)$ has presentation given by generators $\{b_i\}$ in bijective correspondence with $\{a_i\}$ and relations $\sum_{j,k} r_{j,k} b_j b_k$. Priddy is only concerned with Ext and Tor over A in [**SP**] and he identifies a class of such algebras for which the May spectral sequence (4.1.3) collapses at E^2: he defines A as above to be a *Koszul algebra* if $E^0(A)$ satisfies the property that the cohomology algebra $Tor_{E^0(A)}(R,R)$ [**CE, §XI.7**] is generated (as an algebra) by the elements corresponding to the (indecomposable) elements $\{b_i\}$ of $E^0(A)$. The point is that, for such algebras, a degree argument will show that the associated spectral sequence collapses.

A condition is given in [**SP**] which insures that A is a Koszul algebra, viz., the existance of a Poincaré-Birkhoff-Witt (PBW) basis. We refer the interested reader to [**SP**] for the details about this notion. It should also be noted that Koszul algebras play an important rôle in Manin's work [**YM**].

The identification of the differential d^1 in the spectral sequence in [**SP, §3**] uses the following observation. For convenience, write $Tor_S(R,R) = H_*(S)$ for an augmented algebra $S \to R$. We have

Theorem 4.2.1. [SP, §3]: *If A is a Koszul algebra, then there is an injective R-module map*

$$\nabla : H_*(E^0(A)) \to \bar{B}(A)$$

and a differential $d : H_*(E^0(A)) \to H_*(E^0(A))$ such that

$$\bar{\partial} \circ \nabla = \nabla \circ d.$$

The complex $(H_*(E^0(A)), d)$ is homology equivalent to $\bar{B}(A)$. ▮

Priddy calls $(H_*(E^0(A)), d)$ the Koszul complex associated to the Koszul algebra A. It has homology $H_*(A)$.

We note that the map ∇ is obtained as a composite

$$\begin{array}{ccc} H_*(E^0(A)) & \xrightarrow{r^*} & \bar{B}(A) \\ i \searrow & & \nearrow j \\ & \bar{B}(E^0(A)) & \end{array}$$

for injections i, and j and, in fact, it is easy to see that i may be completed to an SDR

$$(H_*(E^0(A)) \underset{p}{\overset{i}{\rightleftarrows}} \bar{B}(A), \nu).$$

Now since we are working over a field, there is a vector space isomorphism $\bar{B}(E^0(A)) \cong \bar{B}(A)$ and we let V denote this vector space. There are differentials $\bar{\partial}^0$ and $\bar{\partial}$ on V corresponding to $(\bar{B}(E^0(A)), \bar{\partial})$ and $(\bar{B}(A), \bar{\partial})$ respectively. With this notation, we see that there is an SDR

$$((H_*(E^0(A)), 0) \underset{p}{\overset{i}{\rightleftarrows}} (V, \bar{\partial}^0), \nu). \tag{4.2.2}$$

and changing the differential $\bar{\partial}^0$ on V to $\bar{\partial}$, we obtain a solution to the corresponding transference problem (2.7) given by the Koszul complex. The main results of this paper in §8 show that, for a more general class of filtered algebras (not just quadratic algebras which are Koszul), we still have complexes which model the bar construction. There will generally be more terms in the perturbation.

4.3. The Quillen-Jennings Theorem.

In conjunction with the following theorem which relates the group ring of a group to the universal enveloping algebra of an associated Lie algebra, the May spectral sequence gives us a spectral sequence for group cohomology as is well-known (e.g.,[**LP,** §**7**])

Let G be a group and p be a prime integer or 0. If $p \neq 0$, let $G_1 = G$ and more generally, for $i \geq 1$, let G_i be the subgroup of G generated by all elements of the form $(x_1(\dots(x_{r-1}, x_r)\dots))y^{p^s}$ where $rp^s \geq n$ [**DQ**]. $\{G_i\}$ is the mod-p lower central series of G. (See [**SJ,** §**5**]). If $p = 0$, let $\{G_i\}$ be the lower central series of G. We may form the graded Lie algebra $gr(G) = \oplus_n G_n/G_{n+1}$ and we may also form the associated graded algebra $E^0(A)$ of the group ring $A = \mathbb{Z}/p\mathbb{Z}(G)$, filtered by powers of the augmentation ideal (using the usual augmentation $\mathbb{Z}/p\mathbb{Z}(G) \to \mathbb{Z}/p\mathbb{Z}$). Quillen's generalization of Jennings' theorem [**SJ**] is

Theorem 4.5.1. [DQ]:

$$\mathcal{U}(gr(G)) \otimes_{\mathbb{Z}} K \cong E^0(A)$$

*where, if $p \neq 0$, the left hand side denotes the mod-p universal enveloping algebra of $gr(G)$ [**NJ**] and $K = \mathbb{Z}/p\mathbb{Z}$, while if $p = 0$, the left hand side denotes the ordinary universal enveloping algebra of $gr(G)$ and $K = \mathbb{Q}$, the rational numbers.* ▮

(4.1.3) and (4.3.1) obviously imply the well-known

Theorem 4.3.2: *For any group G, there is a spectral sequence $\{E^r\}$ with*

$$E^2 = H_*(gr(G), K) \implies H_*(G, K)$$

where if $p \neq 0$, $K = \mathbb{Z}/p\mathbb{Z}$ and if $p = 0$, $K = \mathbb{Q}$. ∎

We note that by the inverse process (3.2.1) of Cartan–Eilenberg, we may easily convert any such "one-sided" theorem for groups and Lie algebras into the corresponding "two-sided" theorem, i.e., into the corresponding theorem for Hochschild (co)homology.

§5. Hochschild (Co)Homology of Various Classes of Algebras

Moving from very specific to rather general, the Hochschild (co)homology for algebras of various kinds is considered in this section.

5.1. The Hochschild–Kostant–Rosenberg Theorem.

The Hochschild homology $H_*(A, M)$ of an algebra A with coefficients in an A-bimodule M is by definition $Tor^{A^e}(M, A)$. By the results of the previous section, we have a resolution $B(A, A) \to A$ of A over A^e and hence, we may calculate $Tor^{A^e}(M, A)$ as the homology of the complex $M \otimes_{A^e} B(A, A)$ with the differential $1_M \otimes \partial$. Similiar remarks apply to the Hochschild cohomology $H^*(A, M) = Ext_{A^e}(M, A)$. Of course, any A^e-projective resolution of A will do for the computation of homology and cohomology. In certain cases, this (co)homology is explicitly known. For example, consider one part of the theorem of Hochschild, Kostant, and Rosenberg for certain kinds of *commutative* R-algebras (it has been pointed out that the theorem holds under less restrictive conditions on R [**JB**]) :

Theorem 5.1.1 [HKR, p. 395]: *Let R be a perfect field, and A a regular affine algebra over R. The natural map*

$$H_1 = H_1(A, A) \to H_*(A, A)$$

of Hochschild homology groups induces an isomorphism

$$E_A(H_1) \to H_*(A, A)$$

where $E_A(H_1) = \Lambda(H_1)$ is the exterior A-algebra built using alternating multilinear maps on H_1 over A. ∎

Perhaps a few remarks are in order with regard to H_1 in the theorem above. Since A is commutative, the enveloping algebra $A^e \cong A \otimes A$ as algebras and the augmentation $\epsilon : A^e \to A$ defined in §3 is just the multiplication map in A. In this case, it is well-known that $Tor_1^{A\otimes A}(A, A) \cong I/I^2$ where I is the augmentation ideal of A^e. Also, I/I^2 is isomorphic to the *Kähler differentials* for A, i.e., the A-module generated by symbols $\{a, da | a \in A\}$ modulo the relations $d(r) = 0$, and $d(ab) = a(db) + (da)b$, for $r \in R$, and $a, b \in A$. We will use this notation in (5.3.1) below.

It is also worth noting that a "one-sided" case of this theorem for a polynomial algebra follows from the the well-known *Koszul resolution* for the maximal ideal $(x_1, \ldots, x_n)$ in the polynomial ring $A = R[x_1, \ldots, x_n]$ mentioned in §2. Using that resolution, it is immediate that we have an isomorphism of homology

$$Tor_*^A(R,R) \cong E_R[u_1, \ldots, u_n]$$

where

$$Tor_1^A(R,R) = R\{u_1, \ldots, u_n\}$$

is the free R-module generated by a set of elements $\{u_1, \ldots, u_n\}$ corresponding to the algebra generators $\{x_1, \ldots, x_n\}$. Here we have used the usual augmentation $\epsilon : A \to R$ given by

$$\epsilon(f) = \begin{cases} 0, & deg(f) > 0 \\ f, & deg(f) = 0. \end{cases}$$

In fact, what we have said about the polynomial algebra holds over $R = \mathbb{Z}$, the *ordinary integers.*

5.2 Sridharan's Work on Algebras A with $E^0(A)$ Polynomial

Our exposition in this section follows that found in Kassel [**CK**]. We wish to point out however, that Kassel goes on to investigate, not only Hochschild (co)homology, but has interesting results on *cyclic homology* (also see [**CK2**].

As we have already remarked, the Poincaré–Birkoff–Witt theorem implies that if the underlying R-module structure of a Lie algebra $\mathcal{G}$ is projective, then the associated graded algebra $E^0(A)$ (with respect to length of monomial) of the universal enveloping algebra A of $\mathcal{G}$ is isomorphic to the symmetric algebra $Sym(\mathcal{G})$ of the underlying R-module of $\mathcal{G}$.

More generally, suppose that $\mathcal{G}$ is a Lie algebra and $f : \mathcal{G} \times \mathcal{G} \to R$ is a 2-cocycle on $\mathcal{G}$ with coefficients in the trivial $\mathcal{G}$-module R. Let

$$\mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I$$

where I is the ideal generated by

$$\{x \otimes y - y \otimes x - [x,y] - \sigma(f(x,y)) | x, y \in \mathcal{G}\}$$

where σ is the unit of the tensor algebra $T(\mathcal{G})$ on the underlying R-module of $\mathcal{G}$. We have

Theorem 5.2.1 [RS §3]: *Suppose that A is a multiplicatively filtered algebra over R and that $F_n(A) = 0$ for $n < 0$, $F_0(A) = R$, and $\bigcup_n F_n(A) = A$.*

If $E^0(A) \cong Sym(A)$ (the symmetric algebra on the underlying R-module structure of A), then

$$A \cong \mathcal{U}_f(\mathcal{G})$$

where $\mathcal{G}$ is a Lie algebra and $f : \mathcal{G} \times \mathcal{G} \to R$ is a 2-cocycle on $\mathcal{G}$ with coefficients in the trivial $\mathcal{G}$-module R. ∎

Now Sridharan goes on in [**RS**] to prove that there is a "perturbed" version of the inverse process of Cartan and Eilenberg (3.2.1) for algebras A which satisfy the hypotheses of the theorem above. To review, the natural quotient map $q : T(\mathcal{G}) \to \mathcal{U}_f(\mathcal{G})$ induces an R-linear map $i_f : \mathcal{G} \to \mathcal{U}_f(\mathcal{G})$ with the following properties: let $A = \mathcal{U}_f(\mathcal{G})$ and $A_0 = \mathcal{U}(\mathcal{G})$. The map $p : \mathcal{G} \to \mathcal{U}_f(\mathcal{G})^e$ defined by $p(x) = i_f(x) \otimes 1 + 1 \otimes i_f(x)$ induces an R-algebra map $\wp : A_0 \to A$ which makes A into a left A_0-module which we denote by ${}_\wp A$. With this notation, we have

Theorem 5.2.2 [RS]: *If $\mathcal{G}$ is free as an R-module and $X \to R$ is a projective resolution of R over A_0, then ${}_\wp A \otimes_{A_0} X \to A$ is a projective resolution of A over A^e (compare (3.2.1)).* ▮

Using the one-sided resolution (2.2) over A_0 in (4.2.2) to play the role of X, we obtain a two-sided resolution ${}_\wp A \otimes_{A_0} X \to A$. The complex is isomorphic to $A \otimes_R \Lambda(\mathcal{G}) \otimes_R A$ with differential given by [**CK**]

Theorem 5.2.3: *Assume the hypotheses of (4.2.2) with X the resolution in (2.2). Then the differential in $A \otimes_R \Lambda(\mathcal{G}) \otimes_R A$ is given by*

$$\begin{aligned}\partial(g \otimes g_1 \wedge \cdots \wedge g_n \otimes g') = &\sum_{i=1}^{n} (-1)^{i-1} g g_i \otimes g_1 \wedge \cdots \wedge \widehat{g}_i \wedge \cdots \wedge g_n \otimes g' \\ &- \sum_{i=1}^{n} (-1)^{i-1} g \otimes g_1 \wedge \cdots \wedge \widehat{g}_i \wedge \cdots \wedge g_n \otimes g_i g' \\ &+ \sum_{j<k} (-1)^{j+k} g \otimes [g_j, g_k] \wedge g_1 \cdots \wedge \widehat{g}_j \wedge \cdots \wedge \widehat{g}_k \wedge \cdots \wedge g_n. \quad (5.2.4)\end{aligned}$$ ▮

Note that is identical to the two-sided Chevalley-Eilenberg complex.

5.3. Some Generalizations of the Theorem of Hochschild, Kostant, and Rosenberg

Suppose that A is an algebra which admits a filtration such as in (5.2.1) except that we only assume that the associated graded algebra $E^0(A)$ is *commutative*. In case $E^0(A)$ satisfies the hypotheses of the HKR-theorem (5.1.1), we have $H_*(E^0(A), E^0(A)) = \Omega$, where we denote the exterior algebra generated by $H_1(A, A)$ by Ω. We recall that, in this case, Brylinski independantly discovered a spectral sequence beginning with $H_*(E^0(A), E^0(A))$ and abutting to $H_*(A, A)$. He gave an identification of the (E^1, d^1) term as we review: there is a well-known construction of a Lie algebra structure $\{\ ,\ \}$ on $E^0(A)$ given by choosing a representative $u(x) \in F_n(A)$ for each homogeneous element $x \in E^0_n(A) = F_n(A)/F_{n+1}(A)$. We the have $\{x, y\} = u(xy - yx)$. In fact, $(E^0(A), \{\ ,\ \})$ is a *Poisson algebra*, but we won't go into that here (this fact plays an important rôle in deformation theory and its application to *quantum groups* [**VD**]).

We also wish to point out that Brylinski goes on to discuss Poisson structures related to manifolds and cyclic homology. We will not touch upon those topics in this paper. See [**JB**] and [**CK1,2**] for more information about these things.

Using the Lie bracket defined above, we have

Theorem 5.3.1 [JB]: *If A is an algebra with filtration such that $E^0(A)$ is commutative, then the associated spectral sequence $\{E^r\}$ satisfies*

$$E^1 = H_*(E^0(A), E^0(A)) = \Omega$$

where $\Omega = E_A((H_1(S,S))$ (5.1.1) and the differential corresponding to d^1 is given by

$$\begin{aligned}\delta(x_0 dx_1 \dots dx_n) = &\sum_{i=1}^{n} (-1)^{i-1}\{x_0, x_i\} dx_1 \dots \widehat{dx_i} \dots dx_n \\ &+ \sum_{j<k} (-1)^{j+k} x_0 d\{x_j, x_k\} dx_1 \dots \widehat{dx_j} \dots \widehat{dx_k} \dots dx_n. \quad \blacksquare\end{aligned} \tag{5.3.2}$$

In the theorem, we have identified $H_1(S,S)$ with Kähler differentials as mentioned just after (5.1.1).

Kassel [**CK,** §**7**] observes that the spectral sequence collapses for the A of theorem (5.2.1), i.e., for A such that $E^0(A) \cong Sym(A)$ and then goes on to discuss the cyclic homology of such algebras. We will only recall the result for Hochschild homology here.

Theorem 5.3.3. [CK, §**7.4]:** *With the hypotheses of (5.3.1), assume in addition that $E^0(A) \cong Sym(A)$, (so that by (4.2.1) $A \cong \mathcal{U}_f(\mathcal{G})$), then the resolution $A \otimes_R \Lambda(\mathcal{G}) \otimes_R A$ (5.2.4) gives rise to a complex $L = A \otimes_R \Lambda(\mathcal{G}) \otimes_R A \otimes_{A^e} A \cong A \otimes_R \Lambda(\mathcal{G})$ (whose homology is $H_*(A,A)$, of course). We have that L is isomorphic to the complex defined by (5.3.2). Consequently, the spectral sequence (5.3.1) collapses, in this case, at E^2.* $\blacksquare$

§6 Splitting Homotopies and Resolutions

This brief section gives a general recursive formula for a splitting homotopy on the bar construction for a class of resolutions.

6.1. Resolutions Which Split Off From the Bar Construction.

Suppose that $\epsilon : A \to R$ is an augmented algebra, $\bar{X}$ is a free R-module, and $X = A \otimes_R \bar{X}$ posesses a differential $d : X \to X$ and a map $X \to R$ which makes (X, d) into a free A-module resolution of R. By the comparison theorem for resolutions, there exists a chain homotopy equivalence $B(A) \to X$. If this map is onto, then as we mentioned in §2, we may complete to an SDR $(X \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi)$. In this section, we will be concerned with examples where an SDR arises in a natural way from a given contracting homotopy $c : X \to X$. In fact, assume that $X \to R$ is a resolution in which we have an explicit contracting homotopy c

$$(R \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} X, c).$$

The one-sided bar construction is an example. The Koszul resolution (§2) is another; the contracting homotopy is given in [**GH**]. These sorts of resolutions are found in a more general context in the Cartan Seminar [**HC**] and are called *constructions*.

To begin, note that A-linear comparison maps $\nabla : X \to B(A)$ and $f : B(A) \to X$ may be defined recursively, using the contracting homotopy c for X, and the standard contracting homotopy s for $B(A)$ as follows: $\nabla(\bar{x}) = s\nabla d\bar{x}$, for $\bar{x} \in \bar{X}$ and $f(\bar{b}) = cf\partial\bar{b}$, for $\bar{b} \in \bar{B}(A)$. May has pointed this out in [**PM**] and has given a straightforward and useful criterion for when the resulting map is one-one. We have

Theorem 6.1.1. [PM, §7]: *As above, let $X = A \otimes_R \bar{X} \to R$ be a free resolution of R over A and let $\nabla : X \to B(A)$ be defined by $\nabla(\bar{x}) = s\nabla d\bar{x}$, for $\bar{x} \in \bar{X}$. If $d(\bar{X}_n) \cap \bar{X}_{n-1} = 0$, then ∇ is one-one.* ∎

In general, both composites ∇f and $f\nabla$ are homotopic to the respective identity maps and we may use the contracting homotopies to explicitly construct these homotopies; however, we shall assume that we are in the case that $f\nabla = 1_X$. We still have the A-linear chain homotopy $\phi : B(A) \to B(A)$ given recursively by $\phi(\bar{b}) = s(\nabla f\bar{b} - \bar{b} - \phi\partial\bar{b})$. Now note that s vanishes on $\bar{B}(A)$ and so this formula reduces to $\phi(\bar{b}) = s(\nabla f\bar{b} - \phi\partial\bar{b})$. In summary, we will assume, in this section, that the recursive definitions

$$\nabla(\bar{x}) = s\nabla d\bar{x}, \quad \text{for } \bar{x} \in \bar{X} \tag{6.1.2}$$

$$f(\bar{b}) = cf\partial\bar{b}, \quad \text{for } \bar{b} \in \bar{B}(A) \tag{6.1.3}$$

$$\phi(\bar{b}) = s(\nabla f\bar{b} - \phi\partial\bar{b}), \quad \text{for } \bar{b} \in \bar{B}(A) \tag{6.1.4}$$

define maps such that $\nabla f = 1_X$ so that we have an explicit splitting homotopy ϕ on the bar construction. This situation is not unusual (see for example, [**HC**], [**GM, appendix**]).

In the following theorem we adopt the notation: if $\bar{b} = \sum [b_{i_1}|\ldots|b_{i_k}] \in \bar{B}(A)$, then

$$[x_1|\ldots|x_n : \bar{b}] = \sum [x_1 \ldots |x_n|b_{i_1}|\ldots|b_{i_k}] \in \bar{B}(A).$$

Lemma 6.1.5: *Assuming the hypotheses above,*

$$\phi[\bar{b}_1|\ldots|\bar{b}_k] = \sum_{i=0}^{k-1} (-1)^i [b_1|\ldots|b_i : s\nabla f[b_{i+1}|\ldots|b_k]].$$

Proof: This is a straightforward induction using the recursive formula for ϕ. ∎

In addition, we have

Lemma 6.1.6: *Assuming the hypotheses above, if X is a finite resolution, i.e., X vanishes above degree n for some n, then for all $m > n$*

$$\phi[\bar{b}_1|\ldots|\bar{b}_m] = (-1)^{m-n}[b_1|\ldots|b_{m-n} : \phi[b_{m-n+1}|\ldots|b_m]].$$

Thus ϕ is completely determined by $\phi[b_1]$, $\phi[b_1|b_2]$, ... , $\phi[b_1|\ldots|b_n]$. The formula follows from this.

Proof: Since $\phi - s\nabla f - s\phi\partial$ on $\bar{B}(A)$ and necessairly, f vanishes on elements of degree greater than n, we have that for $m > n$, $\phi = -s\phi\partial = -s\phi p$, where $p[b_1|\dots|b_m] = b_1[b_2|\dots|b_m]$. ∎

For example, the Koszul resolution is a finite resolution as is the Tate resolution for $\mathbb{Z}$ over the free abelian group $\mathbb{Z}^n$. We will look at this in detail in §9.

§7 Some Complexes Related to Groups and Formal Groups

Groups with underlying set $\mathbb{Z}^n$ and group law a polynomial function $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ are known to be finitely generated torsion free nilpotent.

There is a class of solvable groups which have as underlying set $\mathbb{Z}^n$ and whose group laws are convergent power series $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$.

Similiar remarks apply to finite p-groups; although it seems a bit unusual to think of them this way, they are of the form $(\mathbb{Z}/p\mathbb{Z})^n$ with a polynomial function $\rho : (\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^n$.

More generally, one can consider a formal power series ρ over a ring R which formally satisfies the associative law and formally satisfies the existence of an identity element [**JS**]. These systems are called *formal groups* and there are various cochain complexes which can be associated to a formal group. We examine some of the relationships between these complexes and the ideas presented in the previous sections.

7.1 Some Convergent Group Laws.

The class of finitely generated torsion-free solvable groups properly contains the class of finitely generated torsion-free nilpotent groups. The latter class can be thought of as consisting of (up to isomorphism) all groups of the form $(\mathbb{Z}^n, \rho)$, where $\mathbb{Z}^n$ denotes the n^{th} Cartesian product of the set of integers with itself and the group operation ρ is a polynomial function, $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$. It can be furthermore shown that, for such a group, the function ρ can be taken to have the form

$$\rho(x, y) = x + y + b(x, y) + O(\geq 3) \tag{7.1.1}$$

where $x, y \in \mathbb{Z}^n$, $+$ denotes ordinary addition of n-tuples of integers, $b(x, y)$ denotes the homogeneous degree 2 term of ρ, and $O(\geq 3)$ denotes the terms of degree ≥ 3 (some polynomial function) [**PH**].

Another class of examples is given by choosing a matrix $A \in GL(n, \mathbb{Z})$. If we consider this as a group homomorphism

$$\begin{gathered} \mathbb{Z} \to GL(n, \mathbb{Z}) \\ 1 \mapsto A, \end{gathered}$$

we may form the semi-direct product $\mathbb{Z}^n \times_A \mathbb{Z}$ where the underlying set is just the $\mathbb{Z}^n \times \mathbb{Z}$ and the operation is

$$\begin{aligned} (x, x_{n+1})(y, y_{n+1}) &= (x + A^{x_{n+1}} y, x_{n+1} + y_{n+1}) \\ &= (x, x_{n+1}) + (y, y_{n+1}) + ((A^{x_{n+1}} - I)y, 0). \end{aligned}$$

We will only consider the cases for which $B = log(A)$ exists. Then we may rewrite the operation above as

$$\begin{aligned}(x, x_{n+1})(y, y_{n+1}) &= (x + e^{x_{n+1}B}y, x_{n+1} + y_{n+1}) \\ &= (x, x_{n+1}) + (y, y_{n+1}) + (x_{n+1}By, 0) \\ &+ (\frac{x_{n+1}^2}{2}B^2y + \frac{x_{n+1}^3}{3!}B^3y + \dots, 0).\end{aligned}$$

So we can obtain a convergent power series group law, $\rho\colon \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$, where

$$\begin{aligned}\rho(x, y) &= x + y + b(x, y) + \psi(x, y) \\ b(x, y) &= (x_{n+1}By, 0) \\ \psi(x, y) &= (\frac{x_{n+1}^2}{2}B^2y + \frac{x_{n+1}^3}{3!}B^3y + \dots, 0).\end{aligned} \tag{7.1.2}$$

For example, the fundamental group G of the Klein bottle may be written this way: take $[-1] \in GL(1, \mathbb{Z})$ to form $G = \mathbb{Z}^2$ with operation

$$\begin{aligned}(x_1, x_2)(y_1, y_2) &= (x_1 + (-1)^{x_2}y_1, x_2 + y_2) \\ &= (x_1 + cos(\pi x_2)y_1, x_2 + y_2) \\ &= (x_1 + y_1, x_2 + y_2) - (\frac{\pi^2 x_2^2}{2!}, 0) + (\frac{\pi^4 x_2^2}{4!}, 0) - \dots\end{aligned}$$

Another example which we will examine more closely is $G = \mathbb{Z}^2 \times_\phi \mathbb{Z}$ where

$$\begin{aligned}\phi : \mathbb{Z} &\to SL(2, \mathbb{Z}) \\ 1 &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Working out the group law as above,

$$\begin{aligned}((x_1, x_2), m)((y_1, y_2), n) &= ((x_1, x_2) + \phi(m)(y_1, y_2), m + n) \\ &= ((x_1, x_2) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^m (y_1, y_2), m + n).\end{aligned}$$

We are thinking of the usual action of a 2×2 matrix on an ordered pair of integers. Thinking of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as corresponding to $i = \sqrt{-1}$, we can obtain an isomorphic group by defining the following operation on $\mathcal{G} \times \mathbb{Z}$, where $\mathcal{G}$ is the group of Gaussian integers, $\mathcal{G} = \{a + bi \mid a, b \in \mathbb{Z}\}$:

$$(g_1, m)(g_2, n) = (g_1 + i^m g_2, m + n). \tag{7.1.3}$$

Now note that

$$\begin{aligned} i^n = e^{\frac{\pi}{2} in} &= \sum_0^\infty \frac{(n\pi i)^k}{2^k k!} \\ &= \sum_0^\infty \frac{(n\pi)^{2k} i^{2k}}{2^{2k}(2k)!} + \sum_0^\infty \frac{(n\pi)^{2k+1} i^{2k+1}}{2^{2k+1}(2k+1)!} \\ &= \sum_0^\infty \frac{(-1)^k (n^2\pi^2)^k}{4^k(2k)!} + \frac{n\pi}{2i} \sum_0^\infty \frac{(-1)^k (n^2\pi^2)^k}{4^k(2k+1)!}. \end{aligned}$$

Thus, for a Gaussian integer $y = y_1 + y_2 i$, we have

$$\begin{aligned} i^{x_3} \cdot y = i^{x_3} y_1 + i^{x_3} y_2 i = \\ & y_1 \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k)!} + \frac{x_3 y_1 \pi}{2} i \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k+1)!} \\ & + y_2 i \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k)!} - \frac{x_3 y_2 \pi}{2} \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k+1)!} = \\ & y_1 \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k)!} - \frac{x_3 y_2 \pi}{2} \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k+1)!} \\ & + \{y_2 \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k)!} + \frac{x_3 y_1 \pi}{2} \sum \frac{(-1)^k (x_3^2\pi^2)^k}{4^k(2k+1)!}\} i. \end{aligned}$$

We therefore have

$$\begin{aligned} (x_1 + x_2 i, x_3)(y_1 + y_2 i, y_3) &= ((x_1 + x_2 i) + (y_1 + y_2 i), x_3 + y_3) \\ &+ \{(\frac{-x_3 y_2 \pi}{2}, 0) + (\frac{x_3 y_1 \pi}{2} i, 0)\} \\ &+ O(\geq 3), \end{aligned}$$

and so we obtain a group law ρ on the set $\mathbb{Z}^3$ by defining

$$\begin{aligned} \rho(x, y) &= (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) \\ &= (x_1, x_2, x_3) + (y_1, y_2, y_3) + (\frac{-x_3 y_2 \pi}{2}, \frac{x_3 y_1 \pi}{2}, 0) + O(\geq 3). \end{aligned} \tag{7.1.4}$$

7.2. Formal group laws.

Following [**JS**], we consider collections of n power series in $2n$ variables with coefficients in some ring R, $\rho_i \in R[[x_1, \ldots, x_n, y_1, \ldots, y_n]] = R[[X, Y]]$, $1 \leq i \leq n$ which formally satisfy the associative law, i.e., if $\rho = (\rho_1, \ldots, \rho_n)$, then as formal power series, we have

$$\rho(\rho(X, Y), Z) = \rho(X, \rho(Y, Z)).$$

(We are substituting one power series in another). We also assume that $\bar{0} = (0, \dots, 0)$ is the "identity", i.e., that we have the formal identities:

$$\rho(Z, \bar{0}) = \rho(\bar{0}, Z) = Z.$$

Under these conditions, it can be shown that a "formal inverse" exists. Furthermore, because of the associative law, we have the following construction for a formal group ρ. Let $[\ ,\ ]$ be defined by

$$\begin{aligned} &[\ ,\ ] \colon R^n \otimes R^n \to R^n \\ &[x, y] = b(x, y) - b(y, x) \end{aligned}$$

where $b(x, y)$ denotes the homogeneous degree 2 term of ρ. Then $(R^n, [\ ,\ ])$ is a Lie algebra over R, i.e., the bracket is bilinear, skew-symmetric, and the Jacobi identity holds. We will call this object the Lie algebra of ρ and write it as $\mathcal{L}(\rho)$.

7.3. Some Cochain Complexes Associated to Formal group laws.

There are several cochain complexes that may be associated to the group laws we have been discussing. Some are related to the cogroup structure of the bialgebra associated to the given formal group. In general, given a formal group law ρ, and an extension ring $\widetilde{R}$ of R, we may form the complex

$$\mathcal{C}^i(\rho; \widetilde{R}) = \widetilde{R}[[X_1, \dots, X_i]]$$

(each X_j denotes n variables $x_{i,1}, \dots, x_{i,n}$). We give this complex a differential in analogy with the (dual) bar-contruction differential (for cohomology)

$$\begin{aligned} \delta(f)(X_1, \dots, X_{i+1}) = & f(X_2, \dots, X_{i+1}) \\ & + \sum_j (-1)^j f(X_1, \dots, \rho(X_j, X_{j+1}), \dots, X_{i+1}) \\ & + (-1)^{i+1} f(X_1, \dots, X_i). \end{aligned}$$

This is in analogy with the "functional bar construction cochains" of [**SM**].

Given some of the special group laws such as those corresponding to a nilpotent group, which are actually *polynomials*, we can restrict the "cochains" above to only polynomials. For group laws that are actually *convergent power series*, we might restrict to only cochains which are themselves convergent power series. We will denote these other complexes by $\mathcal{C}^*_{poly}(\rho; \widetilde{R})$ and $\mathcal{C}^*_{\infty}(\rho; \widetilde{R})$ respectively. We will not examine the relationships between these cochain complexes in this paper, but we would like to point out some rather subtle points about them. Indeed, consider the case of the group law $\rho = +$ on $\mathbb{Z}$, i.e., the ordinary integers under addition. One might think that polynomial cochains suffice to compute the group cohomology, but an easy exercise gives that ordinary multiplication of integers $\mu(x, y) = xy$ is a 2-cocycle in $\mathcal{C}^*_{poly}(\rho; \mathbb{Z})$ and in fact it is an element **of order 2** in $H^2(\mathcal{C}^*_{poly}(\rho; \mathbb{Z}))$; but the group cohomology of the integers vanishes above degree 2! In order to capture the group cohomology of $(\mathbb{Z}, +)$, one needs to take polynomial functions which take integers to integers for cochains. These are not just polynomials with $\mathbb{Z}$ coefficients, but are integral linear combinations of binomial coefficient functions such as $f(n) = \binom{n}{2}$, $f(m,n) = \binom{n}{7}\binom{m}{3}$, etc. (We might denote this complex by $\mathcal{C}^*_{binco}(\rho; \widetilde{R})$). The proof of this fact is the first part of an induction showing, among other things, that the whole homotopy category of *nilpotent topological spaces* can be built up out of "binomial k-invariants" over the integers and follows from Ekedahl's work [**TE**]. The case for finitely generated torsion-free nilpotent groups using the machinery of [**LS**], [**GL**], (see especially [**GL,§4.3**]) was announced, but not published [**LL4**]. There is, undoubtedly a strong connection between these ideas and the complex of Grothendiek-Cartan [**HC2**] used in our work [**LP**] but we will not investigate the connections here. The issue of group cohomology will be brought up again in the sections below.

We will leave this section with a few remarks about products and higher order operations on the complexes above. First of all, the pointwise product of formal power series gives rise to a product operation in $\mathcal{C}^*(\rho; \widetilde{R})$: given $f(X_1, \ldots, X_i)$ and $g(X_1, \ldots, X_j)$, take

$$fg(X_1, \ldots, X_{i+j}) = f(X_1, \ldots, X_i)g(X_1, \ldots, X_j).$$

This makes $\mathcal{C}^*(\rho; \widetilde{R})$ into a differential graded algebra. More can be done however. Following the recursive formulas given by Gugenheim and May [**GM, appendix**], we may formally define a "cup-1" product and *explicitly* show that $H^*(\mathcal{C}^*(\rho; \widetilde{R}))$ is a graded commutative algebra. One also has an action of the Steenrod algebra on the cohomology by formally following the recursive construction in [**PM2**]. We will not encounter these statements again in this paper until the last section and then only briefly.

§8 Perturbation Theorems for Resolutions Over Filtered Algebras

In this section, we present a derivation of resolutions for a class of filtered algebras by the use of homological perturbation theory. These resolutions give rise to complexes which can be thought of as giving models for the May spectral sequence (4.1.3). Throughout this section, we assume that A is a multiplicatively filtered algebra.

8.1. Main Theorem.

Suppose that we have a resolution X over $E^0(A)$. We wish to investigate the possibility of altering X in some systematic way to obtain a resolution over A. If we have, for example, an SDR, $X \rightleftarrows B(E^0(A))$, then if we are working over a field, we may think of $B(A)$ as additively (i.e. as a vector space over R) isomorphic to $B(E^0(A))$. We then have a corresponding transference problem. We mentioned something like this in (4.2.2).

Now it is sometimes the case that we have found only a model $\bar{X}$ of the reduced bar construction $\bar{B}(A)$ for an augmented algebra A (so that $H_*(\bar{X})$ is isomorphic to $Tor^A_*(R, R)$). When this is the case, we'd like to be able to alter an SDR $\bar{X} \rightleftarrows \bar{B}(A)$ to an SDR $A \otimes_R \bar{X} \rightleftarrows A \otimes_R \bar{B}(A)$ and even to an SDR $A \otimes_R \bar{X} \otimes_R A \rightleftarrows A \otimes_R \bar{B}(A) \otimes_R A$. We have formal solutions to each of these transference problems using the basic homological perturbation theory of §2:

Theorem 8.1.3: *Suppose that, as R-modules, A and $E^0(A)$ are isomorphic (for example R is a field). Suppose also that A is an augmented algebra. Let V denote the underlying R-module structure of $\bar{B}(A)$ which is isomorphic as an R-module to $\bar{B}(E^0(A))$. Let $\bar{\partial}^0$ and $\bar{\partial}$ denote the differentials on V corresponding to the bar construction differentials in the one-sided bar constructions $B(E^0(A))$ and $B(A)$ respectively. Assume that we have an SDR*

$$((\bar{X}, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (V, \bar{\partial}^0), \phi)$$

for some complex $\bar{X}$. Then, we have the tensor product SDR [**EM2, (3.1)**], [**LS, (2.5)**]

$$((A \otimes_R \bar{X}, 1_A \otimes d) \underset{1 \otimes_A f}{\overset{1_A \otimes \nabla}{\rightleftarrows}} (A \otimes_R V, 1_A \otimes \bar{\partial}^0), 1 \otimes_A \phi).$$

Changing the differential on $1_A \otimes V$ to $\partial = 1_A \otimes \bar{\partial} + \widetilde{\pi}$ (see (3.4.1) and the notation before (3.3.1)), and letting $X = A \otimes_R \bar{X}$, we have a formal solution of the transference problem

$$((X, d') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (V, \partial), \phi')$$

given by equations (2.8). If the sequence (2.8) converges, or if there exists a splitting homotopy ϕ' as in (2.9), which is, in addition, A-linear, then we obtain a resolution (X, d') of R over A which splits off from the one-sided bar construction $B(A)$ (3.2).

Proof: Let $\nabla^0 = 1 \otimes_A \nabla$, $f^0 = 1 \otimes_A f$, $\phi^0 = 1 \otimes_A \phi$, and $d^0 = 1_A \otimes d_{\bar{X}}$. The perturbation $t = \partial - 1_A \otimes \partial^0$ is A-linear as are ∇^0,f^0,ϕ^0, and d^0 so that we obtain A-linear maps using the perturbation formulae:

(8.1.4) $$\partial_\infty = 1_A \otimes d_{\bar{X}} + \phi^0 t \nabla^0 + f^0(t\phi^0)t\nabla^0 + \ldots$$

(8.1.5) $$\nabla_\infty = \nabla^0 + \phi^0 t \nabla^0 + \phi^0(t\phi^0)t\nabla^0 + \ldots$$

(8.1.6) $$f_\infty = f^0 + ft\phi^0 + f(t\phi^0)t\phi^0 + \ldots$$

(8.1.7) $$\phi_\infty = \phi^0 + \phi^0 t\phi^0 + \phi^0(t\phi^0)t\phi^0 + \ldots \quad \blacksquare$$

There are several useful variations of (8.1.3) and we mention a few here. Indeed, if we encounter the hypotheses of (8.1.3), and we are not concerned with a resolution, but just want to model the reduced bar construction, we could either carry out the perturbation of (8.1.3) and then tensor the resulting complex with R over A to reduce it or we could simply use the SDR (8.1.1) and the initiator $\bar{\partial}^0 - \bar{\partial}$, where $\bar{\partial}$ correspond to the differential on V corresponding to the differential in $\bar{B}(A)$.

If we had a model of the reduced bar construction of A which, as above, is an SDR $(\bar{X} \underset{f}{\overset{\nabla}{\rightleftarrows}} \bar{B}(A), \phi)$ and we wanted a resolution over A, then we could proceed as in (8.1.3) above using the initiator $t = \partial - 1_A \otimes \bar{\partial}$ to obtain a (formal) solution to the transference problem.

We may also use this method to model the two-sided bar construction to obtain Hochschild (co)homology. Although the process is completly analogous to that of (8.1.3), we state it seperately. We may think of this as a "perturbed inverse process" (compare (3.2.1) and (5.2.2)).

Theorem 8.1.8: *Suppose that, as R-modules, A and $E^0(A)$ are isomorphic (for example R is a field). Let V denote the underlying R-module structure of $\bar{B}(A)$ which is isomorphic as an R-module to $\bar{B}(E^0(A))$. Let $\bar{\partial}^0$ and $\bar{\partial}$ denote the differentials on V corresponding to the bar construction differentials in the one-sided bar constructions $B(E^0(A))$ and $B(A)$ respectively. Assume that we have an SDR*

$$((\bar{X}, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (V, \bar{\partial}^0), \phi)$$

for some complex $\bar{X}$. Then, we have the tensor product SDR

$$((A \otimes_R \bar{X} \otimes_R A, 1_A \otimes d \otimes_R A) \underset{1 \otimes_A f \otimes_R A}{\overset{1_A \otimes \nabla \otimes_R A}{\rightleftarrows}} (A \otimes_R V \otimes_R A, 1_A \otimes \bar{\partial}^0 \otimes_R A), 1 \otimes_A \phi \otimes_R A).$$

Changing the differential on $A \otimes_R V \otimes_R A$ to

$$\begin{aligned} \partial &= 1_A \otimes \bar{\partial} \otimes 1_A \\ &\quad + [(m \otimes 1)(1 \otimes \pi \otimes 1)(1 \otimes \triangle)] \otimes 1 \\ &\quad + (1 \otimes 1 \otimes m)(1 \otimes 1 \otimes m)(1 \otimes 1 \otimes \pi \otimes 1)(1 \otimes \triangle \otimes 1) \end{aligned}$$

(see (3.5.3)), and letting $X = A \otimes_R \bar{X} \otimes_R A$, we have a formal solution of the transference problem

$$((X, d') \underset{f'}{\overset{\nabla'}{\rightleftarrows}} (V, \partial), \phi')$$

given by equations (2.8). If the sequence (2.8) converges, or if there exists a splitting homotopy ϕ' as in (2.9), which is, in addition, A^e-linear, then we obtain a resolution (X, d') of A over A which splits off of the one-sided bar construction $B(A)$ (3.2). ▮

§9. Applications to Group Laws

Small models for torsion-free nilpotent groups have had a long history which we may think of as beginning with Mal'cev [**AM**]. The first result on a small model for the cohomology of a finitely torsion-free nilpotent group G that we are aware of is the result of Nomizu which we will interpret here in our algebraic context. We go on to review how the methods of §8 were applied to calculate explicit resolutions of $\mathbb{Z}$ over the integral group ring A of a finitely generated nilpotent group. By the inverse process of Cartan and Eilenberg, corresponding results hold for resolutions of A over A^e for Hochschild (co)homlogy. We give the computation of a resolution for the one-parameter family of nilpotent groups presented at this conference. This example points to an important difference between homological perturbation methods for computing resolutions as opposed to the usual methods of calculating minimal resolutions via linear (or, using Groebner bases, certain kinds of non-linear) algebra, viz., we obtain closed formulae for the resolution in terms of the given parameter.

9.1. Models of $\bar{B}(G)$ for G Finitely Generated Torsion-Free Nilpotent

Let G be written in the form $(\mathbb{Z}^n, \rho)$ as in (7.1.1). We then have a corresponding Lie algebra $\mathcal{L}_{\mathbb{R}} = \mathcal{L}(\rho)$ over the extension ring $\mathbb{R}$ of $\mathbb{Z}$. Nomizu gives

Theorem 9.2.1. [KN]: *There is an isomorphism of cohomology algebras*

$$Ext^*_{\mathcal{U}}(\mathbb{R}, \mathbb{R}) \cong Ext^*_{\mathbb{R}(G)}(\mathbb{R}, \mathbb{R}).$$

where $\mathcal{U} = \mathcal{U}(\mathcal{L}_{\mathbb{R}})$ *is the universal enveloping algebra of* $\mathcal{L}_{\mathbb{R}}$. ▮

Note that the *Ext* on the left in the theorem is usually written simply as $H^*(\mathcal{L}_{\mathbb{R}})$ and is called the Lie algebra cohomology of $\mathcal{L}$ and *Ext* on the right in the theorem is usually simply written as $H^*(G; \mathbb{R})$ and is called the *group cohomology* of G. (see (4.3)).

A generalization of Nomizu's theorem was given in [**LP**] which gave the result for subrings of $\mathbb{Q} \subset \mathbb{R}$. The result over $\mathbb{Q}$ may also be found in [**PP**] and [**DS**]. Using these results, and the observations in (4.3), we may also translate the results of [**LP**] into the context of the present paper:

Theorem 9.2.2. [LP]: *For certain subrings* $\widetilde{\mathbb{Z}}$ *of* $\mathbb{Q}$ *depending on the torsion-free nilpotent group* G*, the May spectral sequence for groups (4.3.2) collapses and the associated Chevalley-Eilenberg complex derived from (2.2) for the Lie algebra* $\mathcal{L}_{\mathbb{R}}$ *computes the group cohomology of* G *with coefficients in* $\widetilde{\mathbb{Z}}$. ▮

Much of the homological perturbation theory as applied to resolutions in this paper came about from trying to understand a proper generalization of (9.2.2) over the integers and, has already been mentioned, was inspired by work of Gugenheim and May. A much more general theorem was discovered with Stasheff in [**LS**]. It concerns a certain class of *nilpotent topological spaces*. Again, that paper was motivated by (9.2.2) and the author's earlier attempts in [**LL1**] and [**LL2**]. We will not review the results of those papers here. We wish to point out however that the last two contain some explicit computations and the simplicial methods used in [**LS**] (inspired by Gugenheim's work in [**VG1**]) give rise to a proof of the convergence of the perturbation in the formal solution to the transference problem in (8.3.1) for the case of $A = \mathbb{Z}(G)$ (the group ring over the *integers*) and $R = \mathbb{Z}$.

The computer algebra system Scratchpad was used in [**LL3**] to experiment with the computations involved in generating resolutions of $\mathbb{Z}$ over the integral group rings of finitely generated torsion-free nilpotent groups using homological perturbation. The idea is a straightforward variation of (8.1.3) and we will review it below.

9.3. The Exterior Algebra SDR.

We begin with an explicit and well-known resolution of $\mathbb{Z}$ over the group ring $A = \mathbb{Z}[t^{-1}, t]$ of the free abelian group on one generator t over $\mathbb{Z}$ [**EM2, III.14**], [**HC**]. First, we establish a notation to be used throughout the remainder of the paper. Let

$$[t]_n = \frac{t^n - 1}{t - 1}$$

Note that $[t]_n = \sum_{i=0}^{n-1} t^i \in A$.

A has augmentation ϵ and unit σ, both algebra maps over $\mathbb{Z}$, given by

$$\epsilon(t^n) = 1$$
$$\sigma(n) = nt^0 = n1.$$

We have

Theorem 9.3.1: *Let $A = \mathbb{Z}[t^{-1}, t]$ be the integral group ring of the free abelian group on one generator t. Define an A-linear map $d : A \otimes_A E[u] \to A \otimes_A E[u]$ of degree -1 and a $\mathbb{Z}$ linear map $\varphi : A \otimes_A E[u] \to A \otimes_A E[u]$ of degree $+1$ by*

$$\begin{aligned} d(t^n) &= 0 \\ d(u) &= t - 1 \\ \varphi(t^n) &= [t]_n u \\ \varphi(t^n u) &= 0. \end{aligned}$$

Then $(A \otimes_A E[u], d)$ is an A-free resolution of $\mathbb{Z}$ and extending the augmentation ϵ and unit σ to $(A \otimes_A E[u], d)$ in the obvoius way ($\sigma(n) = n \otimes 1$, and $\epsilon(u) = 0$), we have an SDR [**EM2**]

$$(\mathbb{Z} \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} (A \otimes_A E[u], d), \varphi).$$

Proof: The identity $d\varphi + \varphi d = 1 - \sigma\epsilon$ is easily verified. ▮

As already mentioned, one can tensor strong deformation retractions to produce an SDR

$$(\mathbb{Z} \underset{\epsilon}{\overset{\sigma}{\rightleftarrows}} A \otimes_{\mathbb{Z}} E[u_1, \ldots, u_n], \varphi)$$

where $A = \mathbb{Z}[t_n^{-1}, \ldots, t_1^{-1}, t_1, \ldots, t_n]$ is the integral group ring of the free abelian group on n-generators $t_1, \ldots, t_n$. We have

Theorem 9.3.2: *With the above notation, we have an SDR*

$$(A \otimes_{\mathbb{Z}} E[u_1, \ldots, u_n] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi).$$

Proof: The formulae (6.1.2)-(6.1.4) produce the desired maps and the identity $f\nabla = 1$ is easily verified. ▮

In fact, it was precisely the methods given in (9.3.1) and (9.3.2) that were used to implement the strong deformation retractions in [**LL3**]. Furthermore, the method used in [**LL3**] to solve the transference problem for the resolutions involved there were a variation on (8.1.3) and they agree with the straightforward application given by (8.1.3).

We now present the SDR of (9.3.2) explicitly in the three dimensional case obtained by simply expanding the formulae we have presented so far.

Lemma 9.3.3: *For $n = 3$, and $A = \mathbb{Z}[t_3^{-1}, t_2^{-1}, t_1^{-1}, t_1, t_2, t_3]$, the SDR*

$$(A \otimes_{\mathbb{Z}} E[u_1, u_2, u_3] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi)$$

in (9.3.2) is given as follows: The differential d (a derivation of the exterior algebra) and contracting homotopy φ on $A \otimes_{\mathbb{Z}} E[u_1, u_2, u_3]$ are given by

$$\begin{aligned}
d(p) &= 0, \quad \text{for } p \in A \\
d(u_i) &= t_i - 1 \\
\varphi(t^{i_1} t^{i_2} t^{i_3}) &= [i_1]_{t_1} u_1 + t_1^{i_1} [i_2]_{t_2} u_2 + t_1^{i_1} t_2^{i_2} [i_3]_{t_3} u_3 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_2) &= t_1^{i_1} [t_2]_{i_2} u_2 u_1 + t_1^{i_1} t_2^{i_2} [i_3]_{t_3} u_3 u_1 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_2) &= t_1^{i_1} t_2^{i_2} [i_3]_{t_3} u_3 u_2 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_3) &= 0 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_1 u_2) &= t_1^{i_1} t_2^{i_2} [i_3]_{t_3} u_1 u_2 u_3 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_1 u_3) &= 0 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_2 u_3) &= 0 \\
\varphi(t^{i_1} t^{i_2} t^{i_3} u_1 u_2 u_3) &= 0.
\end{aligned}$$

The inclusion (which is related to the well-known "shuffle product") is given by

$$\begin{aligned}
\nabla(u_i) &= [t_i] \\
\nabla(u_i u_j) &= [t_i|t_j] - [t_j|t_i] \\
\nabla(u_1 u_2 u_3) &= [t_1|t_2|t_3] - [t_1|t_3|t_2] - [t_2|t_1|t_3] \\
&\quad + [t_2|t_3|t_1] + [t_3|t_1|t_2] - [t_3|t_2|t_1].
\end{aligned}$$

The projection is given by

$$\begin{aligned}
f([t^{i_1} t^{i_2} t^{i_3}]) &= t_1^{i_1} t_2^{i_2} [i_3]_{t_3} u_3 + t_1^{i_1} [i_2]_{t_2} u_2 + [i_1]_{t_1} u_1 \\
f([t^{i_1} t^{i_2} t^{i_3} | t^{j_1} t^{j_2} t^{j_3}]) &= -t_1^{i_1 + j_1} t_2^{i_2} [j_2]_{t_2} [i_3]_{t_3} u_2 u_3 - t_1^{i_1} [j_1]_{t_1} t_2^{i_2} [i_3]_{t_3} u_1 u_3 \\
&\quad - t_1^{i_1} [j_1]_{t_1} [i_2]_{t_2} u_1 u_2 \\
f([t^{i_1} t^{i_2} t^{i_3} | t^{j_1} t^{j_2} t^{j_3} | t^{k_1} t^{k_2} t^{k_3}] &= -t_1^{i_1 + j_1} [k_1]_{t_1} t_2^{i_2} [j_2]_{t_2} [i_3]_{t_3} u_1 u_2 u_3. \quad \blacksquare
\end{aligned}$$

Finally, we will not give the homotopy ϕ here but note that it can be worked out using (6.1.5), (6.1.6) and the above formulae.

9.4. A One Parameter Family of Nilpotent Groups.

Consider the set of matrices

$$U_q(\mathbb{Z}^3) = \left\{ \begin{pmatrix} 1 & x & q^{-1}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$

For a fixed $q \in \mathbb{Z}$, these matrices form a nilpotent group and it is known that an arbitrary rank 3 torsion free nilpotent group is isomorphic to one of the groups $U_q(\mathbb{Z}^3)$ for some q. Using the results of §8 and §9, we have

Theorem 9.4.1: *A resolution of $\mathbb{Z}$ over $\mathbb{Z}(U_q(\mathbb{Z}^3))$ is given by the complex*

$$(\mathbb{Z}(U_q(\mathbb{Z}^3)) \otimes_{\mathbb{Z}} E_{\mathbb{Z}}[u_1, u_2, u_3], d)$$

where

$$\begin{aligned}
d(u_i) &= t_i - 1\\
d(u_1u_2) &= (-t_1t_2[q]_{t_3})u_3 + (t_1 - 1)u_2 - (t_2 - 1)u_1\\
d(u_1u_3) &= (t_1 - 1)u_3 - (t_3 - 1)u_1\\
d(u_2u_3) &= t_2 - 1)u_3 - (t_3 - 1)u_2\\
d(u_1u_2u_3) &= (t_1 - 1)u_2u_3 - (t_2 - 1)u_1u_3 + (t_3 - 1)u_1u_2. \quad \blacksquare
\end{aligned}$$

More generally, the same sort of argument applies to any finitely generated torsion-free nilpotent group and we have

Theorem 9.4.2 [LS, (4.1)]: *For any finitely generated torsion-free nilpotent group G, we have an isomorphism $G \cong (\mathbb{Z}^n, \rho)$ where ρ is a polynomial function (7.1.1). Let $A = \mathbb{Z}(G)$ be the augmented group ring over $\mathbb{Z}$. If we consider the transference problem of (8.1.3) for the SDR (9.3.2), then the formal solution (2.8.2) converges. In fact, in this case $t\phi$ is nilpotent in any fixed degree. The formal solution is also a solution in the sense of (2.9).* $\blacksquare$

A few remarks are in order here. First of all, the proof in [**LS**] is given only for the *reduced* complexes. One can use that result to obtain an SDR

$$(E[u_1, \ldots, u_n] \underset{f\infty}{\overset{\nabla_\infty}{\rightleftarrows}} \bar{B}(A), \phi_\infty)$$

and then form the tensor product SDR by simply tensoring A onto the left of these objects and maps. We may then use the initiator $t = \partial - 1_A \otimes \bar{\partial}$ on $A \otimes_R \bar{B}(A)$. Note that this initiator is precisely the twisted tensor product perturbation $\widetilde{\pi}$ (3.3.1) for the universal twisting cochain (3.4.1). The formal solution for this transference problem converges using a straightforward filtration on the bar construction. We will not go into the details here, but we point out that this variation on (8.1.3) was exactly the method used in [**LL3**] for the upper uni-triangular matrix groups over $\mathbb{Z}$ and that the direct method of (8.1.3) gives precisely the same solutions.

By the inverse process of Cartan and Eilenberg, we may obtain a two-sided resolution by a straightforward tensor product (the hypotheses of (3.2.1) are satisfied for a group ring [**CE**]). On the other hand, we could use the two-sided perturbation theorem (8.1.8) directly on the exterior algebra contraction. We will not present these results here, but the interested reader should be able to produce these resolutions using the methods we have presented so far.

Before leaving this section, we want to mention an application to other nilpotent groups. Let $\Gamma[y_1, \ldots, y_n]$ be the divided power algebra in n indeterminants over $\mathbb{Z}/p\mathbb{Z}$.

Let G_1 be the cyclic group $\mathbb{Z}/p\mathbb{Z}$ written multiplicatively and let G_n be the product of G with itself n times. Let $A_0 = \mathbb{Z}/p\mathbb{Z}(G_n)$. Let P be any *finite p-group*, $R = \mathbb{Z}/p\mathbb{Z}$, and $A = R(G)$. By (4.5.1), there is an n such that $E^0(A) = A$. Using the SDR

$$(A \otimes_R \Gamma[y_1, \ldots, y_n] \otimes_R E[x_1, \ldots, x_n] \underset{f}{\overset{\nabla}{\rightleftarrows}} B(A), \phi)$$

given by Eilenberg and MacLane [**EM2**] to set up the transference problem (8.1.3) for P, we obtain a result analogous to (9.4.2). This case is covered by the result in [**LS**]. It would be interesting to have a comparison of these results with the results given by Huebschmann for certain classes of finite p-groups [**JH2**],[**JH3**].

All of these results may be thought of as giving models for the May spectral sequence (4.1.3) and as extensions of Priddy's complex (4.2.1) to the case where the relations in the algebra are not necessarily quadratic (also see (9.5.2) and the remark following it).

9.5. Power Series Group Law.

Consider again the example given in (7.1.3). We worked out a convergent power series group law on $\mathbb{Z}^3$ in (7.1.4). We noted that

$$\rho(x, y) = x + y + \tau(x, y)$$

where τ is a convergent power seires (see (7.1.4). Thus, $G = (\mathbb{Z}^3, \rho)$ is a perturbation of the free abelian group on 3 generators. Using the theory in §8 which gives a formal solution to the perturbation problem and the explicit SDR above, we have

Theorem 9.5.1: *Let G be the solvable group defined by the convergent power series group law (7.1.4). Consider the transference problem in (8.1.3) using the SDR in (9.3.3). There is a solution to the transference problem which gives rise to the resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}(G)$, given by the complex*

$$(\mathbb{Z}(G) \otimes_{\mathbb{Z}} E_{\mathbb{Z}}[u_1, u_2, u_3], d)$$

where

$$\begin{aligned}
d(u_i) &= t_i - 1 \\
d(u_1u_2) &= (t_1 - 1)u_2 - (t_2 - 1)u_1 \\
d(u_1u_3) &= (t_2 - 1)u_3 + u_2 + t_3u_1 \\
d(u_2u_3) &= -(t_1^{-1} - 1)u_3 - t_3u_2 - t_1^{-1}u_1 \\
d(u_1u_2u_3) &= (t_2 - 1)u_2u_3 - (t_1^{-1} - 1)u_1u_3 - (t_1 - t_3)u_1u_2.
\end{aligned}$$

Proof: The proof is actually quite straightforward and consists of finding an inverse of $1 - t\phi$ for the initiator of (8.1.3) using the SDR of (9.3.3) and the form of the group law given by (7.1.3). Remarkably, the inverse can be found rather quickly and the solution shows that $t\phi$ is, in fact, nilpotent. ∎

More than a few comments are in order, but we will limit the discussion in this paper to two observations. First, in retrospect, a filtration of the bar construction can be found for which one can show that the map $t\phi$ from the formal solution is nilpotent in each degree so that the differential for the resolution we have presented is the one given by the formal solution. Second, there is a theorem of Mostow that says that the cohomology of a solvable group (such as the one in (9.5.1)) with coefficients in the real numbers $\mathbb{R}$, is isomorphic to the cohomology of the associated real Lie algebra [**GMo**]. If one were to actually write the (convergent) formal power series group law in (9.5.1) as in (7.1.4), then the Lie algebra of Mostow's theorem is given by the homogeneous degree 2 term (which is necessairly over $\mathbb{R}$ even though the whole function takes integers to integers). Now the formal solution (2.8.1) written out for the initiator using the form (7.1.4) has the *first term of the perturbation for the differential given by the Chevalley-Eilenberg differential (2.2).* We thus have proven that, in this case, if the "perturbation is continued", then we get a *descent of Mostow's theorem to the integers.* These observations are related to the following general observation: if t is the formal perturbation corresponding to a formal group law (as in our example above), then $ft\phi$ is a perturbation of the Chevalley-Eilenberg differential. This fact follows from the observation that ∇ is just given by shuffles and f is essentially just the classical Alexander-Whitney map [**SM,** §**VIII.8**], [**CE,** §**IX.7**].

In summary, if ρ is a formal power series group law (7.2) over a subring R of the complex numbers $\mathbb{C}$ and we allow infinite sums in the group ring in (9.3.1), i.e. work in the completion, then the formulae for the maps ∇, f, and ϕ that result from (9.3.2) allow straightforward extensions to maps in the completed case and we may *formally* apply the formal perturbation formulae (2.8.2) using the formal initiator given (in analogy with (8.1.3)) by the formal group law. We have that generally,

$$\begin{aligned} d_\infty &= ft\nabla + ft\phi t\nabla + f(t\phi)^2 t\nabla + \dots \\ &= (d_{CE} + \dots) + \dots \end{aligned}$$

where d_{CE} is the Chevalley-Eilenberg differential for the corresponding Lie algebra.

It also appears that the first part of the higher terms are related to the formal higher order product structure such as is found in the differentials occuring in [**GM**].

Similiar remarks apply to the case of formal groups over $\mathbb{Z}/p\mathbb{Z}$ using the SDR (9.4.3). The analysis of this situation will be done elsewhere.

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