A SURVEY OF APPLICATIONS OF SURGERY TO KNOT AND LINK THEORY

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1. Introduction

Knot and link theory studies how one manifold embeds in another. Given a manifold embedding, one can alter that embedding in a neighborhood of a point by removing this neighborhood and replacing it with an embedded disk pair. In this way traditional knot theory, the study of embeddings of spheres in spheres, impacts the general manifold embedding problem. In dimension one, the manifold embedding problem *is* knot and link theory.

This article attempts a rapid survey of the role of surgery in the development of knot and link theory. Surgery is one of the most powerful tools in dealing with the question "To what extent are manifolds (or manifold embeddings) uniquely determined by their homotopy type?" As we shall see, roughly speaking, knots and links are determined by their homotopy type (more precisely, Poincaré embedding type) in codimension ≥ 3 and are much more complicated in codimension two. We proceed, largely, from an historical perspective, presenting most of seminal early results in the language and techniques in which they were first discovered. These results in knot theory are among the most significant early applications of surgery theory and contributed to its development. We will emphasize knotted and linked spheres, providing only a brief discussion of more general codimension two embedding questions. In particular, the theory of codimension two embedding, from the standpoint of classifying within a Poincaré embedding type, deserves a long overdue survey paper. The present paper will not fill this void in the literature. Cappell and Shaneson give an excellent introduction to this subject in [CS78].

By no means is this survey comprehensive, and we apologize in advance for the omission of many areas where considerable and important work has been done. For example, we will omit the extensive subject of equivariant knot theory. We will also not include any discussion of the techniques of Dehn surgery that have proven so valuable in the study of three manifolds and classical knots. Furthermore, we will not touch on the related subject of immersion theory, and barely mention singularity theory. We urge the reader to consult one of the many excellent surveys which have covered the early (before 1977) development of codimension two knot theory in more depth. The articles by Cameron Gordon [Gor77] and Kervaire-Weber [KW77] on, respectively, low-dimensional and high-dimensional knot theory are excellent. A detailed discussion of surgery and embedding theory can be found in Ranicki's book, [Ran81]. On the other hand, we are not aware of any previously existing comprehensive survey of recent developments in link theory.

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2. Codimension > 2

Perhaps the first use of surgery techniques in knot theory was in the work of Andre Haefliger. In 1961 Haefliger [Hae61] proved a basic theorem which showed that, for appropriately highly connected manifolds, the isotopy classification of embeddings coincided with the homotopy classification of maps, as long as one was in the *metastable* range of dimensions. More specifically Haefliger showed that if M is a compact manifold, then any q-connected map $f : M^n \to V^m$ (where superscripts denote dimension) is homotopic to an embedding, if $m \geq 2n - q$, and any two homotopic q-connected maps $M^n \to V^m$ are isotopic if $m > 2n - q$. This theorem required, also, the restriction $2m \geq 3(n + 1)$ and $2m > 3(n + 1)$, respectively. In particular, any homotopy n-sphere embeds in S^m and any two such embeddings are homotopic as long as $2m > 3(n + 1)$.

His proof proceeded by examining the singular set of a smooth map and eliminating it by handle manipulations — a generalization of Whitney's method for n-manifolds in $2n$-space. Meanwhile Zeeman in the PL-category, and Stallings in the topological category, had shown, using the technique of *engulfing*, that there were no non-trivial knots as long as $m > n + 2$ [Zee60] [Sta63].

Haefliger, in a seminal paper [Hae62], showed that when $m = \frac{3}{2}(n + 1)$, the analogous smooth result was already false. Here was the first real use of surgery to study embedding problems. In this paper Haefliger developed the technique of *ambient* surgery, i.e. surgery on embedded manifolds, and used this technique to give a classification of knotted $(4k - 1)$-spheres in $6k$-space (which was, shortly after, extended to a classification of $(2k - 1)$-spheres in $3k$-space). He first observed that the set $\Theta^{n,k}$ of h-cobordism classes of embedded homotopy n-spheres in $(n + k)$-space was an abelian group under connected sum (by results of Smale, h-cobordism and isotopy are synonymous if $k > 2$ and $n > 4$). He then showed that $\Theta^{4k-1,2k+1} \cong \mathbb{Z}$ by constructing an invariant in the following manner.

If $K^{4k-1} \subseteq S^{6k}$ is a smooth knot, then choose a framed properly embedded submanifold $N \subseteq D^{6k+1}$ bounded by K. A $2k$-cocycle of N is defined by considering the linking number of any $2k$-cycle of N with a translate of N in D^{6k+1}. The square of this cocycle is the desired invariant. It turned out to be the complete obstruction to ambiently "surgering" N to a disk . A similar argument showed that $\Theta^{4k+1,2k+2} \cong \mathbb{Z}_2$.

In 1964 Levine [Lev65a] used the methods of Kervaire-Milnor's ground-breaking work [KM63] on the classification of homotopy-spheres, together with Haefliger's ambient surgery techniques, to produce a *non-stable* version of the Kervaire-Milnor exact sequences for $k > 2$ and $n > 4$:

$$\cdots \to \pi_{n+1}(G_k, SO_k) \longrightarrow P_{n+1} \xrightarrow{d} \Theta^{n,k} \xrightarrow{\tau} \pi_n(G_k, SO_k) \xrightarrow{\sigma} P_n \to \cdots$$

Here P_n is defined to be $\mathbb{Z}$, if $n \equiv 0 \pmod 4$, $\mathbb{Z}_2$ if $n \equiv 2 \pmod 4$, and 0 if n is odd. G_k is the space of maps $S^{k-1} \to S^{k-1}$ of degree 1. The map d is defined as follows. Choose a proper embedding $N^{n+1} \subseteq D^{n+k+1}$, where N is some framed manifold with spherical boundary and signature or Kervaire invariant a given element $a \in P_{n+1}$. Then $d(a)$ is defined to be the knot $\partial N \subseteq S^{n+k}$ and is independent of the choice of N. If $K \subseteq S^{n+k}$, then $\tau([K])$ is defined from the homotopy class of the inclusion $\partial T \subseteq S^{n+k} - K \simeq S^{k-1}$, where T is a tubular neighborhood of K. This sequence essentially reduced the classification of knots in codimension > 2 to the computation of some homotopy groups of spheres and the relevant J-homomorphisms, modulo some important group extension problems including the infamous Kervaire invariant conjecture.

Shortly after this, Haefliger [Hae66a] produced an alternative classification of knots using triad homotopy groups. He considered the group $C^{n,k}$ of h-cobordism classes of *embeddings* of S^n in S^{n+k}. The relation between $C^{n,k}$ and $\Theta^{n,k}$ is embodied in an exact sequence due to Kervaire:

$$\cdots \to \Theta^{n+1} \longrightarrow C^{n,k} \to \Theta^{n,k} \to \Theta^n \xrightarrow{\partial} C^{n-1,k} \to \cdots$$

Here Θ^n denotes the group of h-cobordism classes of homotopy n-spheres, and ∂ is defined by associating to any $\Sigma \in \Theta^n$ its gluing map h, defined by the formula $\Sigma = D^n \cup_h D^n$, and then considering the embedding $S^{n-1} \xrightarrow{h} S^{n-1} \subseteq S^{n+k-1}$. Haefliger showed that $C^{n,k} \cong \pi_{n+1}(G; G_k, SO)$, where $G = \lim_{q\to\infty} G_q$ and $SO = \lim_{q\to\infty} SO_q$.

All of these results are interconnected by a "braided" collection of exact sequences (see [Hae66a]).

In [Hae66b] Haefliger applied these techniques to the classification of *links* in codimension > 2 and the result was another collection of exact sequences which reduced the classification of links to the classification of the knot components and more homotopy theory. For any collection of positive integers $p_1, \cdots p_r, m$, where $m > p_i + 2$, the set of h-cobordism classes of disjoint embeddings $S^{p_1} + \cdots + S^{p_r} \subseteq S^m$ forms an abelian group under component-wise connected sum. It contains, as a summand, the direct sum $\bigoplus_i C^{p_i, m-p_i}$, representing the *split* links. The remaining summand L^m_p was shown by Haefliger to lie in an exact sequence:

$$\cdots \to A^m_{p+1} \to B^m_{p+1} \to L^m_p \to A^m_p \xrightarrow{W} B^m_p \to \cdots \quad (1)$$

where p stands for the sequence $p_1, \cdots p_r$. The terms A^m_p and B^m_p are made up from homotopy groups of spheres and W is defined by Whitehead products.

After the development of the surgery sequence of Browder, Novikov, Sullivan and Wall [Wal70] these earlier knot and link classification results were given a more concise treatment in [Hab86]. In fact the methods of Browder and Novikov had already been extended to give a surgery-theoretic classification of embeddings of a simply-connected manifold in another simply-connected manifold. A general classification of embeddings in the meta-stable range, using the homotopy theory of the Thom space of the normal bundle was given by Levine in [Lev63]. For any closed simply-connected manifold M^n and vector bundle ξ^k over M, with $n < 2k-3$, which is stably isomorphic to the stable normal bundle of M, there is a one-one correspondence (with some possible exceptions related to the Kervaire invariant problem) between the set of h-cobordism classes of embeddings of M into S^{n+k} and normal bundle ξ and the set $h^{-1}(\omega)$, where $\omega \in H_{n+k}(T(\xi)) \cong \mathbb{Z}$ and $h : \pi_{n+k}(T(\xi)) \to H_{n+k}(T(\xi))$ is the Hurewicz homomorphism. Here, $T(\xi)$ is the Thom space of ξ. Browder, in [Bro66], gives a classification of smooth simply connected embeddings in codimension > 2 in terms of a homotopy-theoretic model of the complement. Here the fundamental notion of a Poincaré embedding first appeared, and was later refined by Levitt [Lev68] and Wall [Wal70].

A Poincaré embedding of manifolds X in Y is a spherical fibration ξ over X, a Poincaré pair (C, B), a homotopy equivalence of B with the total space $S(\xi)$ of ξ, and of Y with the union along B of C and the mapping cylinder of the map $S(\xi) \to X$. C is a homotopy theoretic model for the complement of the embedding. A theorem of Browder (extended by Wall to the non-simply connected case) says that if if X is an m manifold and Y is an n manifold, and $n - m \geq 3$ then a (locally flat) topological or PL embedding determines a unique Poincaré embedding and a Poincaré embedding corresponds to a unique locally flat PL or topological embedding (See, for instance, [Wal70].) For smooth embeddings one must first specify a linear reduction for ξ as well. This extended an earlier result of Browder, Casson, Haefliger and Wall that said that any homotopy equivalence $M^n \to V^{n+q}$

of PL-manifolds is homotopic to an embedding if $q \geq 3$. (The more general result has been sometimes referred to as the Browder, Casson, Haefliger, Sullivan, Wall theorem.) A broad extension of this result to stratified spaces can be found in [Wei94].

3. Knot theory in codimension two

3.1. **Unknotting.** One of the earliest applications of surgery to codimension two knot theory was the unknotting theorem of Levine [Lev65b] which states that a smooth or piecewise-linearly embedded homotopy n-sphere $K \subseteq S^{n+2}$, for $n > 2$, is smoothly isotopic to the standard embedding $S^n \subseteq S^{n+2}$ if and only if the complement $S^{n+2} - K$ is homotopy equivalent to the circle. Earlier Stallings had established that topological locally flat codimension 2 knots, of dimension > 2, whose complements have the homotopy type of a circle, are unknotted [Sta63]. His proof used the method of engulfing. Levine's proof of this fact (in dimensions > 4, extended by Wall [Wal65] to $n = 3$) in the smooth or piecewise-linear category proceeded by showing that one could do ambient surgery on a Seifert surface of the knot to convert it to a disk.

These surgery techniques were later used by Levine, in [Lev70], to give a classification of *simple* odd-dimensional knots of dimension > 1— i.e. knots whose complements are homotopy equivalent to that of the trivial knot below the middle dimension— in terms of the *Seifert matrix* of the knot. The Seifert matrix of a knot $K^{2n-1} \subseteq S^{2n+1}$ is a representative matrix of the Seifert pairing which is defined as follows. Choose any $(n-1)$-connected *Seifert surface* for K, i.e. a submanifold $M^{2n} \subseteq S^{2n+1}$ whose boundary is K. The existence of such M is equivalent to K being simple. The Seifert pairing associated to M is a bilinear pairing $\sigma : H_n(M) \otimes H_n(M) \to \mathbb{Z}$. If $\alpha, \beta \in H_n(M)$ choose representative cycles z, w, respectively and define $\sigma(\alpha, \beta) = \ell k(z', w)$, where ℓk denotes linking number and z' is a translate of z off M in the positive normal direction. Different choices of M give different Seifert matrices but any two are related by a sequence of simple moves called *S-equivalence*. The classification of simple knots is then given by the S-equivalence class of its Seifert matrix.

Classification of simple even-dimensional knots was achieved, in special cases, by Kearton [Kea76] and Kojima [Koj79] and, in full generality, by Farber in [Far84a]. The classification scheme here is considerably more complex than in the odd-dimensional case. For a simple knot $K^{2n} \subseteq S^{2n+2}$ let $X = S^{2n+2} - K$ and $\tilde{X}$ denote the infinite cyclic cover of X. Then the invariants which classify, in Farber's formulation, are: the $\mathbb{Z}[t, t^{-1}]$-modules $A = H_n(\tilde{X}), B = \pi^S_{n+2}(\tilde{X})$ (the stable homotopy group), the map $\alpha : A \otimes \mathbb{Z}_2 \to B$, defined by composition with the non-zero element of $\pi_{n+2}(S^n)$, and two pairings $l : T(A) \otimes T(A) \to \mathbb{Q}/\mathbb{Z}$ ($T(A)$ is the $\mathbb{Z}$-torsion submodule of A) and $\psi : B \otimes_Z B \to \mathbb{Z}_4$ which are defined from Poincaré duality.

This result is, in fact, a consequence of a more general result of Farber's [Far84], [Far80] which gives a homotopy-theoretic classification of *stable* knots, i.e. knots $K^n \subseteq S^{n+2}$ whose complements are homotopy equivalent to that of the trivial knot below dimension $(n+3)/3$. The classification is via the stable homotopy type of a Seifert surface M together with a product structure $u\colon M \wedge M \to S^{n+1}$, representing the intersection pairing, and a map $z\colon \Sigma M \to \Sigma M$ (ΣM is the suspension of M) representing the Seifert pairing, i.e. translation into the complement of M in S^{n+2} combined with Alexander duality. In a somewhat different direction, Lashof-Shaneson [LS69], used the surgery theory of Wall [Wal70] to show that the isotopy class of a knot is determined by the homotopy type of its *complementary pair* $(X, \partial X)$, where X is the complement of the knot, as long as $\pi_1(X) = \mathbb{Z}$.

A specific problem which received some attention was the question of how well the complement of a knot determined the knot (we restrict ourselves to knots of dimension > 1).

Gluck [Glu67] showed that there could be at most two knots with the same complement in dimension 2. Later Browder [Bro67] obtained this result in all dimensions ≥ 5. Lashof and Shaneson extended this to the remaining high dimensional cases, $n = 3, 4$ [LS69]. It followed from Farber's classification that stable knots were determined by their complement, but Gordon [Gor76], Cappell-Shaneson [CS76b] and Suciu [Suc92] constructed examples of knots which were not determined by their complements. These examples all had non-abelian fundamental group and it remains a popular open conjecture that, when $\pi_1(\text{complement}) = \mathbb{Z}$, the knot is determined by its complement.

3.2. **Knot invariants.** Surgery methods were also used to describe the various algebraic invariants associated to knots. For example in [Lev66] Levine gave another proof of Seifert's result characterizing which polynomials could be the Alexander polynomial of a knot (also see [Rol75]). This generalized Seifert's result to a wider array of knot polynomials, defined for higher-dimensional knots as the Fitting invariants of the homology $\mathbb{Z}[t, t^{-1}]$-modules of the canonical infinite cyclic covering of the complement of the knot. In [Ker65a] Kervaire gave a complete and simple characterization of which groups π could be the fundamental group of the complement of a knot of dimension > 2. The proof used plumbing constructions to construct the knot complement with the desired group, and then invoked the Poincaré conjecture to recognize that a given manifold was a knot complement. This last idea at least partially foreshadowed the homology surgery techniques of Cappell and Shaneson of the next decade. The conditions Kervaire obtained were:

(i) $H_1(\pi) \cong \mathbb{Z}$
(ii) $H_2(\pi) = 0$
(iii) π is normally generated by a single element

By replacing condition (ii) by the stronger condition:

(ii') π has a presentation with one more generator than relators

he described a large class of groups which are the fundamental group of the complement of some 2-dimensional knot (the process of *spinning* shows that any 2-knot group is a 3-knot group). Using Poincaré duality in the universal cover of the complement, several people found further properties of 2-knot groups which enabled them to produce examples of 3-knot groups which were not 2-knot groups, but the problem of characterizing 2-knot groups is still open (as is, of course, 1-knot groups). See Farber [Far75], Gutierrez [Gut72], Hausmann and Weinberger [HW85], Hillman [Hil80], Levine [Lev77b], and especially, see Hillman's book [Hil89] for an extensive study of this question. An old example of Fox showed that (ii') was not a necessary condition for 2-knot groups. In [Ker65a] Kervaire also gave a complete characterization of the lowest non-trivial homotopy group of the complement of a knot with $\pi_1(\text{complement}) = \mathbb{Z}$, as a $\mathbb{Z}[t, t^{-1}]$-module. In [Lev77a], Levine gives a complete characterization of the $\mathbb{Z}[t, t^{-1}]$-modules which can arise as any given homology module of the infinite cyclic covering of a knot of dimension > 2 (except for the torsion submodule of H_1).

3.3. **Knot concordance.** In codimension two, the relation of h-cobordism (more often called concordance today) is definitely weaker than isotopy and so the group $\Theta^{n,2}$, known as the knot concordance group, measures this weaker relation. Its computation required drastically different techniques.

The application of surgery techniques in this context was begun by Kervaire. In [Ker65b] he showed that all even-dimensional knots were slice. In [Lev69b] Levine gave an algebraic determination of the odd-dimensional knot concordance group in dimensions > 1 in terms of the *algebraic cobordism* classes of Seifert matrices. Two Seifert matrices A, B are cobordant if the block sum $A \oplus (-B)$ is congruent to a matrix of the form $\left(\begin{smallmatrix} 0 & X \\ Y & Z \end{smallmatrix}\right)$, where X, Y, Z and the zero matrix 0 are all square. It was then shown [Lev69a], using results of Milnor [Mil69], that the knot concordance group is a sum of an infinite number of $\mathbb{Z}, \mathbb{Z}/2$ and $\mathbb{Z}/4$ summands. More detailed information on the structure of this group was obtained by Stoltzfus in [Sto77].

In summation, knot concordance is now reasonably well-understood in dimensions > 1; the (smooth and PL) knot concordance group is periodic of period 4 (except it is a subgroup of index 2 for 3-dimensional knots). The topological knot concordance group preserves this periodicity at dimension 3 and is otherwise the same as the smooth and PL groups (see [CS73]).

3.4. **Homology surgery.** In [CS74], Cappell and Shaneson attacked the problem of classifying codimension two embeddings within a fixed h-Poincaré embedding type. (See [CS78] for a precise definition of an h-Poincaré embedding.) Here, the key idea was to interpret codimension two embedding problems as problems in the classification of spaces up to homology type. The motivating example should illustrate this well.

By the high-dimensional Poincaré conjecture, a manifold with boundary $S^n \times S^1$ is the complement of a knot if and only if it is a homology circle and the fundamental group is normally generated by a single element (the meridian.) Thus, the classification of knot complements is the classification of homology circles, a calculation carried out in [CS74]. (In contrast, Levine's unknotting theorem tells us that only the trivial knot has the *homotopy* type of S^1.) Similarly, a homology cobordism between knot complements (again with extra π_1 condition) extends, by the h-cobordism theorem, to a concordance of knots. Hence the classification of knot concordance reduces to computing the structure group of homology $S^1 \times D^{n+1\prime}s$. A general discussion tying together the various surgery theoretic tools for codimension two placement, known as of 1981, can be found in [Ran81].

Cappell and Shaneson's applications of these techniques were quite rich. For example they showed that concordance classes of embeddings of a simply-connected manifold in a codimension two tubular neighborhood of itself were in one-one correspondence with the knot concordance group of the same dimension and this bijection was produced by adding local knots to the 0-section embedding. (See Matsumoto [Mat73] for related results.) This allowed for a geometric interpretation of the periodicity of knot concordance from a more natural surgery theoretic point of view [CS74], than those given via tensoring knots [KN77], or groups actions [Bre73]. In turn, as an example of how knot theory fertilizes the more general subject of manifold theory, knot theoretic ideas (in particular, branched fibrations) provided a geometric description of Siebenmann periodicity [CW87].

Cappell and Shaneson applied their homology surgery techniques to the study of singularities of codimension two PL-embeddings (i.e. non-locally flat embeddings) in [CS76a] and gave definitive results on the existence of such embeddings as well as an obstruction theory for removing the singularities. A codimension two PL locally flat embedding has a trivial tubular neighborhood. Thus one might hope to study non-locally flat embeddings with isolated singularities via the knot types of the links of the singularities. Indeed, they showed that the classifying space of oriented codimension two thickenings has the knot concordance groups as its homotopy groups. More recently, Cappell and Shaneson studied non-isolated singularities by observing that, with appropriate perversity, the link of a singularity looks like a knot to intersection homology [CS91].

Cappell and Shaneson prove that for closed oriented odd dimensional PL manifolds M^n and W^{n+2}, with $n \geq 3$, a map $f : M \to W$ is homotopic to a (in general, non-locally flat) PL embedding if and only if f is the underlying map of an h-Poincaré embedding. This is often false in even dimensions, but still holds if W is simply connected. In fact, they show the existence of even dimensional spineless manifolds, i.e., manifolds W^{n+2} with the homotopy type of an n manifold and such that W contains no codimension two embedded submanifold within its homotopy type. See [CS78] for an extensive discussion of these and other results and the techniques used to derive them.

3.5. **Four-dimensional surgery and classical knot concordance.** For the case of classical one-dimensional knots it was clear that the classification scheme of Kervaire and Levine must fail but it took some time before it was actually proved by Casson and Gordon [CG76], in a paper that is among the deepest in the literature of knot theory. All the higher-dimensional knot concordance invariants are invariants of knotted circles as well, and knots for which these invariants vanish are often called *algebraically slice*. Casson and Gordon defined secondary slicing obstructions using signatures associated to metabelian coverings of the knot complement, and gave explicit examples of very simple one-dimensional knots that were *algebraically slice* but not (even topologically) slice. These remain among the most obscure invariants in geometric topology, and very little progress has been made in understanding them. Several papers of interest include Gilmer [Gil83], and Letsche [Let95] for traditional Casson Gordon invariants, and results of Cappell and Ruberman [CR88], Gilmer-Livingston [GL92], Ruberman [Rub83], and Smolinsky [Smo86] that investigate the use of Casson Gordon invariants to study *doubly slice* knots in the classical and higher dimensional context.

The knot slice problem seeks to classify the structure set of homology circles, and it seems natural to suppose that the Casson-Gordon invariants manifest the existence of secondary four-dimensional homology surgery invariants. Freedman's work suggests that any secondary obstructions to topological surgery obstruct building Casson handles. It is a central question to relate these ideas, and see what role Casson-Gordon invariants play in the general problem of computing homology structure groups in dimension four, and in creating Casson handles in general.

In a remarkable application of Freedman's topological surgery machine, Freedman has shown that a classical knot is slice with a slicing complement with fundamental group $\mathbb{Z}$ (called $\mathbb{Z}$-slice) if and only if the Alexander module of the knot vanishes [Fre82]. (Donaldson's work implies not all of these knots are smoothly slice, giving counterexamples to the topological ribbon slice problem! Freedman's work predicts that an analogous class of links, called *good boundary links*, are slice. However, such links have free (or nearly so) fundamental groups, and it is still an open question whether topological surgery works for such groups. In fact, Casson and Freedman showed that good boundary links are slice if and only if every four-dimensional normal map with vanishing surgery obstruction is normally cobordant to a homotopy equivalence [CF84]. The Whitehead double of any link, with pairwise vanishing linking numbers zero, is a good boundary link, and the slicing problem for the Whitehead double of the Borromean rings may be the archetypal example on which this problem's solution rests. A discussion of these and other connections between the four-dimensional topological surgery conjecture and the link slice problem can be found in [FQ90].

Among the most important open problems that surgery theory gives hope of answering is the ribbon-slice problem. It is conjectured that a knot is ribbon if and only if it is smoothly slice. In the topological category, one seeks to determine if a knot is slice if and only if it is homotopy ribbon. A knot is *homotopy ribbon* if it is slice by a locally flat, topologically embedded two disk where the inclusion of the complement of the knot to the complement of the slicing disk induces an epimorphism on fundamental group. The Casson-Gordon invariants give potential obstructions to this, as they may detect the failure of this map to induce an epimorphism of fundamental groups. A more complete theory of topological homology surgery in dimension 4 would give deeper invariants, and possibly realization techniques for solving the homotopy ribbon-slice problem. For instance, reducing the classification of classical knot concordance to the four dimensional topological surgery conjecture might reduce this problem to a surgery group computation.

4. Link concordance

4.1. **Boundary links.** Following success in the classification of knot concordance in high dimensions, attention focused on classifying links up to concordance. The knot concordance classification theorems made explicit use of the Seifert surface for the knot. The existence of this Seifert surface meant that S^1 split from the knot complement. A link for which the components bound pairwise disjoint Seifert surfaces is called a *boundary link*. A boundary link complement splits a wedge of circles. It is natural to suspect that concordance of boundary links might be computable using similar techniques to those used to classify knots. In fact, the trivial link gives a nice Poincaré embedding and the classification of concordance of boundary links is, roughly, the classification of homology structures on the trivial link complement.

The arguments of Kervaire used to slice even-dimensional knots were easily seen to slice even-dimensional boundary links as well. Cappell-Shaneson applied their homology surgery machinery to calculate the boundary concordance group of boundary links of dimension > 1 [CS80], where *boundary link concordance* is the natural notion of concordance for boundary links. More precisely, the components of the concordance, together with the Seifert surface systems for the links, are assumed to bound pairwise disjoint, oriented, and embedded manifolds. They prove that in all odd dimensions there exist infinitely many distinct concordance classes of boundary links none of which contain split links. Their argument is somewhat delicate. The homology surgery group which computes boundary concordance of boundary links detects links not *boundary* concordant to a split link. Further arguments were needed to show that these same links were not concordant to split links.

Later, Ko [Ko87] and Mio [Mio87] used Seifert matrices to give an alternative classification of boundary link concordance of boundary links and Duval [DuV86] obtained the classification using Blanchfield pairings. The complete computation of these surgery groups has not been attempted to our knowledge, and remains an interesting open problem. An isotopy classification of simple odd-dimensional (boundary) links was carried out by Liang [Lia77] in terms of Seifert matrices and by Farber [Far91] in terms of the Blanchfield pairing. Farber's result was a special case of a more general classification, carried out in [Far92], of *stable* boundary links using stable homotopy theory and Wall's theory of thickenings.

4.2. **Non-boundary links.** We have seen that the classification of boundary links, up to concordance, followed similar lines to the classification of knot concordance. But the concordance classification of non-boundary links has proven more difficult, requiring new ideas and techniques. With the work of Cappell and Shaneson, attention naturally focused on these two questions:

(1) Are all links concordant to boundary links?
(2) Is boundary link concordance the same as link concordance?

The first question has only recently been answered and the second remains open.

Perhaps the first suggestion of how to proceed appeared in a small concluding section of [CS80], where the authors anticipate and motivate many of the techniques which continue to dominate research on link concordance. The authors suggested that one may study general link concordance (as opposed to boundary link concordance) by considering limiting constructions which serve as a way of measuring the failure of a given link to be a boundary link. We elaborate further.

Boundary links are accessible to surgery techniques because there is a terminal boundary link complement (the trivial link) to which all boundary link complements map by a degree one map. This gives a manifold to which all boundary link complements can be compared. Similarly, a slice complement for the trivial link is terminal among all boundary link slice complements. Since the fundamental group can change dramatically under a homology equivalence (and under a concordance), no simple terminal object exists for general links. Cappell and Shaneson suggested that a limit of link groups might be used to construct such a terminal object for links. This suggestion launched a flurry of research activity.

The missing idea, needed to make Cappell and Shaneson's suggestion work, was discovered by Vogel in an unpublished manuscript, and implemented in a paper of Le Dimet [Dim88]. Vogel suggested that instead of taking the limit through link groups, one should take the limit through spaces of the homology type one seeks to classify, thus constructing a terminal object within a homology class. Homotopy theory had long studied similar limiting constructions, i.e., Bousfield's homology localization of a space [Bou75]. Bousfield's space was far too big for the study of compact manifolds. The Vogel localization was a limit through maps of finite CW complexes with contractible cofiber.

In [Dim88], Le Dimet uses Vogel's idea to classify concordance classes of disk links. A *disk link* is a collection of codimension two disks disjointly embedded in a disk so that the embedding is standard on the boundary [Dim88]. (We believe disk links first appeared in this paper. Links in the 3-disk were later referred to as *string links*.) The inclusion of the meridians (a wedge of circles) of a disk link into the complement is a map of finite complexes with contractible cofiber, and thus becomes a homotopy equivalence after localization. Restricting a homotopy inverse to the boundary of the disk link complement gives a map whose homotopy class is a concordance invariant of the disk link. We will refer to this as *Le Dimet's homotopy invariant.* Le Dimet proved that m component, n-dimensional disks links, modulo disk link concordance, form a group $C_{n,m}$ and that, for $n \geq 2$, this group and his homotopy invariant fit into an exact sequence involving Cappell-Shaneson homology surgery groups. In particular, Le Dimet gives a long exact sequence as follows:

$$\cdots \to C_{n+1,m} \to H_{n+1} \to \Gamma_{n+3} \to C_{n,m} \to H_n \to \cdots$$

Here, $H_n = [\#S^n \times S^1, E(\vee S^1)]$ (homotopy classes of maps) is the home of Le Dimet's homotopy invariant. $E(\vee S^1)$ is the Vogel localization of a wedge of circles, and Γ_{n+3} is a relative homology surgery group involving $\pi_1(E(\vee S^1))$ which acts on $C_{n,m}$.

Concordance classes of links are a quotient set of Le Dimet's disk link concordance group, and so the computation of this group is of fundamental importance. Unfortunately, the Vogel localization is difficult to compute and almost nothing is known about it. Understanding this space and its fundamental group remains among the most central open problems in the study of high-dimensional link concordance. For example, if this space is a $K(\pi, 1)$, then even-dimensional links are always slice.

In [CO90, CO93], Cochran and Orr gave the first examples of higher odd-dimensional links not concordant to boundary links. Although this work was partly motivated by Le Dimet's work (and other sources as well) the paper gave obstructions in terms of localized Blanchfield pairings of knots lying in branched covers of the given link. They obtained examples of 2-torsion and 4-torsion, Brunnian examples (remove one component and the link becomes trivial) as well as other interesting phenomena. They gave similar examples for links in S^3. (Here the interesting problem was to find links with vanishing Milnor $\bar{\mu}$-invariants that are not concordant to boundary links.) All these examples are odd-dimensional and realize non-trivial surgery group obstructions from Le Dimet's sequence. After their work several alternative approaches have provided more examples. (See Gilmer-Livingston [GL92] and Levine [Lev94]. The latter paper investigates the invariance of signatures and the Atiyah-Patodi-Singer invariant under homology cobordism.)

In [Coc87], Cochran began an investigation of homology boundary links and concordance. Homology boundary links, like boundary links, have a rank preserving homomorphism from their link groups to a free group. But unlike boundary links, this homomorphism is not required to take meridians to a generating set. (Using the Pontryagin-Thom construction one can obtain what is called a *singular* Seifert surface system for this class of links (see Smythe [Smy66]). All known examples of higher-dimensional links not concordant to boundary links are sublinks of homology boundary links.

Realizing Le Dimet's homotopy obstruction is a difficult problem about which almost nothing is known. For even-dimensional links it is the sole obstruction to slicing. For links in S^3, Levine showed that it (or equivalently, an invariant he defined independently – see below) vanishes if and only if the link is concordant to a sublink of a homology boundary link [Lev89]. Shortly afterwards, Levine, Mio and Orr proved the same result for links of higher odd dimension [LMO93]. An easy calculation shows Le Dimet's invariant vanishes on even dimensional sublinks of homology boundary links as well, implying these links are always slice! Thus, homology boundary links provide a geometric interpretation for the vanishing of Le Dimet's homotopy invariant.

In [CO94], Cochran and Orr classified homology boundary links. Of particular interest here was a new construction for creating a homology boundary link from a boundary link and a ribbon link with a fixed normal generating set, creating a link with prescribed properties and realizing given surgery invariants. It seems likely that this construction can be generalized, potentially providing examples for a wide class of related problems in knot and link theory.

The work of Vogel and Le Dimet was not unprecedented. In [Coc84], Cochran employed Cappell and Shaneson's suggestion of taking a limit through link groups to classify links of two spheres in S^4. He used the observation that link groups had the homology type of link complements through half the dimensions of the link complement for links in S^4. In fact, Le Dimet's homotopy invariant followed a flurry of mathematical activity in the study of homotopy theoretic invariants of link concordance.

Prior to 1980 Milnor's $\bar{\mu}$-invariants for classical links (equivalently, Massey products) were the only known homotopy theoretic obstructions to slicing a link. They remain among the deepest and most important invariants of knot theory and play an important role in the study of topological surgery in dimension four [FQ90]. In [Sat84], Sato (and Levine, independently) introduced a concordance invariant for higher-dimensional links that generalized the $\bar{\mu}$-invariant, $\bar{\mu}_{1212}$, which detects the Whitehead link in dimension one. These invariants were greatly extended using geometric techniques by Cochran [Coc85, Coc90], and homotopy-theoretic techniques by Orr [Orr87, Orr89]. But the only invariant among these that was not later shown to vanish, or to be roughly equivalent to Milnor's invariants was a single invariant from [Orr89]. This invariant remains obscure and unrealized.

One outgrowth of this study was the formulation of a group theoretic construction called *algebraic closure* by Levine in [Lev89], a smaller version of the nilpotent completion. This work provides a combinatorial description of the fundamental group of a Vogel local space, and has proven useful both in defining new invariants, and as a tool for computing local groups. There are two variations of this construction. For a group π, $\bar{\pi}$ lives in the nilpotent completion of π while the possibly larger group $\hat{\pi}$ is defined by a universal property. $\hat{\pi}$ is the fundamental group of the Vogel localization of any space with fundamental group π. Levine used this latter algebraic closure construction to define a new invariant for certain classical links. First of all, the $\bar{\mu}$-invariants of Milnor can be viewed as living in $\bar{F}$. (This observation allows one to prove a realizability theorem for the $\bar{\mu}$ invariants; one of the conditions for realizability is the vanishing of a class in $H_2(\bar{F})$). One can define a slight generalization of the $\bar{\mu}$ invariants which are just liftings into $\hat{F}$. Then, for (classical) links on which these invariants vanish, a new, possibly non-trivial, concordance invariant lives in $H_3(\hat{F})$. This latter invariant vanishes if and only if Le Dimet's invariant vanishes. It was then proved in [Lev89] that this invariant vanishes if and only if the link is concordant to a sublink of a homology-boundary link, and that every element of $H_3(\hat{F})$ is realized by some link. (Unfortunately we do not know if this homology group is non-zero.) This result suggested the higher-dimensional analogue in [LMO93].

4.3. **Poincaré embeddings again.** In summary, it is still unknown whether all even-dimensional links are slice and whether every higher-dimensional link is concordant to a sublink of a homology boundary link. Both of these problems would be solved by computing Le Dimet's homotopy invariant. But, more generally, we should ask what is the larger role of Vogel local spaces in surgery and embedding theory?

Implicit in Le Dimet's work is the notion that, for codimension two placement and the classification theory of manifolds within a homology type, one should consider a weakened version of Poincaré embedding, where spaces are replaced with their Vogel local counterparts. For the study of high-codimension embeddings all spaces considered are usually simply connected, and therefore already local. For this reason, earlier results did not need this operation of localization. This helps account for both the early progress in high codimension, and the long delay in dealing effectively with the codimension-two case. It is a fundamental problem to develop this theory to its conclusion, and to consider the more general theory for stratified spaces (see [Wei94].) Examples where this is used (at least implicitly) to study general embedding theory can be found in the classification results of Mio, for links with one codimension component [Mio92], and the torus knotting results of Miller [Mil94].

Another problem is to derive the surgery exact sequence for this type of classification. Normal maps with coefficients were classified in [Qui75] and [TW79] for subrings of $\mathbb{Q}$, but a general theory for the localization of an arbitrary ring does not exist at this time.

4.4. **Open problems.** The following list of problems is by no means exhaustive, representing a small subset of difficult problems we think are particularly interesting. They are either problems in surgery theory motivated by knot theory and from whose solution knot theory should benefit, or problems in knot theory that should be approachable through surgery theory.

(1) Are knots determined by their complement when π_1 is $\mathbb{Z}$?
(2) Give an algebraic characterization of 2-knot groups.
(3) Find tools for computing the algebraic closure of a group.
(4) For the free group F, compute the homology of $\bar{F}$ and the algebraic closure of the free group $\hat{F} \cong \pi_1(E(\vee S^1))$. Is $\hat{F} \cong \bar{F}$?
(5) Are all even-dimensional links slice?
(6) Is every odd dimensional link (with vanishing Milnor's $\bar{\mu}$-invariants if $n = 1$) concordant to a sublink of a homology boundary link? Are all higher dimensional links sublinks of homology boundary links?
(7) Is every sublink of a homology boundary link concordant to a homology boundary link?

(8) Compute the homotopy type of the Vogel localization of a wedge of circles, and Le Dimet's homotopy invariant. More generally, find tools for computing the homotopy type of Vogel local spaces.
(9) Find more invariants and, ultimately, compute the homology surgery group classifying boundary link concordance. In particular, if two boundary links are concordant, are they also boundary concordant?
(10) Find more invariants and, ultimately, compute the homology surgery group in LeDimet's exact sequence which classifies sublinks of homology boundary links up to concordance.
(11) Derive a surgery exact sequence for homology structures with coefficients. In particular, classify normal maps.
(12) Find a complete set of invariants for classical knot concordance.
(13) If a classical knot is topologically slice in a homology three disk, is it topologically slice? This problem measures the possible difference between computing concordance of classical knots, and the solution of a homology surgery problem.
(14) Develop a theory of homology surgery in dimension four (at least modulo Freedman's four dimensional topological surgery problem.)
(15) Solve the homotopy ribbon-topological slice problem.
(16) Relate Casson-Gordon invariants to a topological four-dimensional homology surgery machine, and in particular, find the relation to Casson handles.

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