

# Some Properties of the singular words of the Fibonacci word

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## Abstract

The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, formal language, computational complexity, quasicrystals etc.

In this note, we introduce the singular words of the Fibonacci infinite word and discuss their properties. We establish two decompositions of the Fibonacci word in singular words and their consequences. By using these results, we discuss the local isomorphism of the Fibonacci word and the overlap properties of the factors. Moreover, we also give new proofs for the results on special words and the power of the factors.

The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, formal language, computational complexity, quasicrystals etc. See [1, 3, 8, 9, 11]. Moreover, the properties of the subwords of the Fibonacci infinite word have been studied extensively by many authors [2, 4, 5, 6, 9, 10]. In this note, we shall present some new properties of the subwords of the Fibonacci word: as we shall see, the most striking of these properties is that the adjacent singular words of the same order are positively separated.

This note is organized as follows. After recalling some preliminary remarks on the Fibonacci word, we introduce the singular words and discuss their elementary properties. Then we establish two decompositions of the Fibonacci word in singular words (Theorems 1 and 2) and their consequences. By using these results, we discuss the local isomorphism of the Fibonacci word (Theorem 4) and the overlap properties of the factors (Theorem 6). Moreover, we also give new proofs for the results on special words (Theorem 5) and the power of the factors (Theorem 3).

In this note, we use the following definitions and terminology.

Let  $A = \{a, b\}$  be an alphabet of two letters, let  $A^*$  be the free monoid on  $A$ , and let  $F$  be the free group generated by  $A$ . The elements of  $A^*$  are called words. The neutral element of  $A^*$  is called the empty word which we denote by  $\varepsilon$ . Let  $w$  be a word: we denote by  $|w|$  the length of  $w$ , we denote by  $|w|_a$

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(resp.  $|w|_b$ ) the number of letters of  $a$  (resp.  $b$ ) appearing in  $w$ , we denote by  $L(w)$  the vector  $(|w|_a, |w|_b)$ .

An infinite word on  $A$  is a mapping  $\mathbf{x} : \mathbb{N} \rightarrow A$ , and we write  $\mathbf{x} = x_1x_2 \cdots x_n \cdots$ , where  $x_i \in A$ . The set of infinite words is denoted by  $A^\omega$ .

A word  $v$  is a factor of word  $w$  and we write  $v \prec w$ , if there exist  $u, u' \in A^*$ , such that  $w = uvu'$ . We say that  $v$  is a left (resp. right) factor of a word  $w$  and we note that  $v \triangleleft w$  (resp.  $v \triangleright w$ ), if there exists  $u \in A^*$  such that  $w = vu$  (resp.  $w = uv$ ). The notions of left factors and right factors are extended in a natural way to  $A^\omega$ .

Let  $w = x_1x_2 \cdots x_n$ . We denote by  $\bar{w}$  the mirror image of  $w$ ; that is,  $\bar{w} = x_n \cdots x_2x_1$ . If  $w = \bar{w}$ , the word will be called a palindrome, and the set of the palindromes is denoted by  $\mathcal{P}$ . A word  $w \in A^*$  is called primitive if  $u = v^p$ ,  $v \in A^*$ ,  $p > 0$ , implies  $u = v$ .

Let  $w = x_1x_2 \cdots x_n \in A^*$ , and let  $1 \leq k \leq n$ . We define  $C_k(w) = x_{k+1} \cdots x_n x_1 \cdots x_k$ , the  $k$ -th conjugation of the word  $w$ , and we note that  $C(w) = \{C_k(w); 1 \leq k \leq |w|\}$ . By convention,  $C_{-k}(w) = C_{|w|-k}(w)$ .

Now let  $\sigma : A \rightarrow A^*$  be a morphism defined by  $\sigma(a) = ab$ ,  $\sigma(b) = a$ . We define the  $n$ -th iteration of  $\sigma$  by  $\sigma^n(a) = \sigma(\sigma^{n-1}(a))$ ,  $n \geq 2$  and we denote  $F_n = \sigma^n(a)$  (by convention, we define  $\sigma^0(a) = a$ ,  $\sigma^0(b) = b$ ). Then the Fibonacci word  $F_\infty$  is obtained by iterating  $\sigma$  with the letter  $a$  (see [2]).

Let  $w$  be a word. We denote by  $\Omega_n(w)$  the set of factors of  $w$  of length  $n$ , where  $|w| \geq n$ , and we simply note that  $\Omega_n := \Omega_n(F_\infty)$ .

By a natural embedding, we can regard the  $A^*$  as a subset of  $F$ , and if we say that  $w$  is in  $F$ , this means that  $w$  is reduced (see [7] and therein). Let  $w = x_1x_2 \cdots x_n \in A^*$ . We denote by  $w^{-1}$  the inverse word of  $w$ , that is  $w^{-1} = x_n^{-1} \cdots x_2^{-1}x_1^{-1}$ . Let  $w = uv$ ,  $u, v \in A^*$ , then  $wv^{-1} := u$  and  $u^{-1}w := v$  by convention.

One of the motivations of this note is as follows: we know that the Fibonacci word is related closely to the Fibonacci numbers (the Fibonacci number is defined by the recurrence formula  $f_{n+2} = f_{n+1} + f_n$  with the initial condition  $f_{-1} = f_0 = 1$ ). Consider the following decomposition of the Fibonacci word

$$\underline{a} \underline{b} \underline{aa} \underline{bab} \underline{aabaa} \underline{babaabab} \underline{aabaababaabaa} \underline{babaababaabaababab} \cdots$$

that is, the length of the  $n$ -th block in the decomposition is  $f_n$ ,  $n \geq -1$ . Then a question is posed naturally: What are these blocks? As we shall see, Theorem 1 will answer this question completely.

In this note, we shall use the following known facts which can be found in [2, 4, 8, 9].

**Property 1** 1)  $|\sigma^n(a)| = f_n$  and  $|C_n(F_n)| = f_n$ , where  $f_n$  is the  $n$ -th Fibonacci number. That is, all conjugations of  $F_n$  are different each other. In particular, for any  $w \in C(F_n)$ , we have

$$L(w) = L(F_n) = (f_{n-1}, f_{n-2}),$$

and moreover,

$$C(F_n) = \{\bar{w}; w \in C(F_n)\}.$$

2)  $F_{n+1} = F_n F_{n-1}$ .

3) For any  $k \geq 1$ ,  $\sigma^k(F_\infty) = F_\infty$ , that is

$$F_\infty = F_k F_{k-1} F_k F_k F_{k-1} \cdots.$$

4)  $ab$  is a suffix of  $F_n$  for odd positive  $n$ , and  $ba$  is a suffix of  $F_n$  for even positive  $n$ .

5)  $b^2 \not\prec F_\infty$ ,  $a^3 \not\prec F_\infty$ .

6) Any factor of  $F_\infty$  will appear infinitely many times in  $F_\infty$ .

7)  $w \prec F_\infty$  if and only if  $\bar{w} \prec F_\infty$ .

**Remark 1** In this note, we shall only use Property 1 and not the other known results of the Fibonacci word. In particular, we shall again prove Theorems 3 and 5 by using singular words.

Let  $\alpha$  and  $\beta$  be two words. Note that, by Property 1.4), if  $\alpha\beta \triangleright F_n$ , then  $\alpha \neq \beta$ .

**Lemma 1** Let  $n \geq 2$ , and assume that  $\alpha\beta$  is a suffix of  $F_n$ . Then

$$\begin{aligned} F_n &= F_{n-2} F_{n-1} \alpha^{-1} \beta^{-1} \alpha \beta, \\ F_{n-2} F_{n-1} &= F_n \beta^{-1} \alpha^{-1} \beta \alpha. \end{aligned}$$

**Proof.** Note that  $\alpha\beta \triangleright F_n$ , so  $\beta\alpha \triangleright F_{n-1}$  by property 1.4). It is easy to check the case of  $n = 2$  directly. Suppose that the lemma is true for  $n$ . Then, by the hypothesis of the induction, we obtain

$$\begin{aligned} F_{n+1} &= F_n F_{n-1} = F_{n-1} F_{n-2} F_{n-1} = F_{n-1} F_n \alpha^{-1} \beta^{-1} \alpha \beta, \\ F_{n-1} F_n &= F_{n-1} F_{n-2} F_{n-1} \beta^{-1} \alpha^{-1} \beta \alpha = F_{n+1} \beta^{-1} \alpha^{-1} \beta \alpha. \end{aligned}$$

Now let  $|w| = f_n$ . Then by Property 1.3),  $w$  will be a factor of the following words:  $F_n F_n$ ,  $F_n F_{n-1} F_n$ ,  $F_n F_{n-1}$  and  $F_{n-1} F_n$ . If  $w = u F_{n-1} v$  with  $u \triangleright F_n$ ,  $v \prec F_n$  and  $|v| \leq f_{n-2}$ , then  $w \prec F_n F_{n-1} F_{n-2} = F_n F_n$ . On the other hand, evidently,  $F_n F_{n-1} \prec F_n F_n$ , and thus the four cases above will be reduced to the cases  $F_n F_n$  and  $F_{n-1} F_n$ .

On the other hand, by Property 1.1),  $\Omega_{f_n}(F_n F_n) = C(F_n)$ . Therefore it is sufficient to determine the factors of  $F_{n-1} F_n$ .

Let  $\alpha$  and  $\beta$  be two words such that  $\alpha\beta$  is a suffix of  $F_n$ . We denote the word  $\alpha F_n \beta^{-1}$  by  $w_n$ .

**Lemma 2** We have the following:

1)  $w_n \notin C(F_n)$ ;

2)  $\Omega_{f_n}(F_{n-1} F_n) = w_n \cup \{C_k(F_n); 0 \leq k \leq f_{n-2} - 2\}$ . In particular, as a factor,  $w_n$  appears only once in  $F_{n-1} F_n$ .

**Proof.** 1) Since  $\alpha \neq \beta$ ,  $L(w_n) \neq L(F_n)$ , which yields 1).

2) By Lemma 1: if  $\alpha\beta \triangleright F_n$ , then we have  $f_{n-1}F_n = F_nF_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta$ . Since  $F_{n-1} \triangleleft F_n$ , the first  $f_{n-1}$  factors of length  $f_n$  of the word  $F_{n-1}F_n$  are exactly  $C_k(F_n)$ ,  $1 \leq k \leq f_{n-1}-2$ , and the last factor is  $F_n = C_{f_n}(F_n)$ , the  $(f_{n-1}-1)$ -th factor is  $\alpha F_n \beta^{-1} = w_n$ .

As we have seen, for any  $n \geq 1$ , the set  $\Omega_{f_n}$  consists of two parts: the first part consists exactly of all conjugations of  $F_n$ , the other is  $w_n$ . As we shall see,  $w_n$  possesses some interesting properties, which play an important role in the studies of the factors of  $F_\infty$ .

The word  $w_n$  is called the  $n$ -th singular word of the Fibonacci word  $F_\infty$ . For convenience, we define  $w_{-2} = \varepsilon$ ,  $w_{-1} = a$  and  $w_0 = b$ , and we denote by  $\mathcal{S}$  the set of singular words of  $F_\infty$ .

Now we discuss the properties of the singular words:

**Property 2** *We have the following:*

1) *If  $n \geq 1$ , then*

$$L(w_n) = \begin{cases} (f_{n-1} + 1, f_{n-2} - 1) & \text{if } n \text{ is odd;} \\ (f_{n-1} - 1, f_{n-2} + 1) & \text{if } n \text{ is even.} \end{cases}$$

2)  $w_n \not\prec w_{n+1}$ .

3) *If  $\alpha \triangleright w_{n+1}$ , then  $w_{n+2} = w_n w_{n+1} \alpha^{-1} \beta$ .*

4)  $w_n = w_{n-2} w_{n-3} w_{n-2}$ ,  $n \geq 1$ .

5) *For  $n \geq 1$ ,*

$$\begin{aligned} C_{f_{n-1}-1}(F_n) &= w_{n-2} w_{n-1}, \\ C_{f_n-1}(F_n) &= w_{n-1} w_{n-2}; \end{aligned}$$

*In particular,*

$$\begin{aligned} w_{n-2} \prec C_k(F_n) & \text{ if and only if } 0 \leq k \leq f_{n-1} - 1; \\ w_{n-1} \prec C_k(F_n) & \text{ if and only if } f_{n-1} - 1 \leq k \leq f_n - 1. \end{aligned}$$

6) *For  $n \geq 1$ ,  $w_{2n-1} = aauaa$ ,  $w_{2n} = bvb$ , where  $u, v \in A^*$ .*

7) *For  $n \geq 2$ ,  $1 < k < f_n$ , no proper conjugate of  $w_n$  is a subword of  $F_\infty$ .*

8) *For  $n \geq 0$ ,  $w_n^2 \not\prec F_\infty$ .*

9) *For  $n \geq -1$ ,  $w_n$  is a palindrome.*

10)  $w_n$  *is not the product of two palindromes for  $n \geq 2$ .*

11) *If  $n \geq 2$ , then  $w_n$  is primitive.*

12) *For  $n \geq 1$ , we have*

$$w_n = w_n^* \left( \prod_{j=-1}^{n-2} w_j \right) = \left( \prod_{j=-1}^{n-2} w_{n-j-3} \right) w_n^*, \text{ where } w_n^* = \begin{cases} a & \text{if } n \text{ is odd,} \\ b & \text{if } n \text{ is even;} \end{cases}$$

13)  $w_n \not\prec \prod_{j=-1}^{n-1} w_j$ ;

14) *Let  $k \geq -1$  and  $p \geq 1$ , and let  $u = \prod_{j=k}^{k+p} w_j$ . Then  $u \notin \mathcal{S}$ .*

**Proof.** 1) If  $n$  is odd, then  $a \triangleright F_{n-1}$ ,  $b \triangleright F_n$ . Thus

$$L(w_n) = L(aF_nb^{-1}) = (f_{n-1} + 1, f_{n-2} - 1),$$

and the case of  $n$  being even can be proved in the same way.

2) Let  $\alpha \triangleright F_n$ : then  $\beta \triangleright F_{n+1}$ . By the definition of  $w_n$ , it is easy to see that  $w_n \not\prec F_{n+1}$ , so  $w_n \not\prec F_{n+1}\beta^{-1}$ . On the other hand  $w_n = \beta F_n \alpha^{-1} \neq \alpha F_n \beta^{-1}$ . Since  $w_{n+1} = \alpha F_{n+1} \alpha^{-1}$ , thus  $w_n \not\prec w_{n+1}$ .

3) By definition,  $w_{n+2} = \beta F_n \alpha^{-1}$ . Then by Lemma 1, we have

$$w_{n+2} = \beta F_n F_{n+1} \beta^{-1} \alpha^{-1} \beta = w_n w_{n+1} \alpha^{-1} \beta.$$

4) Let  $\alpha \triangleright F_n$ : then  $\alpha \triangleright F_{n-2}$  and  $\beta \triangleright F_{n+1}$ ,  $\beta \triangleright F_{n-1}$ . Thus

$$\begin{aligned} w_{n+1} &= \alpha F_{n+1} \beta^{-1} = \alpha F_{n-1} F_{n-2} F_{n-1} \beta^{-1} \\ &= (\alpha F_{n-1} \beta^{-1})(\beta F_{n-2} \alpha^{-1})(\alpha F_{n-1} \beta^{-1}) \\ &= w_{n-1} w_{n-2} w_{n-1}. \end{aligned}$$

5) Let  $\alpha \triangleright F_n$ : then

$$F_n = F_{n-1} F_{n-2} = (F_{n-1} \alpha^{-1})(\alpha F_{n-2} \beta^{-1}) \beta,$$

and so the results follow from the definitions of singular word and conjugation of word.

6) This follows immediately from the definition of  $w_n$  and 9).

7) and 8) follow from Property 1.5) and Property 2.9).

9) We prove by induction. It is checked directly that the conclusion is true for  $n \leq 2$ . Now suppose that the conclusion is true for  $k \leq n$ . Then, by 5),

$$\overline{w_{n+1}} = \overline{w_{n-1} w_{n-2} w_{n-1}} = \overline{w_{n-1}} \overline{w_{n-2}} \overline{w_{n-1}} = w_{n-1} w_{n-2} w_{n-1} = w_{n+1},$$

that is,  $w_{n+1} \in \mathcal{P}$ .

10) Let  $w_n = uv$ , where  $u, v \in \mathcal{P}$ . Since  $w_n$  is a palindrome, so

$$w_n = \overline{w_n} = \overline{uv} = uv = vu.$$

Therefore, the  $|u|$ -th conjugation of  $w_n$  will be a factor of  $F_\infty$ . Then, by 6), if  $n \geq 2$ , we have  $a^4 \prec F_\infty$ , or  $b^2 \prec F_\infty$ , which will contradict Property 1.5).

11) Let  $w_n = u^p$ , with  $u \in A^*$ , and  $p \geq 2$ . Since  $w_n \in \mathcal{P}$ , so do  $u$  and  $u^{p-1}$ , and hence  $w_n = u^p = uu^{p-1}$  will be a product of two palindromes; but, by 10), that is impossible.

12) It is easy to verify that  $F_n = abF_0F_1 \cdots F_{n-3}F_{n-2}$ . If  $n$  is odd, then  $b \triangleright F_n$ . Therefore

$$\begin{aligned} w_n &= aF_nb^{-1} = aab(aF_1b^{-1})(bF_2a^{-1}) \cdots (bF_{n-3}a^{-1})(aF_{n-2}b^{-1}) \\ &= aw_{-1}w_0w_1 \cdots w_{n-3}w_{n-2}, \end{aligned}$$

and the case of  $n$  being even may be proved in the same manner.

13) If  $w_n \prec \prod_{j=-1}^{n-1} w_j$ , then by 12),  $w_n \prec w_n^* \prod_{j=-1}^{n-1} w_j = w_{n+1}$ , which will contradict 2).

14) Assume that  $u = \prod_{j=k}^{k+p} w_j = w_m$  for some  $m \geq 0$ . Since  $w_{k+p}$  is a factor,

$m > k + p$ . On the other hand, by 12),  $w_m \prec w_{k+p}^* \prod_{j=-1}^{k+p} w_j = w_{k+p+2}$ , so  $m = k + p + 1$ . By 13), this is impossible.

By an argument analogous to that of Property 2.12), we obtain the following result, which answers the question posed in the introduction.

**Theorem 1**  $F_\infty = \prod_{j=-1}^{\infty} w_j$ .

**Proof.** The proof is similar that of Property 2.12).

Now we are going to introduce another decomposition of  $F_\infty$  which will show the positively separate property of the singular words. To this end, we first establish some lemmas.

**Lemma 3** *Let  $w_n w_{n+1} = u_1 u_2 u_3$  (or  $w_{n+1} w_n = u_1 u_2 u_3$ ) with  $0 < |u_1| < f_n$  and  $0 < |u_3| < f_{n+1}$ , then  $u_2 \notin \mathcal{S}$ .*

**Proof.** i) By the condition of the lemma,  $2 < |u_2| < f_{n+2} - 2$ , so  $u_2 \neq w_{n+2}$ .

ii) Let  $\alpha \triangleright F_n$ , then  $w_n w_{n+1} = \beta F_n F_{n-1} \beta^{-1}$ . By Lemma 2,  $w_{n+1} = \alpha F_{n+1} \beta^{-1}$  appears only once in  $F_n F_{n+1}$ . Note that  $|u_3| \geq 1$ , we obtain  $u_2 \neq w_{n+1}$ .

iii) Let  $|u_2| = f_n$  and  $F_{n+1} = F_n F_{n-1}$ ,  $u_2 \prec F_n F_n$ . But by lemma 2,  $w_n \not\prec F_n F_n$ , and thus  $u_2 \neq w_n$ .

iv) Let  $|u_2| = f_{n-1}$ . Since  $w_n w_{n+1} = w_n w_{n-1} w_{n-2} w_{n-1}$ , then we must have  $u_2 \prec \alpha F_n F_{n-1} \alpha^{-1}$ . By using lemma 2, a discussion as in ii) yields  $u_2 \neq w_{n-1}$ .

The other cases will be reduced to one of the four cases above, so by repeating this argument, we prove that, for any  $k \geq 1$ ,  $u_2 \neq w_k$ , that is,  $u_2 \notin \mathcal{S}$ .

Now let  $n \geq 0$  be fixed. We define a substitution  $\phi_n : A \rightarrow A^*$  by  $\phi_n(a) = w_{n+1}$ ,  $\phi_n(b) = w_{n-1}$ . Let  $\Sigma_n = \{w_{n+1}, w_{n-1}\}$ . For a singular word  $w_k$ , we say  $W_k(\Sigma_n) := \phi_n(w_k)$  (if there is no confusion, we simply write  $W_k$ ), the  $k$ -th singular word over  $\Sigma_n$ .

**Lemma 4** *Let  $n \geq 0$  and  $k \geq 1$ . Then we have*

$$\begin{aligned} w_{n+2k} &= w_n x_1 w_n x_2 \cdots w_n x_{f_{2k-2}} w_n, \\ w_{2k+1} &= y_1 w_n y_2 w_n \cdots y_{f_{2k-1}-1} w_n y_{f_{2k-1}}, \end{aligned}$$

where  $x_j, y_j \in \Sigma_n$ . Moreover,

$$x_1x_2 \cdots x_{f_{2k-2}} = W_{2k-2} \text{ and } y_1y_2 \cdots y_{f_{2k-1}} = W_{2k-1}$$

are, respectively, the  $(2k-2)$ -th and  $(2k-1)$ -th singular words over  $\Sigma_n$ .

**Proof.** For any fixed  $n$ , we prove the lemma by induction. We have, by Property 2.4),

$$\begin{aligned} w_{n+2} &= w_n w_{n-1} w_n, \\ w_{n+3} &= w_{n+1} w_n w_{n+1}, \\ w_{n+4} &= w_n w_{n-1} w_n w_{n+1} w_n w_{n-1} w_n, \\ w_{n+5} &= w_{n+1} w_n w_{n+1} w_n w_{n-1} w_n w_{n+1} w_n w_{n+1}, \end{aligned}$$

and hence the conclusion is true for  $k=1, 2$ . Now suppose that the conclusion is true for  $k-1$  and  $k$ . Then

$$\begin{aligned} w_{n+2(k+1)} &= w_{n+2k} w_{n+2k-1} w_{n+2k} \\ &= w_n x_1 \cdots w_n x_{f_{2k-2}} w_n y_1 w_n \cdots w_n y_{f_{2k-3}} w_n \cdots w_n x_{f_{2k-2}}, \end{aligned}$$

since  $x_1x_2 \cdots x_{f_{2k-2}}$  and  $y_1y_2 \cdots y_{f_{2k-3}}$  are, respectively, the  $(2k-2)$ -th and  $(2k-3)$ -th singular words  $W_{2k-2}$  and  $W_{2k-3}$  on  $\Sigma_n$  by the assumption of the induction. So, by Property 2.4),

$$x_1x_2 \cdots x_{f_{2k-2}} y_1y_2 \cdots y_{f_{2k-3}} x_1x_2 \cdots x_{f_{2k-2}} = W_{2k-2} W_{2k-3} W_{2k-2} = W_{2k}$$

is the  $(2k)$ -th singular word. The same discussion gives the proof for  $w_{n+2k+3}$ . From Lemmas 3 and 4, we immediately obtain the following.

**Corollary 1** *Let  $m \geq n+2$ . Then there are exactly  $m-n-2$  factors  $w_n$  appearing in  $w_m$  which are separated by  $w_{n-1}$  and  $w_{n+1}$  as in Lemma 4.*

Let  $n$  be fixed, then by Property 1.6), the word  $w_n$  will appear in  $F_\infty$  infinitely many times. We arrange these words as a sequence  $w_{n,k}$  the  $k$ -th singular word of the order  $n$ .

**Lemma 5** *Let  $F_\infty = \prod_{j=-1}^{\infty} w_j$  be the decomposition as in Theorem 1. Let  $u$  be any singular word of order  $n$  (that is,  $u = w_{n,k}$  for some  $k$ ). Then  $u$  must be contained completely in some  $w_m$ , where  $m \geq n$ .*

**Proof.** i) From Property 2.13),  $w_n \not\leq \prod_{j=-1}^{n-1} w_j$ .

ii) If  $u \prec \prod_{j=-1}^n w_j$ , then by Property 2.12),

$$u \prec \left( w_{n-1}^* \prod_{j=-1}^{n-1} w_j \right) w_n = w_{n+1} w_n,$$

so by Lemma 3,  $u$  must be  $w_n$ .

From i) and ii), we only need to consider  $u \prec \prod_{j=n}^{\infty} w_j$ . Since  $|u| = |w_n|$ , there exists  $m$ ,  $m \geq n$ , such that, either  $u \prec w_m$ , or  $u \prec w_m w_{m+1}$  with  $u \not\prec w_n$  and  $u \not\prec w_{n+1}$ . But by lemma 3, the later case is impossible.

We thus finish the proof from the discussions above.

Now we can state our main result of this note.

**Theorem 2** For any  $n \geq 0$ , we have

$$F_{\infty} = \left( \prod_{j=-1}^{n-1} w_j \right) w_{n,1} z_1 w_{n,2} z_2 \cdots w_{n,k} z_k w_{n,k+1} \cdots$$

where  $z = z_1 z_2 \cdots z_n \cdots$  is the Fibonacci word over  $\Sigma_n$ .

**Proof.** From theorem 1 and lemma 4, we get

$$\begin{aligned} F_{\infty} &= \left( \prod_{j=-1}^{n-1} w_j \right) w_n w_{n+1} \left( \prod_{j=n+2}^{\infty} w_j \right) \\ &= \left( \prod_{j=-1}^{n-1} w_j \right) w_n w_{n+1} (w_n w_{n-1} w_n) (w_{n+1} w_n w_{n+1}) \cdots \\ &\quad (w_n x_1 w_n \cdots x_{f_{2k-2}} w_n) (y_1 w_n y_2 \cdots w_n y_{f_{2k-1}}) \cdots \end{aligned}$$

Note that

i) by lemma 4, lemma 5 and corollary 1, all factor  $w_n$  of  $F_{\infty}$  (or the sequence  $w_{n,k}$ ,  $k \geq 1$ ) appear in the formula above;

ii) by lemma 4,

$$\begin{aligned} x_1 \cdots x_{f_{2k-2}} &= W_{2k-2}, \\ y_1 \cdots y_{f_{2k-1}} &= W_{2k-1}, \end{aligned}$$

thus  $\prod_{j=1}^{\infty} z_j = \prod_{j=-1}^{\infty} W_j$  is the Fibonacci word on  $\Sigma_n$ .

i) and ii) follow the theorem.

The following example illustrates the decomposition of  $F_{\infty}$  of the words  $w_1$ ,  $w_2$ :





**Example 1. Power of the factors.** [2, 5, 6, 9]

**Theorem 3** *We have*

- 1) For any  $n$ ,  $w_n^2 \not\prec F_\infty$ ;
- 2) For  $0 \leq k \leq f_n - 1$ ,  $(C_k(F_n))^2 \prec F_\infty$ ;
- 3) If  $u \prec F_\infty$  with  $f_{n-1} < |u| < f_n$ , then  $u^2 \not\prec F_\infty$ ;
- 4) If  $0 \leq k \leq f_{n-1} - 2$ , then  $(C_k(F_n))^3 \prec F_\infty$ ;
- 5) If  $f_{n-1} - 2 < k < f_n$ , then  $(C_k(F_n))^3 \not\prec F_\infty$ ;
- 6) For any  $u \prec F_\infty$ ,  $u^4 \not\prec F_\infty$ .

**Proof.** 1) It follows from the properties 1.5) and 2.6),  $w_n^2 \not\prec F_\infty$ ;

2) Let  $C_k(F_n) = uv$  with  $F_n = vu$ . Then  $u \triangleright F_n$  and  $v \triangleleft F_n$ . Since

$$(C_k(F_n))^2 = uvuv = uF_nv \prec (F_n)^3,$$

the conclusion  $(C_k(F_n))^2 \prec F_\infty$  will follow from  $F_n^3 \prec F_\infty$ .

3) Suppose that  $w_k$  be the largest singular word contained in  $u$  as in corollary 5, and let  $u = v_1w_kv_2$ . Assume that  $u^2 = v_1w_kv_2v_1w_kv_2 \prec F_\infty$ , then  $w_k \not\prec v_2v_1$ , otherwise by theorem 2 we shall have either  $w_{k+1} \prec v_1$ , or  $w_{k+1} \prec v_2$ , that will be in contradiction with the hypothesis of  $w_k$ . Thus two singular words of the order  $k$  above are adjacent, so by theorem 2 again,  $v_2v_1$  must be either  $w_{k+1}$  or  $w_{k-1}$ . By property 2.5),  $u$  will be either a conjugation of  $F_{k+2}$ , or of  $F_{k+1}$ . But these two cases are impossible because of the hypothesis of  $u$ .

4) Since  $aaba \prec f_\infty$ , so dose  $F_nF_nF_{n-1}F_n$ . Let  $\alpha\beta \triangleright F_{n-1}$ , then by lemma 1, we have

$$F_n^2F_{n-1}F_n = F_n^2F_{n-1}F_{n-2}F_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta = F_n^3F_{n-1}\alpha^{-1}\beta^{-1}\alpha\beta \prec F_\infty,$$

notice that  $F_{n-1} \triangleleft F_n$ , hence if  $0 \leq k \leq f_{n-1} - 2$ , then

$$(C_k(F_n))^3 \prec F_n^3F_{n-1}\alpha^{-1}\beta^{-1} \prec F_\infty.$$

5) Now suppose that  $f_{n-1} - 1 < k < f_n$ , then by property 2.5),  $w_{n-1} \prec C_k(F_n)$ . Let  $C_k(F_n) = uw_{n-1}v$ , then  $vu = w_{n-2}$ , thus

$$(C_k(F_n))^3 = uw_{n-1}w_{n-2}w_{n-1}w_{n-2}w_{n-1}v.$$

Hence if  $(C_k(F_n))^3 \prec F_\infty$ , then the word  $w_{n-1}w_{n-2}w_{n-1} = w_{n+1}$  will have overlap, but by corollary 5, this is impossible.

6) The conclusion follows from an analogous argument with 5).

**Remark 2** *(From theorem 3.2), we see that, any conjugation of  $F_n$ ,  $n \geq 0$ , is not separated positively. This is an important difference between the conjugations of  $F_n$  and  $w_n$ .*

## Example 2. Local isomorphism

Let  $u = u_1u_2\cdots u_n\cdots$  and  $v = v_1v_2\cdots v_n\cdots$  be two infinite words over the alphabet  $\{a, b\}$ . We say that  $u$  and  $v$  are locally isomorphic if any factor (or its mirror image) of  $u$  is also factor of  $v$  and vice versa (By the property 1.7), for the Fibonacci word, we don't need to consider mirror images of the factors). If  $u$  and  $v$  are locally isomorphic, we shall write  $u \simeq v$ . The notion of local isomorphism is very useful in the studies of the energy spectra of one-dimensional quasicrystals [12].

For an infinite word  $u = u_1u_2\cdots u_n\cdots$ , we define the translation  $T(u)$  of  $u$  by  $Tu = u_2u_3\cdots$  and define  $T^k = T(T^{k-1})$  by recurrence. By using the properties of the singular words of the Fibonacci word, we can easily obtain the following results of the local isomorphism of the Fibonacci word.

**Theorem 4** *We have*

- 1) *If we change a finite number of letters of  $F_\infty$ , then the obtained infinite word  $F'_\infty$  is not locally isomorphic to  $F_\infty$ .*
- 2) *Let  $u \in A^*$ , then  $F_\infty \simeq uF_\infty$  if and only if there exists  $m > -1$ , such that  $u \triangleright w_m w_m^*$ , where  $w_m^*$  is defined as in property 2.12).*
- 3) *For any  $k \geq 1$ ,  $T^k(F_\infty) \simeq F_\infty$ .*

**Proof.** 1) Let  $F_\infty = \prod_{j=-1}^{\infty} w_j$  as in theorem 1, because we only change a finite number of letters of  $F_\infty$ , we can find an integer  $m$  and words  $u, v \in A^*$ , such that

$$F'_\infty = uv \prod_{j=m}^{\infty} w_j,$$

where  $|v| = f_{m-1}$ ,  $v \neq w_{m-1}$ . Therefore by corollary 3,  $vw_m \not\prec F_\infty$ , that is  $F_\infty \not\prec F'_\infty$ .

- 2) From theorem 1 and property 2.12), for any  $k > 0$  and  $m \geq 0$ ,

$$w_{2m}aF_\infty = w_{2m}a \left( \prod_{j=-1}^{2m+2k-1} w_j \right) \left( \prod_{j=2m+2k}^{\infty} w_j \right) = w_{2m}w_{2m+2k+1} \left( \prod_{j=2m+2k}^{\infty} w_j \right),$$

then, by corollary 3,  $w_{2m}w_{2m+2k+1} \prec F_\infty$ , that is, for any  $v \prec w_{2m}aF_\infty$ , we can find an integer  $k$ , such that  $v \prec w_{2m}w_{2m+2k+1}$ , so  $v \prec F_\infty$ . The case of  $w_{2m+1}b$  can be proved in the same way. That is, if  $u \triangleright w_m w_m^*$  for some  $m$ , then  $F_\infty \simeq uF_\infty$ . If  $u$  is not a right factor of any  $w_m w_m^*$ , then by the discussion similar that of 1), we see that  $uF_\infty \not\prec F_\infty$ .

- 3) The proof follows from the property 1.6).

## Example 3. Study of special words of $F_\infty$

Berstel [2] introduced the special words of  $F_\infty$  as follows: if  $ua, ub \prec F_\infty$ , then the word  $u$  is called a special word of  $F_\infty$ . The following theorem is due to Berstel [2] which we shall give another proof by using singular word.

**Theorem 5** *A word  $w \prec F_\infty$  is a special word if and only if, for some  $n \geq 0$ ,  $w \triangleright \overline{F_n}$ .*

**Proof.** It is easily checked that, for any  $n \geq 0$ ,  $\overline{F_n}$  is a special word, therefore the theorem is reduced to show that, for any  $n \geq 0$ ,  $|\Omega_n| = n + 1$ .

Now let  $u \prec F_\infty$  and let  $f_k < |u| \leq f_{k+1}$ . By an analogous argument with that for lemma 2, it is readily to see that the word  $u$  must be one of the three following forms:

- $u = sw_nt$ ,  $|st| \leq f_{n-1}$ ;
- $u = sF_nt$ ,  $s, t \neq \varepsilon$ ,  $|st| \leq f_{n-1}$ ,  $s \triangleright F_n$ ,  $t \triangleleft F_n$ ;
- $u = st$ ,  $s \triangleright F_n$ ,  $t \triangleleft F_n$ .

In the first case, by corollary 3, the factor  $s \triangleright w_{n-1}$ , (resp.  $t$ ) are determined uniquely. Moreover, since  $w_n$  has no overlap, if  $s \neq s'$ , then  $sw_nt \neq s'w_nt'$ . Hence there are exactly  $|u| - f_k + 1$  different words  $sw_nt$  which correspond with  $|s| = 0, 1, \dots, n - f_k$ .

In the two later cases, from property 1.1), it is readily to prove that there are exactly  $f_k$  different factors of length  $|u|$  of  $F_k^3$ .

Summarize the discussions above, we get  $|\Omega_{|u|}| = f_k + (|u| - f_k + 1) = |u| + 1$ .

#### Example 4. Overlap of the subwords of the Fibonacci word

In this example, we shall determine the factors which have overlap.

Recall that: Let  $u \prec F_\infty$ , if there exist words  $x, y$  and  $z$  such that  $u = xy = yz$  and  $\hat{u}(y) := uz = xyz \prec F_\infty$ . Then we shall say that the word  $u$  has overlap with the overlap factor  $y$  (or overlap length  $|y|$ ), the word  $\hat{u}(y)$  is called the overlap of  $u$  with the overlap factor  $y$ . We denote by  $\mathcal{O}(F_\infty) = \mathcal{O}$  the set of factors having overlap.

Evidently, if  $u \in \mathcal{O}$ , we have

$$|u| + 1 \leq |\hat{u}(y)| \leq 2|u| - 1, \quad (*)$$

where  $y$  is any overlap factor of  $u$ .

**Lemma 6** *Let  $f_n < |u| \leq f_{n+1}$ , and let  $u \neq w_{n+1}$ , then  $u \in \mathcal{O}$  if and only if  $w_n \not\prec u$ .*

**Proof.** Let  $w_n \prec u$  and write  $u = sw_nt$ . If  $u \in \mathcal{O}$ , notice that  $w_n \notin \mathcal{O}$ , thus overlap of  $u$  must be of the form  $sw_nv w_nt$ . By corollary 4,

$$|sw_nv w_nt| \geq |s| + |t| + 2f_n + f_{n-1} = |u| + f_{n+1} \geq 2|u|,$$

which is in contradiction with the inequality (\*).

Now suppose that  $w_n \not\prec u$ , then discuss as in theorem 5, we have either

- $u = sF_nt$ , where  $s, t \neq \varepsilon$ ,  $|s| + |t| \leq f_{n-1}$ ,  $s \triangleright F_n$ ,  $t \triangleleft F_n$ ;

- or  $u \prec F_n^2$ .

In the first case, if  $|t| = f_n - 1$ , then  $u = w_{n+1} \notin \mathcal{O}$ . Now consider  $|t| < f_n - 1$ . Since  $|s| + |t| \leq f_{n-1}$ ,  $s \triangleright F_n$ ,  $t \triangleleft F_n$ , we can write  $F_n = txs$ . Since  $|t| < f_n - 1$ , by theorem 3.4),

$$(C_{|t|}(F_n))^3 = (xst)^3 = xstxstxst \prec F_\infty,$$

that is,  $u = sF_n t = stxst$  has overlap with factor  $st$ .

In the second case, notice that  $u \prec F_n^2$  and  $|u| > f_n$ , so if we write  $u = st$ , with  $|t| = f_n$ , then  $t = C_k(F_n)$  for some  $k$ , and  $s \triangleright t$ , thus  $u = sxs$ . On the other hand, since  $u = sC_k(F_n) \prec F_n^2$ , so  $sxsxs = s(C_k(F_n))^2 \prec F_n^3 \prec F_\infty$ , that is  $u = sxs$  has overlap with overlap factor  $s$ .

**Lemma 7** *If  $u \in \mathcal{O}$ , then the overlap of  $u$  is unique.*

**Proof.** Let  $f_n < |u| \leq f_{n+1}$ , and let  $w$  be the largest singular word contained in  $u$ . By corollary 6,  $w$  is one of  $w_{n-1}$ ,  $w_n$  and  $w_{n+1}$ . Since  $u \in \mathcal{O}$ ,  $w$  must be  $w_{n-1}$  from lemma 6, so we can write  $u = sw_{n-1}t$ . Now suppose that there are two different overlaps of  $u$ , then  $w_{n-1}$  will appear three times in one of these two overlaps. Since  $w_{n-1} \notin \mathcal{O}$ , this overlap must be of the form  $sw_{n-1}v_1w_{n-1}v_2w_{n-1}t$ , then by an analogous argument with lemma 6, we shall get a contradiction of (\*).

From lemma 7 and the proof of the lemma 6, we obtain immediately

**Corollary 6** *Let  $f_n < |u| \leq f_{n+1}$ , and let  $u \in \mathcal{O}$ , then  $u = vv'v$ , where  $|v|$  is the overlap length.*

Summarize the results above, we have

**Theorem 6** *Let  $f_n < |u| \leq f_{n+1}$  and let  $u \neq w_{n+1}$ ,  $u \prec F_\infty$ , then  $u \in \mathcal{O}$  if and only if  $w_n \not\prec u$ . If  $u \in \mathcal{O}$ , then the overlap of  $u$  is unique and  $u = vv'v$ , where  $v$  is the factor of overlap and  $|v| = |u| - f_n$ .*

*In particular,  $C_k(F_n) \in \mathcal{O}$  if and only if  $0 \leq k \leq f_n - 2$ .*

Note that:

- i)  $f_{n+1} < 2f_n < f_{n+2} < 3f_n < f_{n+3}$ ;
- ii) for any  $k$ ,  $w_{n+1} \not\prec (C_k(F_n))^2$ ;
- iii) for any  $k$ ,  $w_{n+2} \not\prec (C_k(F_n))^3$ .

We get immediately from theorem 6

**Corollary 7** *For any  $k$ ,  $(C_k(F_n))^2 \in \mathcal{O}$ ,  $(C_k(F_n))^3 \in \mathcal{O}$ .*

**Remark 3** *If  $w^2 \not\prec F_\infty$  and  $w$  has no overlap, then the adjacent words of  $w$  will be positively separate. Moreover we can prove that for these words, there is a decomposition similar to the singular words.*

Let  $w = abab$ , by theorem 3.3) and theorem 6,  $w^2 \prec F_\infty$  and  $w \notin \mathcal{O}$ , so  $w$  is separated positively. the following decomposition illustrates the remark above:

$aba(abab)\underline{aaba}(abab)\underline{a}(abab)\underline{aaba}(abab)\underline{aaba}(abab)\underline{a}(abab)\underline{aaba}(abab)\underline{a}(abab)\dots$

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