Normalisation for the fundamental crossed complex of a simplicial set

Ronald Brown* Rafael Sivera

May 31, 2006

Abstract

The algebra of crossed complexes is shown to be sufficiently rich to model the inductive definition of simplices, and so to give a purely algebraic proof of the Homotopy Addition Lemma (HAL) for the boundary of a simplex. This leads to the *fundamental crossed complex* of a simplicial set. The main result is a normalisation theorem for this fundamental crossed complex, analogous to the usual theorem for simplicial abelian groups, but more complicated to set up and prove, because of the complications of the HAL and of the notion of homotopies for crossed complexes.

Introduction

The normalisation theorem for simplicial abelian groups (see, for example Mac Lane [25, §VIII.6]), is of importance in homological algebra and in geometric applications of simplicial theory. It is based on the formula, fundamental in much of simplicial based algebraic topology and homological algebra, that if x has dimension n then

$$\partial x = \sum_{i=0}^{n} (-1)^i \partial_i x, \tag{1}$$

which can be interpreted intuitively as: 'the boundary of a simplex is the alternating sum of its faces'. The setting for this formula is the theory of chain complexes: these are sequences of morphisms of abelian groups (or R-modules) $\partial : A_n \to A_{n-1}$ such that $\partial\partial = 0$. In this theory, each ∂_i is a morphism of abelian groups and the formula (1) is just the alternating sum of morphisms. Thus we have a chain complex (A, ∂). Further, if $(DA)_n$ is for $n \geqslant 0$ the subgroup of A_n generated by degenerate elements, then DA is a contractible subchain complex of (A, ∂). This is the normalisation theorem.

In *homotopy*, rather than homology, theory, there is another and more complicated basic formula, known as the *Homotopy Addition Lemma* (HAL) (or theorem) [22, 29]. It has roughly the same import as (1), namely it gives 'the boundary of a simplex', but it takes account also of:

*Brown was supported by a Leverhulme Emeritus Fellowship, 2002-2004, which also supported collaboration of the authors.

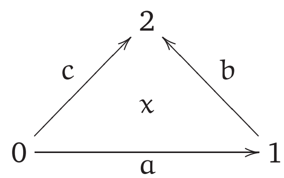
- a *set* of base points (the vertices of the simplex);
- operators of dimension 1 on dimensions $\geqslant 2$;
- nonabelian structures in dimensions 1 and 2.

The set of base points is taken account of through the use of groupoids in dimension 1, while the boundary from dimension 3 to dimension 2 uses crossed modules of groupoids. This leads to basic formulae, which, with our conventions are as follows:

In dimension 2 we have a groupoid rule:

$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x, \qquad \text{(HAL2)}$$

which is represented by the diagram



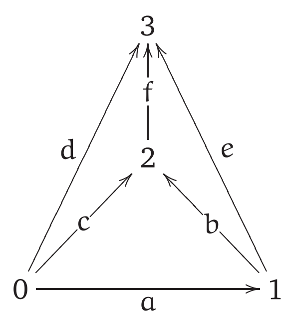
(HAL2-diagram)

and the easy to understand formula (HAL2) says that $\delta_2 x = -c + a + b$.

In dimension 3 we have the nonabelian rule:

$$\delta_3 x = (\partial_3 x)^{u_3} - \partial_0 x - \partial_2 x + \partial_1 x. \qquad \text{(HAL3)}$$

Understanding of this is helped by considering the diagram

$x :=$ 

(HAL3-diagram)

Note that in HAL3 we have an exponent u_3: this is given by $f = \partial_0^2 x$. The necessity for this is that our convention is that each n-simplex x has as base point its last vertex $\partial_0^n x$. Thus the base point of the above 3-simplex x is 3, while the base point of $\partial_0 x$ is 2. The exponent f relocates $\partial_0 x$ to have base point at 3, and so obtains a well defined formula. Note that with the labelling in (HAL3-diagram) we have the groupoid formula

$$-f + (-c + a + b) + f - (-e + b + f) - (-d + a + e) + (-d + c + f) = 0.$$

This is a translation of the rule $\delta_2 \delta_3 = 0$, provided we assume $\delta_2(y^f) = -f + \delta_2 y + f$, which is the first rule for a crossed module.

In dimension $n \geqslant 4$ we have the abelian rule, but still with operators:

$$\delta_n x = (\partial_n x)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i x \quad \text{for } n \geqslant 4, \tag{HAL$\geqslant$ 4}$$

where $u_n x = \partial_0^{n-1} x$. We have some difficulty in drawing a diagram for this! These, or analogous, formulae underly much nonabelian cohomology theory.

The rule $\delta_{n-1}\delta_n = 0$ is straightforward to verify for $n > 4$, through working in abelian groups, but for $n = 4$ we require the second crossed module rule, that for x, y of dimension 2

$$-y + x + y = x^{\delta_2 y}.$$

A consequence is that $\operatorname{Ker} \delta_2$, is central. Hence so also is $\operatorname{Im} \delta_3$, since we have verified $\delta_2\delta_3 = 0$. The type of argument that is used for the $n = 4$ case, [29], and which we shall use later, is the simple:

Lemma 0.1 *If* $c = a + b$ *is a central element in a group, then* $c = b + a$.

Proof

$$c - a - b = -a + c - b = 0. \tag{centrality}$$

□

Thus the setting for the Homotopy Addition Lemma, a fundamental result in homotopy theory, is much more complicated than the chain complexes which are the setting for the boundary formula (1). Yet formulae of these type occur frequently in mathematics, for example in the cohomology of groups, [19], in differential geometry, [24], and in the cohomology of stacks, [6].

The formal structure required for this HAL is known as a *crossed complex of groupoids*, and they form the objects of a category which we write Crs. This category is complete and cocomplete. It contains 'free' objects, satisfying the universal property that a morphism $f : F \to C$ from a free crossed complex is defined by its values on a free basis, subject to certain geometric conditions. In the formulae for the HAL, the ∂_i are *not* morphisms, but x and each $\partial_i x$ is an element of a free basis. The above formulae are not exactly as will be found generally in the literature, but they follow from our conventions given in section 5 for crossed complexes and their 'cylinder object' $\mathcal{I} \otimes C$.

Crossed complexes, but called *group systems*, were first defined by Blakers in [5]. This concept combined into a single structure the fundamental group $\pi_1(X, x)$ and the relative homotopy groups $\pi_n(X_n, X_{n-1}, x), n \geqslant 2$ associated to a filtered space X_*, but only in the reduced case, i.e. when X_0 is a singleton $\{x\}$. We now call this structure the *fundamental crossed complex* $\Pi(X_*)$ of the filtered space (see below). Blakers associated to a reduced crossed complex C a simplicial set which nowadays we would call the *nerve* NC of the crossed complex, [15]; the definition uses the Homotopy Addition Lemma in an essential way, although a detailed and elementary proof of that was available only in 1953, [22].

Blakers' concept was taken up in J.H.C. Whitehead's deep paper 'Combinatorial homotopy theory II' (CHII) [32], in the reduced and free case, under the term 'homotopy system'; this paper is much less read than the previous paper 'Combinatorial homotopy I' (CHI) [31], which introduced the basic concept of *CW*-complex. We give below a full definition of the category Crs of crossed complexes: our viewpoint, following that of CHII, is that Crs should be seen as a basic category for applications in algebraic topology, with better realisability properties, [15, 18], than the more usual chain complexes with a group of operators.

From the point of view of practical algebraic topology, important additional facts on crossed complexes were found in a sequence of papers by Brown and Higgins. These facts are:

(i) The functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ from the category of filtered spaces to crossed complexes preserves certain colimits, [12];

(ii) the category Crs is monoidal closed, [13], with an exponential law of the form

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)). \qquad \text{(exponential law)}$$

(iii) The category Crs has a *unit interval object* written $\{0\} \rightrightarrows \mathcal{I}$, which is essentially just the indiscrete groupoid on two objects $0, 1$, and so has in dimension 1 only one element $\iota : 0 \to 1$. For a crossed complex B, this gives rise to a *cylinder object*

$$\mathsf{Cyl}(B) = (B \rightrightarrows \mathcal{I} \otimes B),$$

and so a homotopy theory for crossed complexes.

(iv) There is a *classifying space functor* $B : \mathsf{Crs} \to \mathsf{Top}$ and a homotopy classification theorem

$$[X, BC] \cong [\Pi X_*, C]$$

for a *CW*-complex X with its skeletal filtration, and crossed complex C, [15].

(v) There is for filtered spaces X_*, Y_* a natural transformation

$$\eta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*),$$

which is an isomorphism if X_*, Y_* are the skeletal filtrations of *CW*-complexes, [15], and more widely, [2].

The first result is a kind of Generalised van Kampen Theorem, GvKT, and has consequences which include the relative Hurewicz Theorem, and nonabelian results in dimension 2, [20], not seemingly obtainable by other means. The fourth result is a homotopy classification theorem in the non simply connected case, and includes many classical results.

It is a considerable work to set up all these results, which require moving to a cubical category, [11], for the proofs. It is planned that the book in preparation, [16], which will give a full account in one place of these main properties, will make these results more accessible and so make them more usable.

Our aim here is more modest. We show how the HAL for a simplex fits neatly into an algebraic pattern in crossed complexes. We define algebraically and inductively an 'algebraic' or 'crossed complex simplex' $a\Delta^n$ by

$$a\Delta^0 = \{0\}, \quad a\Delta^{n+1} = \mathsf{Cone}(a\Delta^n). \qquad (2)$$

Given the conventions for the tensor product, [13], this yields algebraically exactly the HAL given above.

Our main result is an application to the fundamental crossed complex ΠK of a simplicial set K. This is defined to be the free crossed complex on the elements of $K_n, n \geqslant 0$, with boundary given by the homotopy addition lemma HAL. Thus ΠK contains basis elements which are degenerate simplices, of the form $\varepsilon_i y$ for some y. The family of all degenerate simplices have in ΠK a normaliser DK which is a normal subcrossed complex of ΠK. Our main result is:

Theorem 0.2 (Normalisation theorem) *For a simplicial set K, the normal subcrossed complex DK is contractible, the quotient crossed complex $\Pi^\vee K$ is free, and the projection $\Pi K \to \Pi^\vee K$ has a section.*

The last part of the theorem is important in acyclic model arguments. This will be explained elsewhere.

Corollary 0.3 *For a simplicial set K, the projection $\|K\| \to |K|$ from the thick to the standard geometric realisation is a homotopy equivalence.*

Proof We use the Generalised van Kampen Theorem of [12] to give natural isomorphisms

$$\Pi K \cong \Pi(\|K\|_*), \Pi^\vee K \cong \Pi(|K|_*).$$

It is immediate that the projection induces an isomorphism of fundamental groupoids and of the homologies of the universal covers at all base points. □

For further work on crossed complexes, see for example [1, 3, 4, 27, 28]. The first three works refer to crossed complexes as crossed chain complexes. It can be argued that the category Crs gives a linear approximation to homotopy theory: that is, crossed complexes incorporate the fundamental group(oid) and its actions, but do not incorporate, say, higher dimensional Whitehead products, or composition operators. The tensor product, and corresponding notions of an 'algebra', allow for more structure, as in [3, 4].

We assume work on groupoids, for example normal subgroupoids, as in [7, 21].

1 Basic definitions for crossed complexes

We use relative homotopy theory to construct a functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs} \tag{3}$$

where FTop is the category of *filtered spaces*, whose objects

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty$$

consist of a compactly generated topological space X_∞ and an increasing sequence of subspaces $X_n, n \geqslant 0$. The morphisms $f : X_* \to Y_*$ of FTop are maps $f : X_\infty \to Y_\infty$ such that for all $n \geqslant 0$ $f(X_n) \subseteq Y_n$.

The functor Π is given on a filtered space X_* by

$$(\Pi X_*)_n = \begin{cases} X_0 & \text{if } n = 0, \\ \pi_1(X_1, X_0) & \text{if } n = 1, \\ \pi_n(X_n, X_{n-1}, X_0) & \text{if } n \geqslant 2. \end{cases} \tag{4}$$

Here $\pi_1(X_1, X_0)$ is the fundamental groupoid of X_1 on the set X_0 of base points, and $\pi_n(X_n, X_{n-1}, X_0)$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x_0)$ for all $x_0 \in X_0$.

If we write $C_n = (\Pi X_*)_n$, then we find that there is a structure of a family of groupoids over C_0 with source and target maps s, t:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 \\ & \downarrow t & & \downarrow t & & \downarrow t & & s \downarrow\downarrow t \\ & C_0 & & C_0 & & C_0 & & C_0 \end{array}$$

in which: for $n \geqslant 2$, C_n is totally disconnected, i.e. $s = t$; C_1 operates (on the right) on $C_n, n \geqslant 2$, and on the family of vertex groups of C_1 by conjugation; and the axioms are, in addition to the usual operation rules, that:
CC1) $s\delta_2 = t\delta_2$, $\delta_{n-1}\delta_n = 0$;
CC2) δ_n is an operator morphism;
CC3) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC4) for $n \geqslant 3$, C_n is abelian and $\delta_2 C_2$ operates trivially on C_n.
It will be convenient to write all group and groupoid compositions additively, and the operations as x^a. Thus if $a : p \to q$ in dimension 1, then $p = sa$, $q = ta$, and if further $b : q \to r$ then $a + b : p \to r$. If further $n \geqslant 2$ and $x \in C_n(p)$, then $tx = p$ and $x^a \in C_n(q)$.

These laws for ΠX_* reflect basic facts in relative homotopy theory, and also define the objects of the category Crs of crossed complexes. The morphisms $f : C \to D$ of crossed complexes consist of groupoid morphisms $f : C_n \to D_n, n \geqslant 1$, preserving all the structure.

A crossed complex C has a *fundamental groupoid* $\pi_1 C$ defined to be $C_1/(\delta_2 C_2)$. It also has a family of *homology groups* given for $n \geqslant 2$ by

$$H_n(C, p) = \mathrm{Ker}\,(\delta_n : C_n(p) \to C_{n-1}(p))/\mathrm{Im}\,(\delta_{n+1}(C_{n+1}(p) \to C_n(p)).$$

If X_* is the skeletal filtration of a CW-complex X, then there are natural isomorphisms

$$\pi_1(\Pi X_*) \cong \pi_1(X, X_0), \qquad H_n(\Pi X_*, p) \cong H_n(\widetilde{X}_p),$$

where $\widetilde{X}_p$ is the universal cover of X based at p. It follows from this and Whitehead's theorem that if $f : X \to Y$ is a cellular map of CW-complexes X, Y which induces a weak equivalence $\Pi f : \Pi X_* \to \Pi Y_*$, then f is a homotopy equivalence.

The category FTop of filtered spaces is monoidal closed with an exponential law

$$\mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)). \tag{5}$$

Here $(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q$. A standard example of a filtered space is a CW-complex with its skeletal filtration, and among the CW-complexes we have the n-ball E^n with its cell structure

$$E^n = \begin{cases} e^0 & \text{if } n = 0, \\ e^0_\pm \cup e^1 & \text{if } n = 1, \\ e^0 \cup e^{n-1} \cup e^n & \text{if } n > 1. \end{cases} \tag{6}$$

The exponential law for crossed complexes, (exponential law), involves an 'internal hom' crossed complex $\mathsf{CRS}(B, C)$. This is in dimension 0 simply $\mathsf{Crs}(B, C)$, the set of morphisms $B \to C$; in dimension 1 is the groupoid of homotopies between morphisms; and in higher dimensions consists of 'higher homotopies'. The full structure of this is quite complicated. This complexity is also reflected in the structure of the tensor product $A \otimes B$ of crossed complexes A, B: it is generated in dimension n by elements $a \otimes b$ where $a \in A_p, b \in B_q, p + q = n$, but the full list of structure and laws is again quite complex.

A fundamental theorem on the functor Π is the following, whose proof depends on results of [12, 13] and the use of the category of cubical ω-groupoids:

Theorem 1.1 ([15]) *There is a natural transformation*

$$\eta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

which yields an isomorphism if X_*, Y_* are the skeletal filtrations of CW-complexes.

The specific formulae for the tensor product given in [13] are thus related to the cell decomposition of $E^m \times E^n$.

Let $\mathcal{I}$ denote the groupoid with two objects $0, 1$ and exactly one arrow $\iota : 0 \to 1$. We can obtain from this a crossed complex also written $\mathcal{I}$ by extending trivially in higher dimensions than 1. This crossed complex, which is isomorphic to ΠE^1_*, can be given the structure of a *unit interval object*

$$\{0\} \rightrightarrows \mathcal{I}$$

in the category Crs (see for example [23]). This allows us to define homotopies of morphisms $B \to C$ of crossed complexes as morphisms $\mathcal{I} \otimes B \to C$, or, equivalently, as morphisms $\mathcal{I} \to \mathsf{CRS}(B, C)$. The detailed structure of this *cylinder object* $\mathcal{I} \otimes C$, [23], will be given in section 5.

Remark 1.2 From the early days of relative homotopy theory, basic results have been proved by relating the geometries of both cells and cubes. For our purposes, this geometric relation had to be translated into a relation between algebraic theories. This is done by the results of several papers, particularly [11, 13], which give an equivalence of monoidal closed categories between a category of 'cubical ω-groupoids' and the category Crs. In fact many constructions and proofs are clear in the former category, except that both categories are also required for some results. For example the natural transformation of Theorem 1.1 is easy to see in the cubical category. For a survey on crossed complexes and their uses, see [9].

2 Generating complexes

Let C be a crossed complex, and let R_* be a family of subsets $R_n \subseteq C_n$ for all $n \geqslant 0$. We have to explain what is meant by the subcrossed complex $\langle R_* \rangle$ of C *generated* by R_*.

A formal definition of $B = \langle R_* \rangle$ is easy: it is the smallest sub-crossed complex D of C such that $R_n \subseteq D_n$ for all $n \geqslant 0$, and so also is the intersection of all such D. A direct construction is as follows. We set

$$B_0 = R_0 \cup sR_1 \cup \bigcup_{n \geqslant 1} tR_n.$$

Let B_1 be the subgroupoid of C_1 generated by $R_1 \cup \delta_2(R_2)$ and the identities at B_0. Let B_2 be the subcrossed B_1-module of C_2 generated by R_2. For $n \geqslant 3$, let B_n be the sub-B_1-module of C_n generated by $R_n \cup \delta(R_{n+1})$ and the identities at elements of B_0.

Note that this definition is inductive. The usual property of a generating structure holds. Thus if R_* generates C, i.e. $\langle R_* \rangle = C$, and $f, g : C \to D$ are two crossed complex morphisms which agree on R_*, then $f = g$. This is proved by induction.

We say the family R_* is a *generating complex* for a subcrossed complex B of C if for each $n > 0$ the boundaries in C of elements of R_n lie in the subcrossed complex generated by the R_i for $i < n$, and R_* generates B.

3 Normal subcrossed complexes

Definition 3.1 A subcrossed complex N of a crossed complex C is called *normal* in C if:

(i) $N_0 = C_0$;

(ii) N_1 is a totally disconnected normal subgroupoid of C_1;

(iii) N_2 is a normal subgroup of C_2;

(iv) for $n \geqslant 2$, $a \in N_n$ and $c \in C_1$ implies $c^a \in N_n$, when it is defined;

(v) for $n \geqslant 2$, if $a \in N_1$ and $c \in C_n$, then $c - c^a \in N_n$. □

The condition 'totally disconnected' in (ii) can be relaxed, but will be sufficient for our purposes.

Proposition 3.2 *If N is a normal subcrossed complex of the crossed complex C, then the family of quotients C_n/N_n inherits the structure of crossed complex, which we call the quotient crossed complex C/N.*

Let R_* be a family of subsets of the crossed complex C as in the previous section, and such that R_1 is totally disconnected, i.e. just a family of subsets of vertex groups of the groupoid C_1. We say that R_* *normally generates* a subcrossed complex N of C if N is the smallest normal subcrossed complex of C containing R_*, and then we say N is the *normal closure* of R_* in C. We also say R_* is a *normal complex in* C if for each $n > 0$ the boundaries of elements of R_n are in the normal closure of the R_i for $i < n$.

4 Free crossed complexes

A crossed complex F is *free on* R_* if in the first place R_* generates F, and secondly morphisms on F to any crossed complex can be defined inductively by their values on R_*. So in the first instance we have $R_0 = F_0$, and F_1 is the free groupoid on the graph (R_1, R_0, s, t). We assume this concept as known; it is fully treated in [7]. For free crossed complexes, we refer also to [10, 15] for more details.

Secondly, R_2 comes with a function $w : R_2 \to F_1$ given by the restriction of δ_2. We require that the inclusion $R_2 \to F_2$ makes F_2 the free crossed F_1-module on R_2.

By this stage, the fundamental groupoid $\pi_1 F$ is defined; we require that for $n \geqslant 3$, F_n is the free $\pi_1 F$-module on R_n.

A standard fact, due in the reduced case to Whitehead in CHII, is that if X_* is the skeletal filtration of a CW-complex, then ΠX_* is the free crossed complex on the characteristic maps of the cell structure of X_*. This may be proved using the relative Hurewicz theorem, and is also a consequence of the Generalised van Kampen Theorem of [12].

We now give a proposition and a counterexample due in the crossed module case to Whitehead, [33], which illustrate some of the difficulties of working with free crossed modules.

Theorem 4.1 *Let C be the free crossed complex on R_*, and suppose $S_* \subseteq R_*$ generates a subcrossed complex B of C. Let F be the free crossed complex on S_*. Then the induced morphism $F \to C$ is injective if the induced morphism $\pi_1 B \to \pi_1 C$ is injective.*

Proof We use the functor $\Delta : \mathsf{Crs} \to \mathsf{Chn}$ to chain complexes with a groupoid of operators which was introduced in [14], generalising work of Whitehead in CHII. See also [17] for the low dimensional and reduced cases.

First of all, we know that a subgroupoid of a free groupoid is free. Also in dimensions > 2 C_n is the free $\pi_1 C$-module on the basis R_n. So injectivity, under the given condition, is clear in this case.

Thus the only problem is in dimension 2, and here we generalise an argument of Whitehead, [33].

The abelianised groupoids F_2^{ab}, C_2^{ab} are respectively free $\pi_1 F, \pi_1 C$-modules on the bases S_2, R_2. Since the induced morphism on π_1 is injective, so also is the induced morphism $F_2^{ab} \to C_2^{ab}$. But the morphism $C_2 \to C_2^{ab}$ is injective on $\operatorname{Ker} \delta_2 : C_2 \to C_1$, since C_1 is a free groupoid, [32, 17, 14]. So $F \to C$ is injective in dimension 2. □

Example 4.2 Let $X = Y = \{x\}, R = \{a, b\}, S = \{b\}$ where $a = x, b = 1$. The presentations of groups $\langle Y \mid S \rangle, \langle X \mid R \rangle$ determine free crossed modules $\delta_S : C(S) \to F(X), \delta_R : C(R) \to F(X)$. The inclusion $i : S \to R$ determines $C_i : C(S) \to C(R)$. Now $F(X) = F(Y) = C$, the infinite cyclic group, while $C(S)$ is abelian and is the free C-module on the generator b. Also in $C(R)$, $ab = ba$ since $\delta_R b = 1$. So

$$C_i(b^x) = C_i(b)^{\delta_R a} = a^{-1} b a = b = C_i(b),$$

so that C_i is not injective. □

5 Cylinder and homotopies

It is useful to write out first all the rules for the cylinder $\mathrm{Cyl}\,(C) = \mathcal{I} \otimes C$, as a reference. For full details of the tensor product, see [13, 9].

Let C be a crossed complex. The cylinder $\mathcal{I} \otimes C$ is generated by elements $0 \otimes c, 1 \otimes c$ of dimension n and $\iota \otimes c, (-\iota) \otimes c$ of dimension $(n+1)$ for all $n \geqslant 0$ and $c \in C_n$, with the following defining relations for $a \in \mathcal{I}$:

Source and target

$$\begin{aligned} t(a \otimes c) &= ta \otimes tc && \text{for all } \ a \in \mathcal{I}, c \in C \\ s(a \otimes c) &= a \otimes sc && \text{if } \ a = 0, 1, n = 1 \ , \\ s(a \otimes c) &= sa \otimes c && \text{if } a = \iota, -\iota, n = 0 \ . \end{aligned}$$

Relations with operations

$$a \otimes c^{c'} = (a \otimes c)^{ta \otimes c'} \qquad \text{if } \ n \geqslant 2, \ c' \in C_1.$$

Relations with additions

$$a \otimes (c + c') = \begin{cases} (a \otimes c)^{ta \otimes c'} + a \otimes c', & \text{if } a = \iota, -\iota, n = 1, \\ a \otimes c + a \otimes c', & \text{if } a = 0, 1, n \geqslant 1 \text{ or if } a = \iota, -\iota, \ n \geqslant 2, \end{cases}$$

$$(-\iota) \otimes c = \begin{cases} -(\iota \otimes c) & \text{if } n = 0, \\ -(\iota \otimes c)^{(-\iota) \otimes tc} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta(a \otimes c) = \begin{cases} a \otimes \delta c & \text{if } a = 0, 1, \ n \geqslant 2; \\ -ta \otimes c - a \otimes sc + sa \otimes c + a \otimes tc & \text{if } a = \iota, -\iota, \ n = 1; \\ -(a \otimes \delta c) - (ta \otimes c) + (sa \otimes c)^{a \otimes tc} & \text{if } a = \iota, -\iota, \ n \geqslant 2. \end{cases}$$

□

Now we can translate the rules for a cylinder into rules for a homotopy. Thus a homotopy $f^0 \simeq f$ of morphisms $f^0, f : C \to D$ of crossed complexes is a pair (h, f) where h is a family of functions $h_n : C_n \to D_{n+1}$ with the following properties, in which tc for $c \in C$ is c, if $c \in C_0$, is tc, if $c \in C_1$, and is x if $c \in C_n(x), \ n \geqslant 2$. So we require:

$$th_n(c) = tf(c) \qquad \text{for all } c \in C; \tag{7}$$

$$h_1(c + c') = h_1(c)^{fc'} + h_1(c') \qquad \text{if } c, c' \in C_1 \text{ and } c + c' \text{ is defined;} \tag{8}$$

$$h_n(c + c') = h_n(c) + h_n(c') \qquad \text{if } c, c' \in C_n, \ n \geqslant 2 \text{ and } c + c' \text{ is defined;} \tag{9}$$

$$h_n(c^{c_1}) = (h_n c)^{fc_1} \qquad \text{if } c \in C_n, n \geqslant 2, \ c_1 \in C_1, \text{ and } c^{c_1} \text{ is defined.} \tag{10}$$

Then f^0, f are related by

$$f^0(c) = \begin{cases} sh_0c & \text{if } c \in C_0, \\ (h_0sc) + (fc) + (\delta_2 h_1 c) - (h_0tc) & \text{if } c \in C_1, \\ \{fc + h_{n-1}\delta_n c + \delta_{n+1}h_n c\}^{-(h_0tc)} & \text{if } c \in C_n,\ n \geqslant 2. \end{cases} \tag{11}$$

Remark 5.1 Part of the force of this statement is that if (h, f) satisfy properties (7-10), then f^0 defined by (11) is a morphism of crossed complexes.

The following is a substantial result:

Proposition 5.2 ([15]) *If* F, F' *are free crossed complexes, on bases* B_*, B'_**, then* $F \otimes F'$ *is the free crossed complex on the basis* $B \otimes B'$*.*

The proof in [15] uses an inductive construction of free complexes as successive pushouts, and then uses the exponential law and the symmetry of $\otimes$, to show that $\otimes$ preserves colimits on either side.

A consequence, which may also be proved directly, is:

Proposition 5.3 *If* C *is a free crossed complex on a generating family* $B_n, n \geqslant 0$*, then a homotopy* $(h, f) : f^0 \simeq f : C \to D$ *is specified by the values* $fx \in D_n, hx \in D_{n+1}, x \in B_n, n \geqslant 0$ *provided only that the following geometric conditions hold:*

$$\begin{aligned} &sfx = fsx, tfx = ftx, x \in B_1, \delta fx = f\delta x, x \in B_n, n \geqslant 2, \\ &tfx = ftx, x \in B_n, n \geqslant 1, thx = tfx, x \in X_n, n \geqslant 0. \end{aligned} \tag{12}$$

Proof The main fact we need here is that an f-derivation on a free groupoid is uniquely defined by its values on a free basis. But this follows easily from the fact that an f-derivation $h_1 : F_1 \to C_2$ corresponds exactly to a section of a semidirect product construction $F_1 \ltimes C_2 \to F_1$. □

6 Cones and the HAL

The rules for a cylinder simplify nicely on passing to the cone, when it is formed from the cylinder by shrinking the end at 1 to a point.

Definition 6.1 Let C be a crossed complex. The *cone* $\mathrm{Cone}(C)$ is defined by the pushout

$$\begin{array}{ccc} \{1\} \otimes C & \longrightarrow & \{v\} \\ \downarrow & & \downarrow \\ \mathcal{I} \otimes C & \longrightarrow & \mathrm{Cone}\,(C). \end{array}$$

We call v the *vertex* of the cone. □

Proposition 6.2 *If* C *is a crossed complex, then the cone* $\mathrm{Cone}\,(C)$ *on* C *is generated by elements* $0 \otimes c, \iota \otimes c, c \in C_n$ *of dimensions* $n, n+1$ *respectively, and* ν *of dimension* 0 *with the rules:*
Source and target

$$t(a \otimes c) = \begin{cases} 0 \otimes tc, & \text{if } a = 0, \\ \nu & \text{otherwise.} \end{cases}$$

Relations with operations

$$a \otimes c^{c'} = a \otimes c \quad \text{if } \ n \geqslant 2, \ c' \in C_1.$$

Relations with additions

$$a \otimes (c + c') = a \otimes c + a \otimes c'.$$

and

$$(-\iota) \otimes c = \begin{cases} -(\iota \otimes c) & \text{if } n = 0, \\ -(\iota \otimes c)^{-\iota \otimes tc} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta_n(0 \otimes c) = 0 \otimes \delta_n c \qquad \text{if } n \geqslant 2.$$

$$\delta_{n+1}(\iota \otimes c) = \begin{cases} -\iota \otimes sc + 0 \otimes c + \iota \otimes tc & \text{if } n = 1, \\ -(\iota \otimes \delta_n c) + (0 \otimes c)^{\iota \otimes tc} & \text{if } n \geqslant 2. \end{cases}$$

Proposition 6.3 *Let* F *be a free crossed complex on a basis* B_*. *Then* $\mathrm{Cone}(F)$ *is the free crossed complex on* ν *in dimension* 0*, and elements* $0 \otimes b, \iota \otimes b$ *for all* $b \in B_*$*, with boundaries given by proposition 6.2.*

Proof This follows from the inductive definition of a free crossed complex by successive adjunctions, in a similar manner to the proof that a cone on a CW-complex is also a CW-complex. □

We use the above to work out the fundamental crossed complex of the simplex $a\Delta^n$ in an algebraic fashion. We regard $a\Delta^n$ as the cone

$$a\Delta^n = \mathrm{Cone}(a\Delta^{n-1}).$$

The vertices of $a\Delta^1 = I$ are ordered as $0 < 1$. Inductively, we get vertices $\nu_0, \ldots, \nu_n$ of $a\Delta^n$ with $\nu_n = \nu$ being the last introduced in the cone construction, the other vertices ν_i being $(0, \nu_i)$. The fact that our algebraic formula corresponds to the topological one follows from facts stated earlier on the tensor product and on the GvKT.

We now define inductively top dimensional generators of the crossed complex $a\Delta^n$ by, in the cone complex:

$$\sigma^0 = \nu, \ \sigma^1 = \iota, \ \sigma^n = (\iota \otimes \sigma^{n-1}), \ n \geqslant 2,$$

with σ^0 being the vertex of $a\Delta^0$.

Next we need conventions for the faces of σ^n.

We define inductively

$$\partial_i \sigma^n = \begin{cases} \iota \otimes \partial_i \sigma^{n-1} & \text{if } i < n, \\ 0 \otimes \sigma^{n-1} & \text{if } i = n. \end{cases}$$

Theorem 6.4 (Homotopy Addition Lemma) *The following formulae hold, where* $u_n = \iota \otimes \nu_{n-1}$*:*

$$\delta_2 \sigma^2 = -\partial_1 \sigma^2 + \partial_2 \sigma^2 + \partial_0 \sigma^2, \tag{13}$$

$$\delta_3 \sigma^3 = (\partial_3 \sigma^3)^{u_3} - \partial_0 \sigma^3 - \partial_2 \sigma^3 + \partial_1 \sigma^3, \tag{14}$$

while for $n \geqslant 4$

$$\delta_n \sigma^n = (\partial_n \sigma^n)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i \sigma^n. \tag{15}$$

Proof For the case $n = 2$ we have

$$\begin{aligned} \delta_2 \sigma^2 &= \delta_2((\iota \otimes \iota)) \\ &= -\iota \otimes 0 + 0 \otimes \iota + \iota \otimes 1 \\ &= -\partial_1 \sigma^2 + \partial_2 \sigma^2 + \partial_0 \sigma^2. \end{aligned}$$

For $n = 3$ we have:

$$\begin{aligned} \delta_3 \sigma^3 &= \delta_3(\iota \otimes \sigma^2) \\ &= (0 \otimes \sigma^2)^{\iota \otimes \nu_2} - \iota \otimes \delta_2 \sigma^2 \\ &= (0 \otimes \sigma^2)^{u_3} - \iota \otimes (-\partial_1 \sigma^2 + \partial_2 \sigma^2 + \partial_0 \sigma^2) \\ &= (\partial_3 \sigma^3)^{u_3} - \partial_0 \sigma^3 - \partial_2 \sigma^3 + \partial_1 \sigma^3. \end{aligned}$$

We leave the general case to the reader, using the inductive formula

$$\delta_{n+1} \sigma^{n+1} = (0 \otimes \sigma^n)^{\iota \otimes \nu_n} - \iota \otimes \delta_n \sigma^n.$$

The key points that make it easy are the rules on operations and additions of Proposition 6.2. □

Corollary 6.5 *The formula for the boundary of a simplex is as given by the HAL in the Introduction.*

Proof We use the fact that for $n \geqslant 2$, the geometric n-simplex Δ^n may be regarded as the cone $\mathrm{Cone}(\Delta^{n-1})$. Our previous results thus give an isomorphism

$$\Pi \Delta^n_* \cong \mathrm{Cone}(\Pi \Delta^{n-1}_*).$$

Since $\Delta^1_* = E^1_*$, the HAL now follows from theorem 6.4. □

7 The fundamental crossed complex of a simplicial set

Definition 7.1 We define $\Pi(K)$ the *fundamental crossed complex of the simplicial set* K as the free crossed complex having the elements of K_n as generators in dimension n and boundary maps given by the Homotopy Addition Lemma. In detail this gives the crossed complex $\Pi(K)$ as follows:

1. The objects are the vertices of K: $\Pi(K)_0 = K_0$;
2. The groupoid $\Pi(K)_1$ is the free groupoid associated to the directed graph K_1. So it has a free generator $x : \partial_1 x \to \partial_0 x$ for each $x \in K_1$;
3. The crossed module $\Pi(K)_2 \to \Pi(K)_1$ is the free $\Pi(K)_1$-crossed module generated by the map $\delta_2 : K_2 \to \Pi(K)_1$ given by
$$\delta_2 x = -\partial_1 x + \partial_2 x + \partial_0 x$$
for all $x \in K_2$.
4. For all $n \geqslant 3$, $\Pi(K)_n$ is the free $\Pi(K)_1$-module with generators K_n and boundary given by
$$\delta_n x = \begin{cases} (\partial_3 x)^{u_3} - \partial_0 x - \partial_2 x + \partial_1 x & \text{if } n = 3, \\ (\partial_n x)^{u_n} + \sum_{i=0}^{n-1} (-1)^{n-i} \partial_i x & \text{if } n \geqslant 4. \end{cases}$$

This construction is natural, giving a fundamental crossed complex functor of simplicial sets
$$\Pi : \mathsf{Simp} \to \mathsf{Crs}. \qquad \square$$

Remark 7.2 There are two notions of realisation of a simplicial set K, usually written $\|K\|$, and $|K|$. In the first the only identifications are along faces, and in the second the degenerate simplices are also factored out. Each realisation is a *CW*-complex with its skeletal filtration, and the Generalised van Kampen Theorem of [12], shows that there is a canonical isomorphism $\Pi K \cong \Pi(\|K\|_*)$.

8 Statement of the problem

It is standard to define the associated chain complex (A, ∂) of a simplicial abelian group A to be A_n in dimension $n \geqslant 0$ with boundary
$$\partial = \sum_{i=1}^{n} (-1)^i \partial_i .$$
Let $(DA)_n$ for $n \geqslant 0$ be the subgroup of A_n generated by the degenerate elements. Since $\partial_i \varepsilon_i = \partial_{i+1} \varepsilon_i = 1$, an easy calculation shows that $\partial(DA)_n \subseteq (DA)_{n-1}$ and so (DA, ∂) is a subchain complex of (A, ∂).

In the nonabelian case, we have more problems, but the formulae cope well with them. For the rest of this section, K is a simplicial set.

Proposition 8.1 *Let E_* be the set of degenerate elements in ΠK. Then E_* is a normal complex in ΠK.*

Proof By the rules $\partial_i \varepsilon_i = \partial_{i+1}\varepsilon_i = 1$, and the Homotopy Addition Lemma, we get immediate cancellation in $\delta_n \varepsilon_i y$ for $0 < i < n-1$ but not necessarily for $i = 0, n-1$, because of the operators, and the nonabelian structures in dimensions 1,2. Thus terms of concern are:

$$\begin{aligned}
\delta_2 \varepsilon_0 y &= -y + \varepsilon_0 \partial_1 y + y, \\
\delta_3 \varepsilon_0 y &= (\varepsilon_0 \partial_2 y)^{\partial_0 y} + (-y - \varepsilon_0 \partial_1 y + y), \\
\delta_3 \varepsilon_2 y &= (y)^{\varepsilon_0 \partial_0^2 y} - \varepsilon_1 \partial_0 y - y + \varepsilon_1 \partial_0 y, \\
&= ((y)^{\varepsilon_0 \partial_0^2 y} - y) + (y - \varepsilon_1 \partial_0 y - y) + \varepsilon_1 \partial_0 y,
\end{aligned}$$

and for $n \geqslant 4$

$$\delta_n \varepsilon_{n-1} y = (y)^{\varepsilon_0 \partial_0^{n-1} y} - y + \text{terms involving } \varepsilon_{n-2}.$$

This proves the result in view of the definitions in section 3. □

Definition 8.2 For any $k \geqslant 0$ we define a normal subcrossed complex $D_k K$ of ΠK to be K_0 in dimension 0 and in higher dimensions to be normally generated by the degenerate elements $\varepsilon_i y$ for $i < \min\{k, n-1\}, y \in K_{n-1}, n > 0$. Also, we define the *degeneracy subcomplex* $DK = \bigcup_k D_k K$, i.e. $(DK)_n = \bigcup_k (D_k K)_n$ for all $n \in \mathbb{N}$. □

Definition 8.3 We define the 0-normalised crossed complex of K to be

$$\Pi^{0N} K = (\Pi K)/D_0 K.$$

Our first result is:

Theorem 8.4 *The projection* $p^0 : \Pi K \to \Pi^{0N} K$ *has a section* q *such that* $qp^0 \simeq 1$.

The proof will occupy the rest of this section.

We first need a lemma, which will be used later as well.

Lemma 8.5 *Let* $h_1 : (\Pi K)_1 \to (\Pi K)_2$ *be a derivation. Then for* $x \in K_2$ *we have*

$$h_1 \delta_2 x = -(h_1 \partial_1 x)^{\delta_2 x} + (h_1 \partial_2 x)^{\partial_0 x} + h_1 \partial_0 x.$$

Proof

$$\begin{aligned}
h_1 \delta_2 x &= h_1(-\partial_1 x + \partial_2 x + \partial_0 x) \\
&= (h_1(-\partial_1 x + \partial_2 x))^{\partial_0 x} + h_1 \partial_0 x \\
&= (h_1(-\partial_1 x))^{\partial_2 x})^{\partial_0 x} + (h_1 \partial_2 x)^{\partial_0 x} + h_1 \partial_0 x \\
&= -(h_1 \partial_1 x)^{\delta_2 x} + (h_1 \partial_2 x)^{\partial_0 x} + h_1 \partial_0 x.
\end{aligned}$$

□

Lemma 8.6 *If* $h : \psi \simeq 1 : \Pi K \to \Pi K$ *is given by* $h_0 = \varepsilon_0$ *in dimension* 0*, and in dimension 1 by* h_1 *is* ε_0 *or* ε_1 *on the free basis given by* K_1*, then* ψ *is given in dimensions* 0, 1 *by*

$$\psi x = \begin{cases} x & \text{if } \dim x = 0, \\ x - \varepsilon_0 \partial_0 x & \text{if } \dim x = 1, \end{cases}$$

and hence $\phi \varepsilon_0 y = 0_y$ *for all* $y \in K_0$.

Proof The case $\dim x = 0$ is clear. For the case $\dim x = 1$ and for ε_0 we have

$$\begin{aligned} \psi x &= \varepsilon_0 s x + x + \delta_2(-\varepsilon_0 x) - \varepsilon_0 t x \\ &= \varepsilon_0 \partial_1 x + x - (-x + x - \varepsilon_0 \partial_0 x) \\ &= x - \varepsilon_0 \partial_0 x. \end{aligned}$$

and for ε_1 we have

$$\begin{aligned} \psi x &= 0_{sx} + x + \delta_2(-\varepsilon_1 x) - 0_{tx} \\ &= x + (x - x - \varepsilon_0 \partial_0 x) \\ &= x - \varepsilon_0 \partial_0 x. \end{aligned}$$

□

Now we define simultaneously a morphism $\psi : \Pi K \to \Pi K$ and a homotopy $h : \psi \simeq 1$ such that $\psi(D_0 K)$ is trivial.

Proposition 8.7 (0-normalisation) *Let* K *be a simplicial set. Then a homotopy* $(h, 1)$ *may be defined on generators from* K *by* $h_n = (-1)^n \varepsilon_0$*, yielding* $h : \psi \simeq 1$ *where* ψ *is given on generators by*

$$\psi(x) = \begin{cases} x & \text{if } \dim x = 0, \\ x - \varepsilon_0 \partial_0 x & \text{if } \dim x = 1, \\ (x - \varepsilon_0 \partial_0 x)^{-\varepsilon_0 t x} & \text{if } \dim x > 1. \end{cases}$$

This ψ *satisfies*

1.- $\psi(\varepsilon_0 x) = 0_{tx}$ *for all* $x \in K$.

2.- The induced map $\bar{\psi} : \Pi^{0N} K \to \Pi K$ *satisfies* $p_0 \bar{\psi} = 1$ *and* $\psi = \bar{\psi} p_0 \simeq 1$*. Thus* p_0 *is a homotopy equivalence.*

Proof To verify the formula for ψ requires working out a formula for $\overline{\varepsilon}_0 \delta_n x - \delta_{n+1} \varepsilon_0 x$, where $\overline{\varepsilon}_0$ is the derivation or operator morphism defined by ε_0 on generators, and we also have to use the crossed module rules.

Thus for $x \in K_2$, we have by Lemma 8.5:

$$\overline{\varepsilon}_0 \delta_2 x = -(\varepsilon_0 \partial_1 x)^{\delta_2 x} + (\varepsilon_0 \partial_2 x)^{\partial_0 x} + \varepsilon_0 \partial_0 x$$

while

$$\begin{aligned}\delta_3\varepsilon_0 x &= (\partial_3\varepsilon_0 x)^{\partial_0^2\varepsilon_0 x} - x - \varepsilon_0\partial_1 x + x\\ &= (\varepsilon_0\partial_2 x)^{\partial_0 x} + (-\varepsilon_0\partial_1 x)^{\delta_2 x}\\ &= (-\varepsilon_0\partial_1 x)^{\delta_2 x} + (\varepsilon_0\partial_2 x)^{\partial_0 x}, \quad \text{by centrality of } \delta_3\varepsilon_0 x\end{aligned}$$

From this we get

$$-\delta_3\varepsilon_0 x + \overline{\varepsilon}_0\delta_2 x = \varepsilon_0\partial_0 x.$$

More easily, we have for $n \geqslant 3$ and $x \in K_n$

$$\delta_{n+1}\varepsilon_0 x = (\varepsilon_0\partial_n x)^{\partial_0^{n-1}x} + \sum_{i=2}^{n}(-1)^{n+1-i}\partial_i\varepsilon_0 x$$

and

$$\overline{\varepsilon}_0\delta_n x = (\varepsilon_0\partial_n x)^{\partial_0^{n-1}x} + \sum_{i=0}^{n-1}(-1)^{n-i}\varepsilon_0\partial_i x$$

so that

$$\overline{\varepsilon}_0\delta_n x - \delta_{n+1}\varepsilon_0 x = (-1)^n\varepsilon_0\partial_0 x.$$

With these computations we get $h : \psi \simeq 1$ where ψ is the morphism given in the statement. Hence $\psi(\varepsilon_0^n v) = 0_v$ for all $n \geqslant 1$, and in fact $\psi\varepsilon_0 x = 0_{tx}$ for all $x \in K$. From this we easily deduce that $\psi(\Pi^0 K)$ is the trivial subcomplex on K_0. The morphism ψ then defines a morphism $\bar{\psi} : \Pi^{0N}K \to \Pi K$ satisfying $\bar{\psi}p_0 = 1$.

The homotopy $\overline{\varepsilon}_0$ gives also $p_0\bar{\psi} \simeq 1$. Thus $\bar{\psi}$ is a homotopy equivalence (actually a deformation retract). □

Proposition 8.8 *The crossed complex $\Pi^{0N}K$ is isomorphic by $\bar{\psi}$ to the (free) subcrossed complex of ΠK on the elements of K not of the form $\varepsilon_0 y$ for $y \in K_{n-1},\ n \geqslant 1$.*

Proof This follows from theorem 4.1. □

Remark 8.9 An advantage of working in the 0-normalised complex is that certain awkward exponents, which would vanish or not appear in the usual abelian case, now disappear in the 0-normalised complex. For example if $y \in K_1$ we have

$$\begin{aligned}\delta_2\varepsilon_1 y &= -\partial_1\varepsilon_1 y + \partial_2\varepsilon_1 y + \partial_0\varepsilon_1 y\\ &= -y + y + \varepsilon_0\partial_0 y\\ &= 0_{ty} \qquad \text{mod } \varepsilon_0.\end{aligned} \qquad \begin{aligned}\delta_2\varepsilon_0 y &= -\partial_1\varepsilon_0 y + \partial_2\varepsilon_0 y + \partial_0\varepsilon_y\\ &= -y + \varepsilon_0\partial_1 y + y\\ &= 0_{sy} \qquad \text{mod } \varepsilon_0.\end{aligned}$$

□

9 The normalised fundamental crossed complex of a simplicial set

Now we are able to define, in analogy with Mac Lane [25, §VIII.6], some further homotopies on $\Pi^{0N}(K)$ to obtain the normalisation theorem. We can model more closely the classical case on this 0-normalised crossed complex. Note that if $x \in K_n$ we write also x for the corresponding elements of both ΠK and $\Pi^{0N}K$.

Definition 9.1 For any $k \geqslant 0$ we define a subcrossed complex $D_k K \subseteq \Pi^{0N}K$ as follows:

- $(D_k K)_0 = (\Pi^{0N}K)_0 = K_0$.
- $(D_k K)_1$ is trivial, i.e. consists only of identities.
- $(D_k K)_n$ is normally generated by $\varepsilon_i y$ for $y \in K_{n-1}$, $i \leqslant k$ and $i \leqslant n-1$.

Also, we define the *degeneracy subcomplex* $DK = \bigcup_k D_k K$, i.e. $(DK)_n = \bigcup_k (D_k K)_n$ for all $n \in \mathbb{N}$. □

Now we define a sequence of homotopies from the identity to morphisms of crossed complexes sending $D_k K$ into $D_{k-1}K$ and leaving fixed the elements up to dimension $k-1$. Then, the composition of these morphisms is well defined and kills all the degeneracy subcomplex. Let us formalise this sketch.

Definition 9.2 For any $k \geqslant 0$ we define a homotopy $(\tau^k, 1) : \Pi^{0N}K \to \Pi^{0N}K$ given on the free basis $x \in K_n$ by

$$\tau^k x = \begin{cases} 0_{tx} & \text{if } n < k, \\ (-1)^{n+k}\varepsilon_k x & \text{if } n \geqslant k. \end{cases}$$ □

Therefore, for any $k \geqslant 0$ the homotopy τ^k defines a morphism of crossed complex, $\phi^k : \Pi^{0N}K \to \Pi^{0N}K$ such that $\tau^k : \phi^k \simeq 1$. Clearly $\phi^0 = \psi$. For $n \geqslant 1$ this map is given when $x \in K_n$ by

$$\phi^k x = \begin{cases} x & \text{if } n < k, \\ x + (-1)^{k+n-1}\overline{\varepsilon}_k \delta_n x + (-1)^{k+n}\delta_{n+1}\varepsilon_k x & \text{if } k \leqslant n. \end{cases}$$

where $\overline{\varepsilon}_i$ is the extension of ε_i on the basis to a derivation or operator morphism as appropriate.

Proposition 9.3 $\phi^k : \Pi^{0N}K \to \Pi^{0N}K$ *satisfies*

(i) $\phi^k D_j K \subseteq D_j K$ *when* $j < k$, *and*

(ii) $\phi^k D_k K \subseteq D_{k-1} K$.

Proof
(i) By the definition of ϕ^k we have to prove the inclusion only in the case $k \leqslant n$. In this case the generators of $(D_j K)_n$ are elements $\varepsilon_i x$ for $i \leqslant \min\{j, n-1\}$, so the definition of ϕ^k is

$$\phi^k \varepsilon_i x = \varepsilon_i x + (-1)^{k+n-1}\varepsilon_k \delta_n \varepsilon_i x + (-1)^{k+n}\delta_{n+1}\varepsilon_k \varepsilon_i x.$$

Therefore, since $\varepsilon_i x \in D_j K$, which is a subcrossed complex, we have that $\delta_n \varepsilon_i x \in D_j K$. So $\delta_n \varepsilon_i x$ can be written as a combination of $\varepsilon_p y$ with $y \in K_{n-2}$, $p \leqslant \min\{j, n-2\}$. Therefore, since we have

$$\varepsilon_k \varepsilon_p = \varepsilon_p \varepsilon_{k-1} \qquad \text{if } k > p$$

we have that $\varepsilon_k \delta_n \varepsilon_i x \in D_j K$.

On the other hand, for the same reason we have $\delta_{n+1}\varepsilon_k \varepsilon_i \in D_j K$. Therefore, $\phi^k \varepsilon_i x \in D_j K$.

(ii) Now let us prove $\phi^k D_k K \subseteq D_{k-1}K$. Since $(D_k K)_1$ is trivial we have to prove this inclusion only for generators of dimension $n \geqslant 2$.

We first deal with the case $n = 2$. Suppose then $x \in K_2$. Then

$$\begin{aligned} \delta_3 \varepsilon_1 x &= (\partial_3 \varepsilon_1 x)^{\partial_0 x} - \varepsilon_0 \partial_0 x - x + x \\ &= (\partial_3 \varepsilon_1 x)^{\partial_0 x} \qquad \mod \varepsilon_0. \\ \overline{\varepsilon}_1 \delta_2 x &= (-\varepsilon_1 \partial_1 x)^{\delta_2 x} + (\varepsilon_1 \partial_2 x)^{\partial_0 x} + \varepsilon_1 \partial_0 x \end{aligned}$$

so that $\mod \varepsilon_0$ and by centrality

$$\begin{aligned} \phi^1 x &= x + \overline{\varepsilon}_1 \delta_2 x - \delta_3 \varepsilon_1 x \\ &= x - (\varepsilon_1 \partial_1 x)^{\delta_2 x} + \varepsilon_1 \partial_0 x. \end{aligned}$$

Now it is clear that $\mod \varepsilon_0$, $x = \varepsilon_1 y$ implies $\phi^1 x = 0$.

Let $\varepsilon_i y \in (D_k K)_n$, where $i \leqslant \min\{k, n-1\}$.

If $i < k$ then $\varepsilon_i y \in D_{k-1}K$ and so $\phi^k \varepsilon_i y \in D_{k-1}K$ by (i).

It only remains to prove $\phi^k \varepsilon_k y \in D_{k-1}K$ for $y \in K_{n-1}$.

We have already done the case of $n \leqslant 2$. In general

$$\phi^k \varepsilon_k y = \varepsilon_k y + (-1)^{k+n-1} \varepsilon_k \delta_n \varepsilon_k y + (-1)^{k+n} \delta_{n+1} \varepsilon_k \varepsilon_k y,$$

for $y \in K_{n-1}$ with $n > 2$, and, in this case, $(D_{k-1}K)_n$ is abelian. We can write,

$$\varepsilon_k \delta_n \varepsilon_k y = \varepsilon_k (\partial_n \varepsilon_k y)^{\partial_0^{n-1} \varepsilon_k y} + \sum_{j=0}^{n-1} (-1)^{n-j} \varepsilon_k \partial_j \varepsilon_k y$$

and

$$\delta_{n+1} \varepsilon_k \varepsilon_k y = (\partial_{n+1} \varepsilon_k \varepsilon_k y)^{\partial_0^{n} \varepsilon_k \varepsilon_k y} + \sum_{j=0}^{n} (-1)^{n+1-j} \partial_j \varepsilon_k \varepsilon_k y.$$

Therefore $\phi^k(D_k K) \subseteq D_{k-1}K$ follows from

$$\varepsilon_k \partial_j \varepsilon_k y = \begin{cases} \varepsilon_{k-1}\varepsilon_{k-1}\partial_j y & \text{if } j < k \\ \varepsilon_k y & \text{if } j = k, k+1 \\ \varepsilon_k \varepsilon_k \partial_{j-1} y & \text{if } j > k+1 \end{cases}$$

and on the other hand,

$$\partial_j \varepsilon_k \varepsilon_k y = \begin{cases} \varepsilon_{k-1}\varepsilon_{k-1}\partial_j y & \text{if } j < k \\ \varepsilon_k y & \text{if } j = k, k+1, k+2 \\ \varepsilon_k \varepsilon_k \partial_{j-1} y & \text{if } j > k+2. \end{cases}$$

□

Now we define $\phi = \phi_0 \phi^1 \cdots \phi^k \cdots : \Pi^{0N} K \to \Pi^{0N} K$.

Notice that since $\phi^k x = x$ for $k > \dim x$, this composite is finite in each dimension.

Proposition 9.4 $\phi DK = 0$.

Proof We have $(DK)_0 = 0$ and for $n > 0$, $(DK)_n$ is generated by $\varepsilon_i y$ where $y \in K_{n-1}$ and $i \leqslant n-1$. Therefore,

$$\phi \varepsilon_i y = \phi^0 \phi^1 \cdots \phi^n \varepsilon_i y$$

If $i = n-1$ we have that $\varepsilon_i y \in (D_n K)_n$. So,

$$\phi^n \varepsilon_i y \in D_{n-1}K, \quad \phi^{n-1}\phi^n \varepsilon_i y \in D_{n-2}K, \quad \cdots, \quad \phi^0 \cdots \phi^n \varepsilon_i y \in D_0 K.$$

If $i < n-1$ we have that $\varepsilon_i y \in (D_i K)_n$. Therefore, since $\phi^j D_i K \subseteq D_i K$ for $i < j$ we have $\phi^{i+1} \cdots \phi^n \varepsilon_i y \in D_i K$. So, as above , $\phi^0 \cdots \phi^n \varepsilon_i y \in D_0 K$. □

Definition 9.5 We define the *normalised fundamental crossed complex of the simplicial set* K by

$$\Pi^\vee K = \frac{\Pi^{0N} K}{DK}.$$

Theorem 9.6 *The quotient morphism*

$$p : \Pi^{0N} K \to \Pi^\vee K$$

is a homotopy equivalence with a section q*. Further,* $\Pi^\vee K$ *has free generators given by the images of the non degenerate elements of* K.

This follows as for the 0-normalised case in the previous section.

Remark 9.7 One reason for needing the two fundamental crossed complexes is that when we apply this to the singular simplicial set SX of a topological space X we find that ΠSX is free on models in the sense of acyclic models, but the normalised complex $\Pi^\vee SX$ is not. However because $\Pi^\vee SX$ is 'smaller' than $\Pi^\vee SX$ we often want to use the normalised complex. Thus the equivalence between the two is of practical importance. □

The crossed complex $\Pi^\vee K$ homotopy equivalent to ΠK can be described as freely generated by the non degenerate simplices of K, with boundary maps given by forgetting the degenerate parts. In this sense, we get two alternative descriptions of $\Pi^\vee K$:

Proposition 9.8 *The normalised fundamental crossed complex* $\Pi^\gamma K$ *of a simplicial set* K *is given by the coend*

$$\Pi^\gamma K = \int_n K_n \times \Pi\boldsymbol{\Delta}^n,$$

taken over the category of all simplicial operators.

This is a formal translation of the definition. Notice that ΠK is the analogous coend but over the category of simplicial face operators.

Proposition 9.9 *The normalised fundamental crossed complex of a simplicial set* K *is the fundamental crossed complex of the realisation* $|K|$.

Proof This crossed complex is the same as that given in the previous Proposition, as is shown by an application of the GvKT of [12] applied to a CW-complex covered by a family of subcomplexes. □

References

[1] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press.

[2] H. J. Baues and R. Brown, *On the relative homotopy groups of the product filtration and a formula of Hopf*, J. Pure Appl. Algebra 89 (1993) 49-61.

[3] H. J. Baues and D. Conduché, *On the tensor algebra of a nonabelian group and applications*, K-Theory 5 (1991/92) 531–554.

[4] H.-J. Baues and A. Tonks, *On the twisted cobar construction*, Math. Proc. Cambridge Philos. Soc., 121, (1997), 229–245.

[5] A. L. Blakers, *Some relations between homotopy and homology groups*, Ann. Math. 49 (1948) 428-461.

[6] L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque, 225 (1994) 1-160.

[7] R. Brown, *Topology and Groupoids*, (2006) Booksurge LLC; (revised and retitled version of previous editions: McGraw Hill, Maidenhead, 1968; Ellis Horwood Ltd., Chichester, 1988).

[8] R. Brown, *Groupoids and crossed objects in algebraic topology*, Homology, homotopy and applications, 1, (1999), 1–78.

[9] R. Brown, *Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems*, Proceedings of the Fields Institute Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories, September 23-28, Fields Institute Communications 43 (2004) 101-130.

[10] R. Brown and M. Golasinski, *A model structure for the homotopy theory of crossed complexes*, Cah. Top. Géom. Diff. Cat. 30 (1989) 61-82.

[11] R. Brown and P. J. Higgins, *The algebra of cubes*, J. Pure Appl. Alg., 21, (1981), 233–260.

[12] R. Brown and P. J. Higgins, *Colimit theorems for relative homotopy groups*, J. Pure Appl. Alg, 22, (1981), 11–41.

[13] R. Brown and P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes,*, J. Pure Appl. Alg, 47, (1987), 1–33.

[14] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33–57.

[15] R. Brown and P. J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Camb. Phil. Soc. 110 (1991) 95-120.

[16] R. Brown, P.J. Higgins and R. Sivera, *Nonabelian algebraic topology*, (in preparation). Part I downloadable.

[17] R. Brown and J. Huebschmann, *Identities among relations*, in *Low dimensional topology*, (ed. R. Brown and T.L. Thickstun, Cambridge University Press, 1982), London Math. Soc. Lecture Note Series 48, 153-202.

[18] R. Brown, E. Moore, T. Porter and C. Wensley, *Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups*, Georgian Math. J., 9, (2002), 623–644.

[19] R. Brown and A. Razak Salleh, *Free crossed resolutions of groups and presentations of modules of identities among relations*, LMS J. Comput. Math., 2, (1999), 28–61 (electronic), ISSN 1461-1570.

[20] R. Brown and C.D.Wensley, *Computation and homotopical applications of induced crossed modules*, J. Symbolic Computation 35 (2003) 59-72.

[21] P.J. Higgins, *Categories and Groupoids*, van Nostrand, (1971), reprinted as Theory and Applications of Categories Reprint, 2005.

[22] Hu, S.-T., *The homotopy addition theorem*, Ann. of Math. (2), 58 (1953) 108–122.

[23] H.K. Kamps and T. Porter, *Abstract and simple homotopy theory*, World Scientific.

[24] A. Kock, *Combinatorics of curvature and the Bianci identity*, Theory and Applications of Categories, 2 (1996) 62-89.

[25] S. Mac Lane, 1967, *Homology*, number 114 in Grundlehren, Springer.

[26] S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41–48.

[27] E. J. Moore, 2001, *Graphs of Groups: Word Computations and Free Crossed Resolutions*, Ph.D. thesis, University of Wales, Bangor.

[28] A. P. Tonks, 1993, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, Bangor.

[29] G.W. Whitehead, *Homotopy theory*, Springer.

[30] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2), 42, (1941), 409–428.

[31] [CHI] J. H. C. Whitehead, *Combinatorial Homotopy I*, Bull. Amer. Math. Soc., 55, (1949), 213–245.

[32] [CHII] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453–496.

[33] J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math., 72, (1950), 1–57.