Integral inequalities

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Basic remark: If $f : [a, b] \to \mathbb{R}$ is (Riemann) integrable and nonnegative, then

$$\int_a^b f(t)dt \geq 0.$$

Equality occurs if and only if $f = 0$ almost everywhere (a.e.)

When f is continuous, $f = 0$ a.e. if and only if $f = 0$ everywhere.

Important Consequence: Monotony of integral,

$$f \leq g \quad \text{implies} \quad \int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

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In Probability Theory, integrable functions are *random variables*. Most important inequalities refer to:

$$M(f) = \frac{1}{b-a}\int_a^b f(t)dt \quad (\textit{mean value of } f)$$

$$Var(f) = M\left((f - M(f))^2\right) \quad (\textit{variance of } f)$$

$$= \frac{1}{b-a}\int_a^b f^2(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2.$$

Theorem 1 *Chebyshev's inequality: If $f, g : [a, b] \to \mathbb{R}$ have the same monotony, then*

$$\frac{1}{b-a}\int_a^b f(t)g(t)dt \geq \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right);$$

if f, g have opposite monotony, then the inequality should be reversed.

Application: Let $f : [a, b] \to \mathbb{R}$ be a differentiable function having bounded derivative. Then

$$Var(f) \leq \frac{(b-a)^2}{12} \cdot \sup_{a \leq x \leq b} \left|f'(x)\right|^2.$$

Theorem 2 (*The Mean Value Theorem*). *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function and* $g : [a,b] \to \mathbb{R}$ *be a non-negative integrable function. Then there is* $c \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)\,dx = f(c) \cdot \int_a^b g(x)\,dx.$$

Theorem 3 (*Boundedness*). *If* $f : [a,b] \to \mathbb{R}$ *is integrable, then* f *is bounded,* $|f|$ *is integrable and*

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(t)dt\right| &\le \frac{1}{b-a}\int_a^b |f(t)|\,dt \\ &\le \sup_{a\le t\le b} |f(t)|\,. \end{aligned}$$

Remark 4 *If* f' *is integrable, then*

$$\begin{aligned} 0 \le \frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le \frac{b-a}{3}\sup_{a\le x\le b}\left|f'(x)\right|. \end{aligned}$$

Remark 5 *Suppose that f is continuously differentiable on $[a,b]$ and $f(a) = f(b) = 0$. Then*

$$\sup_{a \leq t \leq b} |f(t)| \leq \frac{b-a}{2} \int_a^b \left| f'(t) \right| dt.$$

Theorem 6 (*Cauchy-Schwarz inequality*).

If $f, g : [a,b] \to \mathbb{R}$ are integrable, then

$$\left| \int_a^b f(t) g(t) dt \right| \leq \left(\int_a^b f^2(t) dt \right)^{1/2} \left(\int_a^b g^2(t) dt \right)^{1/2}$$

with equality iff f and g are proportional a.e.

Special Inequalities

Young's inequality. Let $f : [0, a] \to [0, f(a)]$ be a strictly increasing continuous function such that $f(0) = 0$. Using the definition of derivative show that

$$F(x) = \int_0^x f(t)\,dt + \int_0^{f(x)} f^{-1}(t)\,dt - xf(x)$$

is differentiable on $[0, a]$ and $F'(x) = 0$ for all $x \in [a, b]$. Find from here that

$$xy \le \int_0^x f(t)\,dt + \int_0^y f^{-1}(t)\,dt.$$

for all $0 \le x \le a$ and $0 \le y \le f(a)$.

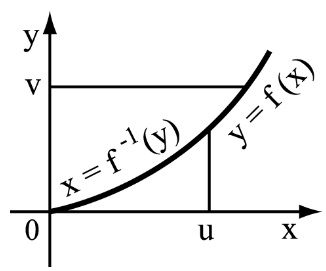

Figure 1: The geometric meaning of Young's inequality.

Special case (corresponding for $f(x) = x^{p-1}$ and $f^{-1}(x) = x^{q-1}$) : *For all* $a, b \geq 0 p, q \in (1, \infty)$ *and* $1/p + 1/q = 1$, then

$$\begin{aligned} ab &\leq \frac{a^p}{p} + \frac{b^q}{q} \quad \textit{if } p, q \in (1, \infty) \textit{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ ab &\geq \frac{a^p}{p} + \frac{b^q}{q} \quad \textit{if } p \in (-\infty, 1) \backslash \{0\} \textit{ and } \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

The equality holds if (and only if) $a^p = b^q$.

Theorems 7 and 8 below refer to arbitrary measure spaces (X, Σ, μ).

Theorem 7 (*The Rogers-Hölder inequality for* $p > 1$). Let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, and let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then fg is in $L^1(\mu)$ and we have

$$\left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu \tag{1}$$

and

$$\int_X |fg| \, d\mu \leq \|f\|_{L^p} \|g\|_{L^q} . \tag{2}$$

Thus

$$\left| \int_X fg \, d\mu \right| \le \|f\|_{L^p} \, \|g\|_{L^q} \, . \tag{3}$$

The above result extends in a straightforward manner for the pairs $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$. In the complementary domain, $p \in (-\infty, 1) \backslash \{0\}$ and $1/p + 1/q = 1$, the inequality sign should be reversed.

For $p = q = 2$, we retrieve the *Cauchy-Schwarz inequality*.

Proof. If f or g is zero μ-almost everywhere, then the second inequality is trivial. Otherwise, using the Young inequality, we have

$$\frac{|f(x)|}{\|f\|_{L^p}} \cdot \frac{|g(x)|}{\|g\|_{L^q}} \le \frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{\|g\|_{L^q}^q}$$

for all x in X, such that $fg \in L^1(\mu)$. Thus

$$\frac{1}{\|f\|_{L^p} \, \|g\|_{L^q}} \int_X |fg| \, d\mu \le 1$$

and this proves (2). The inequality (3) is immediate. ■

Remark 8 (*Conditions for equality*). *The basic observation is the fact that*

$$f \geq 0 \text{ and } \int_X f \, d\mu = 0 \text{ imply } f = 0 \ \mu\text{-almost everywhere.}$$

Consequently we have equality in (1) if, and only if,

$$f(x)g(x) = e^{i\theta} \, |f(x)g(x)|$$

for some real constant θ and for μ-almost every x.

Suppose that $p, q \in (1, \infty)$. In order to get equality in (2) it is necessary and sufficient to have

$$\frac{|f(x)|}{\|f\|_{L^p}} \cdot \frac{|g(x)|}{\|g\|_{L^q}} = \frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{\|g\|_{L^q}^q}$$

almost everywhere. The equality case in Young's inequality shows that this is equivalent to $|f(x)|^p \, / \, \|f\|_{L^p}^p = |g(x)|^q \, / \, \|g\|_{L^q}^q$ almost everywhere, that is,

$$A \, |f(x)|^p = B \, |g(x)|^q \text{ almost everywhere}$$

for some nonnegative numbers A and B.

If $p = 1$ and $q = \infty$, we have equality in (2) if, and only if, there is a constant $\lambda \geq 0$ such that $|g(x)| \leq \lambda$ almost everywhere, and $|g(x)| = \lambda$ for almost every point where $f(x) \neq 0$.

Theorem 9 (*Minkowski's inequality*). For $1 \leq p < \infty$ and $f, g \in L^p(\mu)$ we have

$$||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p} . \tag{4}$$

Proof. For $p = 1$, this follows immediately from $|f + g| \leq |f| + |g|$. For $p \in (1, \infty)$ we have

$$\begin{aligned} |f + g|^p &\leq (|f| + |g|)^p \leq (2 \sup \{|f|, |g|\})^p \\ &\leq 2^p (|f|^p + |g|^p) \end{aligned}$$

which shows that $f + g \in L^p(\mu)$.

According to the Rogers-Holder inequality,

$$
\begin{aligned}
||f+g||_{L^p}^p &= \int_X |f+g|^p \, d\mu \\
&\le \int_X |f+g|^{p-1} \, |f| \, d\mu + \int_X |f+g|^{p-1} \, |g| \, d\mu \\
&\le \left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} + \\
&\quad + \left(\int_X |g|^p \, d\mu \right)^{1/p} \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} \\
&= (||f||_{L^p} + ||g||_{L^p}) \, ||f+g||_{L^p}^{p/q} \, ,
\end{aligned}
$$

where $1/p + 1/q = 1$, and it remains to observe that $p - p/q = 1$. ■

Remark 10 *If $p = 1$, we obtain equality in (4) if, and only if, there is a positive measurable function φ such that*

$$f(x)\varphi(x) = g(x)$$

almost everywhere on the set $\{x : \, f(x)g(x) \neq 0\}$.

If $p \in (1, \infty)$ and f is not 0 almost everywhere, then we have equality in (4) if, and only if, $g = \lambda f$ almost everywhere, for some $\lambda \geq 0$.

Landau's inequality. Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function. Put $M_k = \sup_{x \geq 0} \left| f^{(k)}(x) \right|$ for $k = 0, 1, 2$. If f and f'' are bounded, then f' is also bounded and

$$M_1 \leq 2\sqrt{M_0 M_2}.$$

Proof. Notice that

$$f(x) = f(x_0) + \int_{x_0}^{x} \left(f'(t) - f'(x_0) \right) dt + f'(x_0)(x - x_0).$$

■

The case of functions on the entire real line.

Extension to the case of functions with Lipschitz derivative.

Inequalities involving convex functions

Hermite-Hadamard inequality: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \quad \text{(HH)}$$

with equality only for affine functions.

The geometric meaning.

The case of arbitrary probability measures. See [2].

Jensen's inequality: If $\varphi : [\alpha, \beta] \longrightarrow [a, b]$ is an integrable function and $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous convex function, then

$$f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(x)\,dx\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta f\left(\varphi(x)\right)\,dx. \quad \text{(J)}$$

The case of arbitrary probability measures.

An application of the Jensen inequality:

Hardy's inequality: Suppose that $f \in L^p(0,\infty)$, $f \geq 0$, where $p \in (1,\infty)$. Put

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0.$$

Then

$$\|F\|_{L^p} \leq \frac{p}{p-1}\,\|f\|_{L^p}$$

with equality if, and only if, $f = 0$ almost everywhere.

The above inequality yields the norm of the averaging operator $f \rightarrow F$, from $L^p(0,\infty)$ into $L^p(0,\infty)$.

The constant $p/(p-1)$ is best possible (though untainted). The optimality can be easily checked by considering the sequence of functions $f_n(t) = t^{-1/p} \cdot \chi_{(0,n]}(t)$.

A more general result (also known as Hardy's inequality): *If f is a nonnegative locally integrable function on $(0,\infty)$ and $p, r > 1$, then*

$$\int_0^\infty x^{p-r} F^p(x) dx \le \left(\frac{p}{r-1}\right)^p \int_0^\infty t^{p-r} f^p(t)\, dt. \tag{5}$$

Moreover, if the right hand side is finite, so is the left hand side.

This can be deduced (via rescaling) from the following lemma (applied to $u = x^p$, $p > 1$, and $h = f$).

Lemma. *Suppose that $u : (0,\infty) \to \mathbb{R}$ is convex and increasing and h is a nonnegative locally integrable function. Then*

$$\int_0^\infty u\left(\frac{1}{x}\int_0^x h(t)\, dt\right) \frac{dx}{x} \le \int_0^\infty u(h(x))\, \frac{dx}{x}.$$

Proof. In fact, by Jensen's inequality,

$$\int_0^\infty u\left(\frac{1}{x}\int_0^x h(t)dt\right)\frac{dx}{x} \le \int_0^\infty \left(\frac{1}{x}\int_0^x u(h(t))\,dt\right)\frac{dx}{x}$$
$$= \int_0^\infty \frac{1}{x^2}\left(\int_0^\infty u(h(t))\chi_{[0,x]}(t)\,dt\right)dx$$
$$= \int_0^\infty u(h(t))\left(\int_t^\infty \frac{1}{x^2}\,dx\right)dt$$
$$= \int_0^\infty u(h(t))\,\frac{dt}{t}. \quad \blacksquare$$

Exercises

1. Prove the inequalities

$$1.43 < \int_0^1 e^{x^2} dx < \frac{1+e}{2};$$

$$2e < \int_0^1 e^{x^2} dx + \int_0^1 e^{2-x^2} dx < 1 + e^2;$$

$$1 < \frac{1}{e^2(e-1)} \int_e^{e^2} \frac{x}{\ln x} dx < \frac{e}{2}.$$

2. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function having bounded derivative. Prove that

$$Var(f) \leq \frac{(b-a)^2}{12} \cdot \sup_{a \leq x \leq b} \left| f'(x) \right|^2$$

where $Var(f)$ represents the variance of f.

Hint: Put $M = \sup_{a \leq x \leq b} |f'(x)|$. Then apply the Chebyshev inequality for the pair of functions $f(x) + Mx$ and $f(x) - Mx$ (having opposite monotony).

3. If f' is integrable, then

$$0 \leq \frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{3}\sup_{a\leq x\leq b}\left|f'(x)\right|.$$

Hint: Consider the identity

$$(b-a)\,f(x) = \int_a^b f(t)dt + \int_a^x (t-a)\,f'(t)dt - \int_x^b (b-t)f'(t)dt.$$

4. Suppose that f is continuously differentiable on $[0,1]$. Prove that

$$\sup_{0\leq x\leq 1}|f(x)| \leq \int_0^1 \Big(|f(t)| + |f'(t)|\Big)\,dt$$

and

$$|f(1/2)| \leq \int_0^1 \left(|f(t)| + \frac{1}{2}|f'(t)|\right)dt.$$

3. Suppose that f is continuously differentiable on $[a, b]$ and $f(a) = f(b) = 0$. Then

$$\sup_{a \le t \le b} |f(t)| \le \frac{1}{2} \int_a^b \left| f'(t) \right| dt.$$

5. For $t > 1$ a real number, consider the function

$$f : (1, \infty) \to \mathbb{R}, \quad f(x) = x^t.$$

i) Use the Lagrange Mean Value Theorem to compare $f(7) - f(6)$ with $f(9) - f(8)$;

ii) Prove the inequality $7^t + 8^t < 6^t + 9^t$;

iii) Compute $\int_1^2 7^t dt$.

iv) Conclude that $\frac{6 \cdot 7}{\ln 7} + \frac{7 \cdot 8}{\ln 8} < \frac{5 \cdot 6}{\ln 6} + \frac{8 \cdot 9}{\ln 9}$.

6. Consider the sequence $(a_n)_n$ defined by the formula

$$a_n = \int_0^1 \frac{dx}{\underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2x}}}}_{n \text{ sqr}}}.$$

Prove that

$$\frac{1}{2} \leq a_n \leq \frac{1}{\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n-1 \text{ sqr}}} \text{ for all } n \geq 1$$

and find the limit of the sequence $(a_n)_n$.

7. Infer from the Cauchy-Schwarz inequality that

$$\ln(n+1) - \ln n < \frac{1}{\sqrt{n(n+1)}} \quad \text{for } n \text{ natural}$$

and

$$\int_0^{\pi/2} \sin^{3/2} x dx < \sqrt{\frac{\pi}{3}}.$$

8. Prove the inequalities:

$$\int_0^1 2^{x^2} dx \leq 3/2; \quad \left(\int_0^{\pi} e^{\sin x} dx\right)\left(\int_0^{\pi} e^{-\sin x} dx\right) \geq \pi^2.$$

9. Compute $\lim_{n\to\infty} \int_n^{n+1} x \sin \frac{1}{x} dx$ and $\lim_{x\to\infty} \int_{2x}^{3x} \frac{t^2}{e^{t^2}} dx$.

10. (The Bernoulli inequality). i) Prove that for all $x > -1$ we have

$$(1+x)^\alpha \geq 1+\alpha x \quad \text{if } \alpha \in (-\infty, 0) \cup (1, \infty)$$

and

$$(1+x)^\alpha \leq 1+\alpha x \quad \text{if } \alpha \in [0, 1];$$

equality occurs only for $x = 0$.

ii) The substitution $1 + x \to x/y$ followed by a multiplication by y leads us to Young's inequality (for full range of parameters).

11. (The integral analogue of the AM-GM inequality). Suppose that $f : [a, b] \to (0, \infty)$ is a continuous function. Prove that

$$e^{\frac{1}{b-a}\int_a^b \ln f(x)dx} \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

12. (Ostrowski's inequality). Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function. Prove that

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \sup_{a\le}$$

13. (Z. Opial). Let $f : [0, a] \to \mathbb{R}$ be a continuously differentiable function such that $f(0) = 0$. Prove that

$$\int_0^a f(x)dx = \int_0^a (a-x) f'(x) dx$$

and infer from this formula the inequalities:

$$\left| \int_0^a f(x)dx \right| \le \frac{a^2}{2} \sup_{0\le x\le a} \left| f'(x) \right|$$

$$\int_0^a |f(x)| \left| f'(x) \right| dx \le \frac{a}{2} \int_0^a \left| f'(x) \right|^2 dx.$$

References

[1] Constantin P. Niculescu, *An Introduction to Mathematical Analysis*, Universitaria Press, Craiova, 2005.

[2] C. P. Niculescu and L.-E. Persson, *Convex Functions and their Applications. A Contemporary Approach.* CMS Books in Mathematics **23**, Springer Verlag, 2006.