ADJOINT POLYNOMIALS OF BRIDGE-PATH AND BRIDGE-CYCLE GRAPHS
AND CHEBYSHEV POLYNOMIALS

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Abstract

The chromatic polynomial of a simple graph $G$ with $n > 0$ vertices is a polynomial $\sum_{k=1}^{n} \alpha_k(G)x(x-1) \cdots (x-k+1)$ of degree $n$, where $\alpha_k(G)$ is the number of $k$-independent partitions of $G$ for all $k$. The adjoint polynomial of $G$ is defined to be $\sum_{k=1}^{n} \alpha_k(G^c)x^k$, where $G^c$ is the complement of $G$. We find explicit formulas for the adjoint polynomials of the bridge-path and bridge-cycle graphs. Consequence, we find the zeros of the adjoint polynomials of several families of graphs.

Key words. Adjoint polynomial, Bridge graph, Chebyshev polynomial

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1. Introduction

Let $G$ be a connected simple graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. As usual, the degree of a vertex of a graph is defined as the number of edges incident to the vertex. An independent set of a graph $G$ is a subset of $V(G)$ such that no two vertices in the subset represent an edge of $G$. A partition $\{V_1, V_2, \ldots, V_k\}$ of the vertex set of $G$, where $k$ is a positive integer, is called a $k$-independent partition if each $V_i$ is a nonempty independent set of $G$. We denote the number of $k$-independent partitions of $G$ by $\alpha_k(G)$. Then the chromatic polynomial of $G$ is given by

$$P(G, u) = \sum_{k=1}^{n} \alpha_k(G)u(u-1) \cdots (u-k+1).$$

Let $N_k(G)$ be the number of spanning subgraphs of $G$ with exactly $k$ components, each of which is complete. The adjoint polynomial of $G$ is defined to be the polynomial

$$h(G, u) = \sum_{k=1}^{n} N_k(G)u^k.$$

Evidently $N_k(G) = \alpha_k(G^c)$, where $G^c$ is the complement of $G$. For $x, y \in V(G)$, let $G \cdot xy$ be the graph obtained from $G$ by identifying the vertices $x$ and $y$ and replacing multi-edges by single ones. For $xy \in E(G)$, let

$$E'(xy) = \{xu \in E(G)|u \neq y, yu \notin E(G)\} \cup \{yv \in E(G)|v \neq x, xv \notin E(G)\}.$$

For any subset $S$ of $E(G)$, we define $G - S$ to be the spanning subgraph of $G$ with edge set $E(G) - S$. If $xy \in E(G)$, define $G - xy = G - \{xy\}$. For $x \neq y$, define $G \ast xy = (G - E'(xy)) \cdot xy.$
Let families of graphs. path graph and cycle graph. As well as, we find the zeros of the a djoint polynomials of several new and bridge-tree graphs, respectively. As consequence, we obtain all the results of [3] in the cases of

\[ h(G, u) = h(G - xy, u) + h(G \star xy, u). \]

When \( xy \) is an edge of \( G \) which is not contained in any triangle of \( G \), then (1.1) can be written as

\[ h(G, u) = h(G - xy, u) + uh(G - \{x, y\}, u). \]

The adjoint polynomial (and the chromatic polynomial) are studied extensively, see [1, 2, 5, 8, 10]. So far, the chromatic polynomials of very few graphs may be computed and closed formulas of which are given. Chromaticity of the complements of paths and cycles were studied in [4, 6, 3], the formulas for adjoint polynomial of the path graph \( P_n \) with \( n \) vertices and of the cycle \( C_n \) graph with \( n \) vertices were given. Also, the zeros of these two adjoint polynomials are studied in [3].

In next sections we find explicit formulas for the adjoint polynomials of the bridge-path, bridge-cycle and bridge-tree graphs, respectively. As consequence, we obtain all the results of [3] in the cases of path graph and cycle graph. As well as, we find the zeros of the adjoint polynomials of several new families of graphs.

2. Bridge-path graph

Let \( \{G_i\}_{i=1}^d \) be a set of finite pairwise disjoint graphs with \( v_i \in V(G_i) \). The bridge-path graph

\[ \text{BP}(G_1, G_2, \ldots, G_d) = \text{BP}(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d) \]

of \( \{G_i\}_{i=1}^d \) with respect to the vertices \( \{v_i\}_{i=1}^d \) is the graph obtained from the graphs \( G_1, \ldots, G_d \) by connecting the vertices \( v_i \) and \( v_{i+1} \) by an edge for all \( i = 1, 2, \ldots, d - 1 \), see Figure 1.

![Figure 1. The bridge-path graph](image)

In order to compute the adjoint polynomial of the bridge-path graph \( \text{BP}(G_1, G_2, \ldots, G_d) \) we need the following definition and notation. Let \( A \) be any set of integer numbers, a sparse subset of \( A \) is a subset of \( A \) in which there are no two consecutive elements. Let \( G \) be any graph and let \( v \in V(G) \) be any vertex of \( G \). We denote the graph obtained from \( G \) by deleting the vertex \( v \) by \( G \setminus v \).

**Theorem 2.1.** The adjoint polynomial of the bridge-path graph \( \text{BP}(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d) \) is given by

\[
\sum_{S \in S_d} u^{|S|} \prod_{j \in S'} h_j \prod_{j \in S} h_j' h_{j-1}' = \sum_{s \geq 0} \left( \sum_{2 \leq i_2 < i_3 < \cdots < i_s \leq d} \prod_{j \in \{i_1, i_1-1, \ldots, i_s, i_s-1\}} h_j \prod_{j=1}^d h_j' h_{j-1}' \right) u^s,
\]

where \( S_d \) is the set of all sparse subsets of \( \{2, 3, \ldots, d\} \) and \( S' = \{d \setminus \{j - 1, j \mid j \in S\} \} \), \( h_i = h(G_i, u) \), \( h_i' = h(G_i \setminus v_i, u) \), and we say that \( i \ll j \) if and only if \( i + 1 < j \).
Proof. We give two proofs.

Inductive proof. We denote the adjoint polynomial of the bridge-path graph \( G = BP(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d) \) by \( F_d \). If we consider the edge \( v_d - v_{d+1} \) in the graph \( G \), then (1.1) gives

\[
F_d = h_d F_{d-1} + uh'_d h'_{d-1} F_{d-2},
\]

for all \( d \geq 3 \). By direct calculations we can state that \( F_1 = h_1 \) and \( F_2 = h_2 h_1 + uh'_2 h'_1 \). Define \( h_0 = 1 \).

Now we proof by induction that

\[
F_d = \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s.
\]

It is not hard to verify this for \( d = 0, 1, 2 \). Assume the claim holds for \( 0, 1, \ldots, d \) and let us prove it for \( d + 1 \). By (2.1) and induction hypothesis we obtain that

\[
F_d = h_d F_{d-1} + uh'_d h'_{d-1} F_{d-2}
\]

\[
= h_d \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d-1} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s + uh'_d h'_{d-1} \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d-2} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s
\]

\[
= \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d-1} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s + \sum_{s \geq 1} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s = d} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s,
\]

If we define the last sum as zero for \( s = 0 \), then we obtain that

\[
F_d = \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d} \prod_{j=1}^{s} h_{j} \prod_{j<i} h'_j h'_{i_j-1} \right) u^s,
\]

which completes the proof.

Combinatorial proof. Let \( G = BP(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d) \), since we are considering a partition \( S \) of \( G \) in which every block is a complete graph, for a vertex \( v_i \) there are only two possibilities: either it belongs to a block of size \( 2 \) consisting in an edge \( v_i v_{i+1} \) or \( v_{i-1} v_i \), or it belongs to a block of a partition of \( G_i \) in which every block is a complete graph. Let \( S \) be the set of all indices \( j \) corresponding to the endpoints of the edges \( v_{j-1} v_j \) forming a block in the partition. For a given set \( S \) of this kind, the contribution to the adjoint polynomial \( h(G, u) \) is

\[
u^{S'} \prod_{j \in S'} h_j \prod_{j \in S} h'_j h'_{j-1},
\]

where \( S' = \{ d \backslash \{ j, j-1 \mid j \in S \} \} \). By the definitions, a set \( S \) is any sparse subset of \( \{2, 3, \ldots, d\} \). Hence, by summing over all possibilities of \( S \), we obtain

\[
\sum_{S \in S_d} u^{S'} \prod_{j \in S'} h_j \prod_{j \in S} h'_j h'_{j-1},
\]

where \( S_d \) is the set of all sparse subsets of \( \{2, 3, \ldots, d\} \).
Let $G$ be any finite graph and $v \in V(G)$. Define $BP_d(G, v)$ to be the bridge-path graph $BP_{d} (G, \ldots, G; v, \ldots, v)$. The above theorem for $BP_d(G, v)$ gives
\[
h (BP_d(G, v), u) = \sum_{s \geq 0} \sum_{2 \leq i_1 \ll i_2 \ll \cdots \ll i_s \leq d} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) u^s.
\]
On the other hand,
\[
\binom{d-s}{s} = \sum_{2 \leq i_1 \ll i_2 \ll \cdots \ll i_s \leq d} 1,
\]
which is true since the number of vectors $(i_1, i_2, \ldots, i_s)$ such that $2 \leq i_1 \ll i_2 \ll \cdots \ll i_s \leq d$ equals the number of vectors $(j_1, j_2, \ldots, j_s)$ such that $1 \leq j_1 \ll j_2 \ll \cdots \ll j_s \leq d - s$ (define $j_m = i_m - m$), which it is given by $\binom{d-s}{s}$. Hence,
\[
h (BP_d(G, v), u) = \sum_{s \geq 0} \binom{d-s}{s} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) u^s,
\]
where $\binom{a}{b}$ is defined to be zero for $a < b$. Now, by using the fact that the $n$-th Chebyshev polynomial of the second kind $U_n(x)$ (see [9]) is given by
\[
U_n(x) = \sum_{k \geq 0} \binom{n-k}{k} (-1)^k (2x)^{n-2k},
\]
we obtain an explicit formula for the adjoint polynomial of the bridge-path graph $BP_d(G, v)$ as follows.

**Theorem 2.2.** Let $G$ be any finite graph and $v \in V(G)$. Then for all $d \geq 0$,
\[
h (BP_d(G, v), u) = (i \sqrt{u})^d h^d(G \setminus v, u) U_d \left( \frac{h(G, u)}{2i \sqrt{u} h(G \setminus v, u)} \right),
\]
where $i^2 = -1$ and $U_d(t)$ is the $d$-th Chebyshev polynomial of the second kind.

**Example 2.3.** Let $G$ be a graph with one vertex $v$. By definitions we have $h(G, u) = u$ and $h(G \setminus v, u) = 1$. Thus Theorem 2.2 gives
\[
h (BP_d(G, v), u) = (i \sqrt{u})^d U_d \left( \frac{\sqrt{u}}{2i} \right) = \sum_{s \geq 0} \binom{d-s}{s} u^{d-s},
\]
as showed in [3].

**Example 2.4.** More generally, assume that $G = P_m$ be a path graph on $m$ vertices $v_1, v_2, \ldots, v_m$ such that the degree of $v_1$ is 1. By Example 2.3 we have $h(G, u) = (i \sqrt{u})^m U_m \left( \frac{\sqrt{u}}{2i} \right)$ and $h(G \setminus v, u) = (i \sqrt{u})^{m-1} U_{m-1} \left( \frac{\sqrt{u}}{2i} \right)$. Thus Theorem 2.2 gives
\[
h (BP_d(G, v_1), u) = (i \sqrt{u})^m U_{m-1}^d \left( \frac{\sqrt{u}}{2i} \right) U_d \left( \frac{U_m \left( \frac{\sqrt{u}}{2i} \right)}{2U_{m-1} \left( \frac{\sqrt{u}}{2i} \right)} \right).
\]
Now we study the zeros of the adjoint polynomial of the graph $BP_d(G, v)$. 

Theorem 2.5. Let $G$ be a graph with a vertex $v$. For all $d \geq 0$,
\[ h(BP_d(G, v), u) = \prod_{j=1}^{\lfloor d/2 \rfloor} \left( h^2(G, u) + 2uh(G\setminus v, u) + 2uh(G\setminus v, u) \cos \frac{2j\pi}{d+1} \right). \]

Proof. Theorem 2.2 and the fact that the $d$-th Chebyshev polynomial of the second kind is given by
\[ U_d(x) = 2^n \prod_{j=1}^{d} (x - \cos \frac{j\pi}{d+1}) \] give
\[ h(BP_d(G, v), u) = \prod_{j=1}^{\lfloor d/2 \rfloor} \left( h^2(G, u) - 2i\sqrt{uh(G\setminus v, u)} \cos \frac{j\pi}{d+1} \right), \]
which is equivalent to
\[ h(BP_d(G, v), u) = \prod_{j=1}^{\lfloor d/2 \rfloor} \left( h^2(G, u) + 4uh(G\setminus v, u) \cos^2 \frac{j\pi}{d+1} \right). \]
Using the identity $2\cos^2 t = 1 + \cos 2t$, we get the desired result. \qed

Example 2.6. Let $G$ be a graph with one vertex $v$. By definition we have $h(G, u) = u$ and $h(G\setminus v, u) = 1$. Thus Theorem 2.5 gives
\[ h(BP_d(G, v), u) = \prod_{j=1}^{\lfloor d/2 \rfloor} \left( u^2 + 2u + 2u \cos \frac{2j\pi}{d+1} \right), \]
which shows that the zeros of the adjoint polynomial $h(P_n, u)$ are
\[ 0, 0, \ldots, 0 - 2 - 2 \cos \frac{2j\pi}{d+1}, \quad j = 0, 1, \ldots, \lfloor d/2 \rfloor, \]
as showed in [3].

![Figure 2. The graph $D_n$](image-url)

Now, let $G_1$ be a cycle on three vertices $v_1, v'_1, v''_1$ and let $G_i, i = 2, 3, \ldots, d$, be a graph with one vertex $v_i$. Let $D_d = BP(G_1, \ldots, G_d; v_1, \ldots, v_d)$. By applying (1.1) with respect to the edge $v_1v'_1$ of the cycle and Example 2.3, we have
\[ h(D_d, u) = h(BP_d(G, v), u) + h(BP_d(G, v), u) \]
\[ = \frac{1}{u} h(BP_{d+1}(G, v), u) \]
\[ = u(i\sqrt{u})^{d-1}U_{d+3} \left( \frac{\sqrt{u}}{2i} \right). \]
Hence the zeros of the adjoint polynomial of the graph $D_d$, see Figure 2, are given by
\[ 0, 0, \ldots, 0 - 2 - 2 \cos \frac{2j\pi}{d+4}, \quad j = 1, 2, \ldots, \lfloor (d+3)/2 \rfloor. \]
Now, let $G_1$ be a cycle on three vertices $v_1, v'_1, v''_1$, let $G_i$, $i = 2, 3, \ldots, d - 1$, be a graph with one vertex $v_i$, and let $G_d$ be a cycle on three vertices $v_d, v'_d, v''_d$. Let $E_d = BP(G_1, \ldots, G_d; v_1, \ldots, v_d)$. By applying (1.1) with respect to the edge $v_1v'_1$ of the cycle, we have that

$$h(E_d, u) = h(D_{d+2}, u) + h(D_{d+1}, u) = -u(i\sqrt{u})^d \left( \frac{\sqrt{u}}{i} U_{d+5} \left( \frac{\sqrt{\pi}}{2t} \right) - U_{d+4} \left( \frac{\sqrt{\pi}}{2t} \right) \right)$$

$$= (i\sqrt{u})^{d+2} U_{d+6} \left( \frac{\sqrt{u}}{2t} \right)$$

Hence the zeros of the adjoint polynomial of the graph $E_d$, see Figure 3, are given by

$$\left\{ 0, 0, \ldots, 0, -2 - 2 \cos \frac{2j\pi}{d+7}, \quad j = 1, 2, \ldots, [(d+5)/2] \right\}$$

**Example 2.7.** Let $G$ be a graph with two vertex $v$ and $w$ and edge $vw$. By definitions we have $h(G, u) = u + u^2$ and $h(G\backslash v, u) = u$. Thus Theorem 2.5 gives

$$h(BP_d(G, v), u) = \prod_{j=1}^{[d/2]} \left( u^4 + 2u^3 + 3u^2 + 2u + \cos \frac{2j\pi}{d+1} \right),$$

which shows that the zeros of the adjoint polynomial $h(P_n, u)$ are

$$\left\{ 0, 0, \ldots, 0, -1 \pm i \sqrt{2 + 2 \cos \frac{2j\pi}{d+1}}, \quad j = 0, 1, \ldots, [d/2] \right\}.$$

### 3. Bridge-cycle graph

Let $\{G_i\}_{i=1}^d$ be a set of finite pairwise disjoint graphs with $v_i \in V(G_i)$. The **bridge-cycle graph**

$$BC(G_1, G_2, \ldots, G_d) = BC(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$$

of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is the graph obtained from the graphs $G_1, \ldots, G_d$ by connecting the vertices $v_i$ and $v_{i+1}$ by an edge for all $i = 1, 2, \ldots, d - 1$ and connecting the vertices $v_1$ and $v_d$ by an edge, see Figure 4.

**Theorem 3.1.** The adjoint polynomial of the bridge-path graph $BC(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$ is given by $F(G_1, \ldots, G_d) + uh'_1h''_1F(G_2, \ldots, G_{d-1})$, where

$$F(G_1, \ldots, G_d) = \sum_{s \geq 0} \left( \sum_{2 \leq i_1 < i_2 < \cdots < i_s \leq d} \prod_{j \in [d] \backslash \{i_1, i_1-1, \ldots, i_s, i_s-1\}} h_j \prod_{j=1}^{s} h_{i_j}h'_{i_{j-1}} \right) u^s,$$

$h_i = h(G_i, u)$, $h'_i = h(G_i\backslash v_i, u)$, and we say that $i \ll j$ if and only if $i + 1 < j$. 
Figure 4. The bridge-cycle graph

Proof. We denote the adjoint polynomial of the bridge-path graph \( G = BC(G_1, \ldots, G_d; v_1, \ldots, v_d) \) by \( B_d \). If we consider the edge \( v_1 v_d \) in the graph \( G \), then (1.1) gives

\[
B_d = F_d + uh'_d F'_{d-2},
\]

for all \( d \geq 2 \), where \( F_d \) is the adjoint polynomial of the bridge-path graph \( BC(G_1, \ldots, G_d; v_1, \ldots, v_d) \) and \( F'_{d-2} \) is the adjoint polynomial of the bridge-path graph \( BC(G_2, \ldots, G_{d-1}; v_1, \ldots, v_{d-1}) \). The proof can be completed by using Theorem 2.1.

Let \( G \) be any finite graph and \( v \in V(G) \). Define \( BC_d(G, v) \) to be the bridge-cycle graph \( BC(G_1, \ldots, G_d; v_1, \ldots, v_d) \). The above theorem for \( BC_d(G, v) \) gives \( h(BC_d(G, v), u) = F_d + uh^2(G \setminus v, u)F_{d-2} \), where \( F_d \) is given by Theorem 2.2. Hence

\[
h(BC_d(G, v), u) = \sum_{s \geq 0} \binom{d-s}{s} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) + \sum_{s \geq 0} \binom{d-2-s}{s} h^{d-2-2s}(G, u) h^{2s+2}(G \setminus v, u) u^{s+1}.
\]

Let \( G \) be any finite graph and \( v \in V(G) \). Define \( BC_d(G, v) \) to be the bridge-cycle graph \( BC(G_1, \ldots, G_d; v_1, \ldots, v_d) \). The above theorem for \( BC_d(G, v) \) gives \( h(BC_d(G, v), u) = F_d + uh^2(G \setminus v, u)F_{d-2} \), where \( F_d \) is given by Theorem 2.2. Hence

\[
h(BC_d(G, v), u) = \sum_{s \geq 0} \binom{d-s}{s} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) + \sum_{s \geq 1} \binom{d-1-s}{s-1} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) u^{s-1} + \sum_{s \geq 0} \frac{d}{d-s} \binom{d-s}{s} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) u^{s},
\]

where \( \binom{a}{b} \) is assumed to be 0 whenever \( a, b < 0 \) or \( a < b \). Thus, we can state the following result.

**Corollary 3.2.** Let \( G \) be any finite graph and \( v \in V(G) \). Then for all \( d \geq 0 \),

\[
h(BC_d(G, v), u) = \sum_{s \geq 0} \frac{d}{d-s} \binom{d-s}{s} h^{d-2s}(G, u) h^{2s}(G \setminus v, u) u^{s}.
\]

The adjoint polynomial of the bridge-cycle graph can be expressed in terms of Chebyshev polynomials of the second kind as follows. From (4.1) and Theorem 2.5 we obtain that
Corollary 3.3. Let $G$ be any finite graph and $v \in V(G)$. Then for all $d \geq 0$,

$$h(BC_d(G, v), u) = (i\sqrt{u})^d h^d(G \setminus v, u) \left[ U_d \left( \frac{h(G, u)}{2i\sqrt{uh(G \setminus v, u)}} \right) - U_{d-2} \left( \frac{h(G, u)}{2i\sqrt{uh(G \setminus v, u)}} \right) \right],$$

where $i^2 = -1$ and $U_d(t)$ is the $d$-th Chebyshev polynomial of the second kind.

In order to find the zeros of the adjoint polynomial of the bridge-cycle graph, we need the trigonometric representation of the Chebyshev polynomials, namely $U_d(x) = \frac{\sin((d+1)\theta)}{\sin \theta}$ with $x = \cos \theta$. Let $x = \cos \theta$, then

$$U_d(x) - U_{d-2}(x) = \frac{\sin((d+1)\theta)}{\sin \theta} - \frac{\sin((d-1)\theta)}{\sin \theta} = \frac{\sin((d+1)\theta) - \sin((d-1)\theta)}{\sin \theta} = 2 \cos(d\theta) = 2T_d(\cos \theta),$$

where $T_d(t)$ is the $d$-th Chebyshev polynomial of the first kind (see [9]). This implies the following result.

Theorem 3.4. Let $G$ be any finite graph and $v \in V(G)$. Then for all $d \geq 0$,

$$h(BC_d(G, v), u) = 2(i\sqrt{u})^d h^d(G \setminus v, u)T_d \left( \frac{h(G, u)}{2i\sqrt{uh(G \setminus v, u)}} \right),$$

where $i^2 = -1$ and $T_d(t)$ is the $d$-th Chebyshev polynomial of the first kind.

Example 3.5. Let $G$ be a graph with one vertex $v$. By Theorem 3.4 we have that

$$h(BC_d(G, v), u) = 2(i\sqrt{u})^d T_d \left( \frac{\sqrt{u}}{2i} \right).$$

Thus $h(BC_d(G, v), u) = 0$ if and only if $u = 0$ (with multiplicity $\lceil d/2 \rceil$) or $u = -4 \cos^2 \left( \frac{(2k-1)\pi}{2d} \right)$ for some $k = 1, \ldots, \lceil d/2 \rceil$. Hence, the zeros of the polynomial $h(BC_d(G, v), u)$ are

$$0, 0, \ldots, 0, -2 - 2 \cos \left( \frac{(2k-1)\pi}{d} \right), \quad k = 1, \ldots, \lfloor d/2 \rfloor,$$

as have been shown in [3].

Example 3.6. Let $G$ be a graph with two vertices $v$ and $w$ and one edge $vw$. By Theorem 3.4 we have that

$$h(BC_d(G, v), u) = 2(i\sqrt{u})^d u^d T_d \left( \frac{1 + u}{2i\sqrt{u}} \right).$$

Thus $h(BC_d(G, v), u) = 0$ if and only if $u = 0$ (with multiplicity $d + \lceil d/2 \rceil$) or $1 + u = 2i \sqrt{u} \cos \left( \frac{(2k-1)\pi}{2d} \right)$ for some $k = 1, \ldots, \lfloor d/2 \rfloor$. Hence, the zeros of the polynomial $h(BC_d(G, v), u)$ are

$$0, 0, \ldots, 0, -1 \pm \sqrt{15 + 8 \cos \left( \frac{(2k-1)\pi}{d} \right) + \cos^2 \left( \frac{(2k-1)\pi}{d} \right)}, \quad k = 1, \ldots, \lfloor d/2 \rfloor.$$

4. Further results

Let \( \{G_i\}_{i=1}^d \) be a set of finite pairwise disjoint graphs with \( v_i \in V(G_i) \) and let \( v_0 \notin \bigcup_{i=1}^d V(G_i) \). The bridge-tree graph

\[
T(G_1, G_2, \ldots, G_d) = T(G_1, G_2, \ldots, G_d; v_0, v_1, v_2, \ldots, v_d)
\]

of \( \{G_i\}_{i=1}^d \) with respect to the vertices \( \{v_i\}_{i=1}^d \) is the graph obtained from the graphs \( G_1, \ldots, G_d \) by connecting any vertex \( v_i \) with the vertex \( v_0 \), see Figure 5.

![Figure 5. The bridge-tree graph](image)

**Theorem 4.1.** The adjoint polynomial of the tree graph \( T(G_1, G_2, \ldots, G_d; v_0, v_1, v_2, \ldots, v_d) \) is given by

\[
u \prod_{j=1}^d h(G_j, u) \left( 1 + \sum_{j=1}^d \frac{h(G_j \setminus v_j, u)}{h(G_j, u)} \right).
\]

**Proof.** We give two proofs.

**Inductive proof.** We denote the adjoint polynomial of the bridge-tree graph \( G = T(G_1, \ldots, G_d; v_0, v_1, \ldots, v_d) \) by \( T_d \). If we consider the edge \( v_0v_d \) in the graph \( G \), then (1.1) gives

\[
T_d = h(G_d, u)T_{d-1} + uh(G_d \setminus v_d, u)h(G_{d-1}, u) \cdots h(G_1, u),
\]

for all \( d \geq 1 \), where \( T_0 \) is defined as \( u \). By simple induction on \( d \) we complete the proof.

**Combinatorial proof.** Let us write an equation for \( T_d \). Since we are considering a partition of \( G \) in which every block is a complete graph, for a vertex \( v_0 \) there are only two possibilities: either it forms a single block, or it belongs to a block formed of an edge \( v_0v_d \). The contribution of the first case is given by \( u \prod_{j=1}^d h(G_j, u) \), and the contribution of the second case is given by \( uh(G_j \setminus v_j, u) \prod_{k=1, k \neq j}^d h(G_k, u) \).

Hence,

\[
T_d = u \prod_{j=1}^d h(G_j, u) + u \sum_{j=1}^d \left( h(G_j \setminus v_j, u) \prod_{k=1, k \neq j}^d h(G_k, u) \right) = u \prod_{j=1}^d h(G_j, u) \left( 1 + \sum_{j=1}^d \frac{h(G_j \setminus v_j, u)}{h(G_j, u)} \right),
\]

as required. \( \square \)

Let \( G \) be any finite graph, \( v \in V(G) \) and \( v_0 \notin V(G) \). Define \( T_d(G, v) \) to be the bridge-tree graph \( T(G, \ldots, G; v_0, v, \ldots, v) \). The above theorem for \( T_d(G, v) \) gives

\[
h(T_d(G, v), u) = uh^d(G, u) \left( 1 + d \frac{h(G \setminus v, u)}{h(G, u)} \right).
\]
For instance, if $G$ has only one vertex $v$, then $h(T_d(G, v), u) = u^d(u + d)$. More generally, let $G$ be the path graph $P_m$ with $m$ vertices where we assume that the vertex $v \in V(P_m)$ has degree 1. Then

$$h(T_d(P_m, v), u) = uh_d(P_m, u) \left(1 + d \frac{h(P_{m-1}, u)}{h(P_m, u)}\right),$$

which, by Example 2.3, implies that

$$h(T_d(P_m, v), u) = u(i \sqrt{u})^{m(d-1)}U_{m-1}^{d-1} \left[\left(i \sqrt{u}\right)^m U_m \left(\frac{\sqrt{u}}{2i}\right) + d\left(i \sqrt{u}\right)^{m-1} U_{m-1} \left(\frac{\sqrt{u}}{2i}\right)\right].$$

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References