# GENERALIZED VANDERMONDE DETERMINANTS.

THOMAS ERNST

Contents



**Abstract**
The purpose of this paper is to present a new expression for the generalized Vandermonde determinant [3], [19], and thus for the Schur function. We also obtain an equivalence relation on the set of all generalized Vandermonde determinants.

**Keywords**: generalized Vandermonde determinant; Schur function; equivalence relation; minor

## 1. Introduction

This paper is organized as follows: In this section we give a general background. In section 2 we prove an expression for a generalized Vandermonde determinant with two deleted rows and finally in section 3 we prove an expression for an arbitrary generalized Vandermonde determinant.

The generalized Vandermonde determinants are intimately connected to the symmetric group as was outlined in [10]. The basic theory of the symmetric group was developed by Young and Frobenius in the first two decades of the twentieth century. A partition [10] of $m \in \mathbb{N}$ is any finite sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n),\ m = \sum_{j=1}^{n} \lambda_j \equiv |\lambda| \tag{1}$$

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of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n \geq 0$$

containing only finitely many non-zero terms such that the weight of $\lambda$

$$\sum_{j=1}^{l(\lambda)} \lambda_j = m, \tag{2}$$

where $l(\lambda)$, the number of parts $> 0$ of $\lambda$, is called the length of $\lambda$. We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeros at the end. Let $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$ be a monomial, and consider the polynomial $a_\alpha$ obtained by antisymmetrizing $z^\alpha$:

$$a_\alpha \equiv a_\alpha(x_1, \ldots, x_n) \equiv \sum_{w \in S_n} \epsilon(w) w(z^\alpha) = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \vdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \vdots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \vdots & x_n^{\alpha_n} \end{vmatrix}, \tag{3}$$

where $\epsilon(w)$ is the sign of the permutation $w$. Given partitions $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ of $m$ and $\delta : (n-1, n-2, \ldots, 1, 0)$ of $\binom{n}{2}$, the generalized Vandermonde determinant is defined by

$$a_{\lambda+\delta} \equiv \begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \vdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \vdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \vdots & x_n^{\lambda_n} \end{vmatrix}. \tag{4}$$

The Schur function $s_\lambda$, defined by

$$s_\lambda \equiv \frac{a_{\lambda+\delta}}{a_\delta}, \tag{5}$$

is a quotient of two homogeneous skew-symmetric polynomials and is thus a homogeneous symmetric polynomial [10].

The Schur functions are particularly relevant to discussions of the quantum Hall effect [17], [15]; to the characters of irreducible representations of $U(n)$ [21, p. 213], [17]; to the characters of $Gl(n, \mathbb{C})$, which can be expressed in terms of Schur functions [14],[1, p. 237], [20, ch. VII.6] ; to the characters of $Sp(2n+1, \mathbb{C})$ [11]; to the characters of $Sp(2n, \mathbb{R})$ [6]; and to the characters of the simple Lie algebras $sl(n, \mathbb{C})$ and $su(n)$, which have the same representations [4]. Recently, Schur functions have been used in $q$-calculus [13],[8]. Further generalized Vandermonde determinants occur in Sato theory, where the variables are partial differential operators [12], [2].

There is a rich literature on Schur functions, e.g. [7], [9], [18], but I have been unable to find a similar expression to the one presented in this paper there.

## 2. Generalized Vandermonde determinants with two deleted rows.

In this section we will prove a general equation for a generalized Vandermonde determinant with two deleted rows in terms of the elementary symmetric polynomials $e_n$. We will henceforth use $k_j$ as summation indices and we will use both $\lambda$ and $l_j$ (which denotes the deleted rows) to characterize the generalized Vandermonde determinant.

Lemma 2.1.

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix} = \left( \prod_{1\leq j<i\leq n} (x_i - x_j) \right) e_{n-l}(x_1, \dots, x_n). \tag{6}$$

Proof. By the properties of the roots of an equation [16], we know that the Vandermonde determinant

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_{n+1}^n \end{vmatrix} = \left( \prod_{1\leq j<i\leq n} (x_i - x_j) \right) \left( \prod_{1\leq j\leq n} (x_{n+1} - x_j) \right) =$$

$$= \left( \prod_{1\leq j<i\leq n} (x_i - x_j) \right) (-1)^n \sum_{l=0}^{n} (-1)^l e_{n-l}(x_1, \dots, x_n) x_{n+1}^l =$$

$$= \sum_{l=0}^{n} (-1)^{l+n} \left( \prod_{1\leq j<i\leq n} (x_i - x_j) e_{n-l}(x_1, \dots, x_n) \right) x_{n+1}^l.$$

On the other hand, an expansion with respect to column $n+1$ gives

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_{n+1}^n \end{vmatrix} = \sum_{l=0}^{n} (-1)^{l+n} (-1)^{\binom{n}{2}} a_{\lambda+\delta} x_{n+1}^l.$$

Equating coefficients of $x_{n+1}^l$, we are done.

Expand the determinant

$$\Xi = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_{n+1}^{n+1} \end{vmatrix} \tag{7}$$

with respect to column $n+2$. The minors $\Xi_{0,m}$ are defined by (8):

$$\Xi = \sum_{m=0}^{n+1} x_{n+1}^m (-1)^{m+n+1} \Xi_{m+1} \quad . \tag{8}$$

Starting with (6), the strategy in this chapter will be to express these minors in two different ways to obtain an equation which gives an expression for the generalized Vandermonde determinants in terms of multiple sums of elementary symmetric functions.

*Remark* 1. The $y$ in (7) is a dummy variable, which is used in the computations. The $x_j$ are the variables that will enter in the generalized Vandermonde determinants.

To simplify notation, we introduce the following operator $\Theta_{n,l}^N$:

DEFINITION 1. Let $0 \leq N \leq l \leq n$, put $k = (k_1, \dots, k_N)$, $1 \leq k_j \leq n, 1 \leq j \leq N$ and let $U_{n,N}$ be the subset of $\{1, \dots, n\}^N$, where no repetitions are allowed. Then

$$\begin{aligned} \Theta_{n,l}^N \equiv & \sum_{k \in U_{n,N}} \\ & \left[ \prod_{j=1}^{N-1} (x_1 \dots \prod_{i=1}^{j} \widehat{x_{k_i}} \dots x_n) \right] (x_1 \dots \prod_{i=1}^{N} \widehat{x_{k_i}} \dots x_n)^2 \times \\ & \times e_{n-l}(x_1, \dots, \widehat{x_{k_1}}, \dots, \widehat{x_{k_N}}, \dots, x_n) \times \\ & \times (-1)^{k_1 + \dots + k_N + I(\pi)_N} \left( \prod_{\substack{1 \leq j < i \leq n, \\ i,j \neq \{k_1, \dots, k_N\}}} (x_i - x_j) \right), \end{aligned} \tag{9}$$

where $I(\pi)_N$ is the number of inversions of the permutation [5] $\pi = (k_1, \dots, k_N)$, where the $k_j$ are counted in increasing order as $1, \dots, N$. In particular,

$$\Theta_{n,l}^0 \equiv - \left( \prod_{1 \leq j < i \leq n} (x_i - x_j) \right) x_1 \dots x_n e_{n-l}(x_1, \dots, x_n); \tag{10}$$

$$\Theta^1_{n,l} \equiv \sum_{k=1}^{n} (-1)^k (x_1 \dots \widehat{x_k} \dots x_n)^2 \times \times \left( \prod_{1 \le j < i \le n, i,j \ne \{k\}} (x_i - x_j) \right) e_{n-l}(x_1, \dots, \widehat{x_k}, \dots, x_n); \tag{11}$$

$$\Theta^n_{n,n} \equiv (-1)^n \left( \prod_{1 \le j < i \le n} (x_i - x_j) \right). \tag{12}$$

THEOREM 2.2. *Let $2 \le l \le n+1$. Then*

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \sum_{k=1}^{n} (x_1 \dots \widehat{x_k} \dots x_n)^2 \times \times \left( \prod_{1 \le j < i \le n, i,j \ne \{k\}} (x_i - x_j) \right) e_{n-l+1}(x_1, \dots, \widehat{x_k}, \dots, x_n) \times \times (-1)^{k+1} \equiv -\Theta^1_{n,l-1}. \tag{13}$$

Proof. Our aim is first to expand the minor $\Xi_2$ with respect to row 1, and then expand all the minors but the first one with respect to column 1 and use (6).

(14)

$$\Xi_2 \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \left( \prod_{1 \le j < i \le n} (x_i - x_j) \right) (x_1 \ldots x_n)^2 +$$

$$+ \sum_{k=1}^{n} (-1)^k \begin{vmatrix} y^2 & x_1^2 & \cdots & \widehat{x_k^2} & \cdots & x_n^2 \\ y^3 & x_1^3 & \cdots & \widehat{x_k^3} & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_k^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix} =$$

$$= \left( \prod_{1 \le j < i \le n} (x_i - x_j) \right) (x_1 \ldots x_n)^2 + \sum_{k=1}^{n} (x_1 \ldots \widehat{x_k} \ldots x_n)^2 \times$$

$$\times \sum_{l=2}^{n+1} (-1)^{l+k} y^l \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_k} & \cdots & x_n \\ x_1^2 & \cdots & \widehat{x_k^2} & \cdots & x_n^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_k^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & \widehat{x_k^{n-1}} & \cdots & x_n^{n-1} \end{vmatrix} =$$

$$\overset{\text{by}(6)}{=} \left( \prod_{1 \le j < i \le n} (x_i - x_j) \right) (x_1 \ldots x_n)^2 + \sum_{l=2}^{n+1} \sum_{k=1}^{n} y^l e_{n-l+1}$$

$$(x_1, \ldots, \widehat{x_k}, \ldots, x_n)(-1)^{k+l} (x_1 \ldots \widehat{x_k} \ldots x_n)^2 \left( \prod_{\substack{1 \le j < i \le n, \\ i,j \ne \{k\}}} (x_i - x_j) \right).$$

Now expand the determinant $\Xi_2$ with respect to column 1.

$$\Xi_2 = \left( \prod_{1 \le j < i \le n} (x_i - x_j) \right) (x_1 \ldots x_n)^2 +$$

$$+\sum_{l=2}^{n+1}(-1)^{l+1}y^l \begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix}.$$

Finally equate the coefficients of $y^l$.

THEOREM 2.3. *Let* $3 \leq l \leq n+1$. *Then*

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ x_1^3 & \cdots & x_n^3 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = \tag{15}$$

$$= \sum_{k \in U_{n,2}} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n) e_{n-l+1}\left(x_1, \ldots, \widehat{x_{k_1}}, \ldots\right.$$

$$\left. \ldots, \widehat{x_{k_2}}, \ldots, x_n\right) (-1)^{1+k_1+k_2+I(\pi)_2} (x_1 \ldots \widehat{x_{k_1}} \ldots \widehat{x_{k_2}} \ldots x_n)^2 \times$$

$$\times \left( \prod_{1 \leq j < i \leq n, i,j \neq \{k_1,k_2\}} (x_i - x_j) \right) \equiv -\Theta_{n,l-1}^2.$$

PROOF. Expand the determinant $\Xi_3$ with respect to row 1.

$$\Xi_3 \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_n \\ y^3 & x_1^3 & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (x_1 \ldots x_n) \begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix} +$$

$$+\sum_{k_1=1}^{n}(-1)^{k_1} \begin{vmatrix} y & x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n \\ y^3 & x_1^3 & \cdots & \widehat{x_{k_1}^3} & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1}. \end{vmatrix}.$$

Expand this last determinant with respect to column 1 and use (6) and (13).

$$\Xi_3 = \left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\neq\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k_1=1}^{n}(-1)^{k_1+l}\begin{vmatrix} x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n\\ x_1^3 & \cdots & \widehat{x_{k_1}}^3 & \cdots & x_n^3\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^l} & \cdots & \widehat{x_{k_1}^l} & \cdots & \widehat{x_n^l}\\ \vdots & \cdots & \vdots & \vdots & \cdots\\ x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1}\end{vmatrix}=$$
$$=\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\neq\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k_1=1}^{n}(-1)^{k_1+l}x_1\cdots\widehat{x_{k_1}}\cdots x_n\times \tag{16}$$
$$\times\begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1\\ x_1^2 & \cdots & \widehat{x_{k_1}}^2 & \cdots & x_n^2\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_1}^{l-1}} & \cdots & \widehat{x_n^{l-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_1^n & \cdots & \widehat{x_{k_1}^n} & \cdots & x_n^n\end{vmatrix}\overset{\text{by}(13)}{=}$$
$$=\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n\; e_{n-1}(x_1,\ldots,x_n)+$$
$$+\,y\sum_{k_1=1}^{n}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^3\left(\prod_{1\le j<i\le n,i,j\neq\{k_1\}}(x_i-x_j)\right)(-1)^{k_1}+$$
$$+\sum_{l=3}^{n+1}y^l\sum_{k\in U_{n,2}}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)\times$$
$$\times\, e_{n-l+1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_2}},\ldots,x_n)(-1)^{l+k_1+k_2+I(\pi)_2}\times$$
$$\times\,(x_1\ldots\widehat{x_{k_1}}\ldots\widehat{x_{k_2}}\ldots x_n)^2\left(\prod_{1\le j<i\le n,i,j\neq\{k_1,k_2\}}(x_i-x_j)\right).$$

The factor $(-1)^{I(\pi)_2+1}$ comes from a renumbering of the summation indices. By expanding the determinant $\Xi_3$ with respect to column 1 we get

$$\Xi_3 = \left(\prod_{1\le j<i\le n}(x_i - x_j)\right) x_1 \ldots x_n \; e_{n-1}(x_1,\ldots,x_n) +$$

$$+ (-1)y \begin{vmatrix} 1 & \cdots & 1 \\ x_1^3 & \cdots & x_n^3 \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} + \tag{17}$$

$$+ \sum_{l=3}^{n+1} (-1)^{l+1} y^l \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ x_1^3 & \cdots & x_n^3 \\ \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_n^l} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix}.$$

The theorem now follows by equating the coefficients of $y^l$.

We can now state a general theorem for a generalized Vandermonde determinant with two deleted rows.

THEOREM 2.4. *Let* $0 < l_1 < l_2 < n+1$. *Then*

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_1}} & \cdots & \widehat{x_n^{l_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_2}} & \cdots & \widehat{x_n^{l_2}} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (-1)^{\binom{l_1+1}{2}} \Theta_{n,l_2-1}^{l_1}. \tag{18}$$

For the proof we need to prove the following lemma for $\Xi_l$ by induction.

LEMMA 2.5. *Let* $2 \le l \le n+2$. *Then*

$$\Xi_l = \sum_{k_1=0}^{l-2} y^{k_1} (-1)^{k_1+\binom{k_1+1}{2}} \Theta_{n,l-2}^{k_1} + \sum_{k_1=l}^{n+1} y^{k_1} (-1)^{k_1+1+\binom{l}{2}} \Theta_{n,k_1-1}^{l-1}. \tag{19}$$

PROOF. The lemma is true for $l=2$ by (14) and for $l=3$ by (16). Assume that the induction hypothesis is true for $\Xi_{l-1}$.

$$\Xi_{l-1}=\sum_{k_1=0}^{l-3} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-3}^{k_1}+\sum_{k_1=l-1}^{n+1}(-1)^{k_1+1+\binom{l-1}{2}}y^{k_1}\Theta_{n,k_1-1}^{l-2}. \tag{20}$$

On the other hand an expansion of $\Xi_{l-1}$ with respect to column 1 gives

$$\begin{aligned}
\Xi_{l-1}=&\left(\prod_{1\le j<i\le n}(x_i-x_j)\right)x_1\ldots x_n\ e_{n-l+3}(x_1,\ldots,x_n)\\
&+\sum_{k_1=1}^{l-3}(-1)^{k_1}y^{k_1}\begin{vmatrix} 1 & \cdots & 1\\ x_1 & \cdots & x_n\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}}\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}\end{vmatrix}\\
&+\sum_{k_1=l-1}^{n+1}(-1)^{k_1+1}y^{k_1}\begin{vmatrix} 1 & \cdots & 1\\ x_1 & \cdots & x_n\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}}\\ \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}}\\ \vdots & \ddots & \vdots\\ x_1^{n+1} & \cdots & x_n^{n+1}.\end{vmatrix}.
\end{aligned} \tag{21}$$

Equating the coefficients for $y^{k_1}$ of the two last equations gives first (10) and (13), then for $1 < k_1 < l-2$

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (-1)^{\binom{k_1+1}{2}} \Theta^{k_1}_{n,l-3}, \tag{22}$$

and finally for $l-2 < k_1 < n+2$

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1}} & \cdots & \widehat{x_n^{k_1}} \\ \vdots & \ddots & \vdots \\ x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (-1)^{\binom{l-1}{2}} \Theta^{l-2}_{n,k_1-1}. \tag{23}$$

The verification of the induction hypothesis is completed by expanding $\Xi_l$ with respect to row 1.

$$\Xi_l \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y & x_1 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{y^{l-1}} & \widehat{x_1^{l-1}} & \cdots & \widehat{x_n^{l-1}} \\ \vdots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} \end{vmatrix} = (x_1 \ldots x_n) \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix} +$$

$$+ \sum_{k_2=1}^{n} (-1)^{k_2} \begin{vmatrix} y & x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ y^2 & x_1^2 & \cdots & \widehat{x_{k_2}^2} & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{y^{l-1}} & \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_2}^{l-1}} & \cdots & \widehat{x_n^{l-1}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_2}^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix}.$$

Expand the last determinant with respect to column 1 and use (13), (22) and (23).

$$\Xi_l = \sum_{k_1=0}^{1} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1} + \sum_{k_2=1}^{n}\sum_{k_1=2}^{l-2}(-1)^{k_2+k_1+1}y^{k_1}\times$$

$$\times x_1\cdots\widehat{x_{k_2}}\cdots x_n \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_{k_2}^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix} + \sum_{k_2=1}^{n}\sum_{k_1=l}^{n+1}$$

$$(-1)^{k_2+k_1}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n \times \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-2}} & \cdots & \widehat{x_{k_2}^{l-2}} & \cdots & \widehat{x_n^{l-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix} =$$

$$= \sum_{k_1=0}^{1} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1} + \sum_{k_2=1}^{n}\sum_{k_1=2}^{l-2}(-1)^{k_2+1+k_1+\binom{k_1+1}{2}}\times$$

$$\times y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\Theta_{n-1,l-3}^{k_1-1}[x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n]+$$

$$+\sum_{k_2=1}^{n}\sum_{k_1=l}^{n+1}(-1)^{k_2+k_1+\binom{l-1}{2}}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\times$$

$$\times\Theta_{n-1,k_1-2}^{l-2}[x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n] = \sum_{k_1=0}^{l-2} y^{k_1}(-1)^{k_1+\binom{k_1+1}{2}}\Theta_{n,l-2}^{k_1}+$$

$$+\sum_{k_1=l}^{n+1} y^{k_1}(-1)^{k_1+1+\binom{l}{2}}\Theta_{n,k_1-1}^{l-1}.$$

Finally equation (18) follows from (22) or (23).

We are now going to prove an equation for a generalized Vandermonde determinant with an arbitrary number of rows deleted. This equation will be a natural generalization of theorem 2.4!

## 3. Generalized Vandermonde determinants with any number of deleted rows.

We start with a definition of some numbers which will be used throughout this chapter:

DEFINITION 2. Let $\{b_{\lambda,j,t,u}\}_{j=1}^{t}$ be natural numbers, $t, u \in \mathbb{N}$, which satisfy the equation

$$b_{\lambda,t,t,u} - 2 + \sum_{j=1}^{t-1}(b_{\lambda,j,t,u} - 1) = u. \tag{24}$$

These numbers also satisfy the inequalities

$$b_{\lambda,j,t,u} \geq 1, 1 \leq j \leq t-1;\ b_{\lambda,t,t,u} \geq 2. \tag{25}$$

The maximum value of $b_{\lambda,j,t,u}$ is in fact equal to the jumps in degree (for $j = 1, \ldots, t$) of the generalized Vandermonde determinant as the following equation shows:

$$max(b_{\lambda,j,t,u}) = \lambda_{n-j} + 1 - \lambda_{n-j+1},\ j = 1, \ldots, t. \tag{26}$$

To compute the $b_{\lambda,j,t,u}$, we apply the following procedure: First $b_{\lambda,1,t,u}$ gets a maximal value, then $b_{\lambda,2,t,u}$, etc. until the 'increment' $u$ is exhausted.

*Remark* 2. The following theorem defines an equivalence relation $E$ on the set $a_{\lambda+\delta}$ of all generalized Vandermonde determinants. The equivalence class $E_{n+s,l_{s-1}+1}$ is defined by the following two criteria:

1. The highest power in the determinant is $n + s - 1$.
2. 

$$\sum_{j=1}^{l_{s-1}-s+2} b_{\lambda,j,l_{s-1}-s+2,u} = l_{s-1} + 1. \tag{27}$$

Any two generalized Vandermonde determinants which belong to the same equivalence class $E_{n+s,l_{s-1}+1}$ can be transformed to each other by the method shown in the following proof.

*Example* 1. The following table gives an example of how to compute $I(\pi)_3 + \sum_{i=1}^{3} k_i$ mod 2 (n=5):

| $k_1$ | $k_2$ | $k_3$ | $I(\pi) + \sum_i k_i$ | $k_1$ | $k_2$ | $k_3$ | $I(\pi) + \sum_i k_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 3 | 4 | 1 | 1 |
| 1 | 2 | 4 | -1 | 3 | 4 | 2 | -1 |
| 1 | 2 | 5 | 1 | 3 | 4 | 5 | 1 |
| 1 | 3 | 2 | -1 | 3 | 5 | 1 | -1 |
| 1 | 3 | 4 | 1 | 3 | 5 | 2 | 1 |
| 1 | 3 | 5 | -1 | 3 | 5 | 4 | -1 |
| 1 | 4 | 2 | 1 | 4 | 1 | 2 | -1 |
| 1 | 4 | 3 | -1 | 4 | 1 | 3 | 1 |
| 1 | 4 | 5 | 1 | 4 | 1 | 5 | -1 |
| 1 | 5 | 2 | -1 | 4 | 2 | 1 | 1 |
| 1 | 5 | 3 | 1 | 4 | 2 | 3 | -1 |
| 1 | 5 | 4 | -1 | 4 | 2 | 5 | 1 |
| 2 | 1 | 3 | -1 | 4 | 3 | 1 | -1 |
| 2 | 1 | 4 | 1 | 4 | 3 | 2 | 1 |
| 2 | 1 | 5 | -1 | 4 | 3 | 5 | -1 |
| 2 | 3 | 1 | 1 | 4 | 5 | 1 | 1 |
| 2 | 3 | 4 | -1 | 4 | 5 | 2 | -1 |
| 2 | 3 | 5 | 1 | 4 | 5 | 3 | 1 |
| 2 | 4 | 1 | -1 | 5 | 1 | 2 | 1 |
| 2 | 4 | 3 | 1 | 5 | 1 | 3 | -1 |
| 2 | 4 | 5 | -1 | 5 | 1 | 4 | 1 |
| 2 | 5 | 1 | 1 | 5 | 2 | 1 | -1 |
| 2 | 5 | 3 | -1 | 5 | 2 | 3 | 1 |
| 2 | 5 | 4 | 1 | 5 | 2 | 4 | -1 |
| 3 | 1 | 2 | 1 | 5 | 3 | 1 | 1 |
| 3 | 1 | 4 | -1 | 5 | 3 | 2 | -1 |
| 3 | 1 | 5 | 1 | 5 | 3 | 4 | 1 |
| 3 | 2 | 1 | -1 | 5 | 4 | 1 | -1 |
| 3 | 2 | 4 | 1 | 5 | 4 | 2 | 1 |
| 3 | 2 | 5 | -1 | 5 | 4 | 3 | -1 |

The following examples show how to compute the $b_{\lambda,j,t,u}$.

*Example* 2. We start with the determinant defined by $\lambda = (5,5,0)$. To compute it we move backwards from the determinant defined by $\lambda = (2,2,0,0,0,0)$, which has $u = 0$, and the $b$s are easy to calculate. Transform to $\lambda = (3,3,0,0,0)$, which has $b_{\lambda,1,3,1} = 1$, $b_{\lambda,2,3,1} = 1$, $b_{\lambda,3,3,1} = 3$. Transform to $\lambda = (4,4,0,0)$, which has $b_{\lambda,1,2,2} = 1$, $b_{\lambda,2,2,2} = 4$. And finally transform to $\lambda = (5,5,0)$, which has $b_{\lambda,1,1,3} = 5$.

*Example* 3. We start with the determinant defined by $\lambda = (4,2,0)$. To compute it we move backwards from the determinant defined by $\lambda = (2,0,0,0,0)$, which has $u = 0$, and the $b$s are easy to calculate. Transform to $\lambda = (3,1,0,0)$, which has $b_{\lambda,1,3,1} = 1$, $b_{\lambda,2,3,1} = 2$, $b_{\lambda,3,3,1} = 2$. And finally transform to $\lambda = (4,2,0)$, which has $b_{\lambda,1,2,2} = 3$, $b_{\lambda,2,2,2} = 2$.

THEOREM 3.1. *Let* $0 < l_1 < l_2 < \ldots < l_s < n+s-1$. *Further assume that we have deleted* $s$ *rows from the original Vandermonde determinant. For convenience put*

$$N \equiv l_{s-1} - s + 2. \tag{28}$$

*Then*

$$(-1)^{\binom{n}{2}} a_{\lambda+\delta} \equiv \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_1}} & \cdots & \widehat{x_n^{l_1}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_n^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_n^{n+s-1} \end{vmatrix} = (-1)^{\binom{N+1}{2}} \times \tag{29}$$
$$\times \sum_{k \in U_{n,N}} \prod_{j=1}^{N} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_n)^{b_{\lambda,j,N,s-2}} \times$$
$$\times\, e_{n-l_s+s-1}(x_1, \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_N}}, \ldots, x_n) \times$$
$$\times\, (-1)^{k_1+\ldots+k_N+I(\pi)_N} \left( \prod_{\substack{1 \le j < i \le n, \\ i,j \ne \{k_1,\ldots,k_N\}}} (x_i - x_j) \right).$$

PROOF. We introduce an 'induction variable' $m$, which goes from 1 to $s-2$. This variable counts the number of times we move backwards in the same equivalence class $E_{n+s,l_{s-1}+1}$. For each $m$ the exponent of the extracted monomial decreases and the number of variables in the new determinant decreases by one. The induction hypothesis is true for $m = 0$ by theorem (2.4).

Assume that the theorem is true for $m-1$.

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-1-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{s-m}}} & \cdots & \widehat{x_{n+s-1-m}^{l_{s-m}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-1-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-1-m}^{n+s-1} \end{vmatrix} = \tag{30}$$

$$= (-1)^{\binom{l_{s-1}+2-m}{2}} \sum_{k\in U_{n+s-1-m,l_{s-1}+1-m}}$$

$$\prod_{j=1}^{l_{s-1}+1-m} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_{n+s-1-m})^{b_{\lambda,j,l_{s-1}+1-m,m-1}} \times$$

$$\times\, e_{n-l_s+s-1}(x_1, , \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_{l_{s-1}+1-m}}}, \ldots, x_{n+s-1-m}) \times$$

$$\times (-1)^{k_1+\ldots+k_{l_{s-1}+1-m}+I(\pi)_{l_{s-1}+1-m}} \left( \prod_{\substack{1\le j<i\le n+s-1-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}+1-m}\}}} (x_i - x_j) \right).$$

An expansion of the same determinant with respect to the last column gives

$$\sum_{l=0}^{l_{s-1-m}} (-1)^{n+s-m+l} x_{n+s-1-m}^{l} \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l}} & \cdots & \widehat{x_{n+s-2-m}^{l}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-2-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-2-m}^{n+s-1} \end{vmatrix} + \tag{31}$$

$$+ G(x_{n+s-1-m}),$$

where $G(x_{n+s-1-m})$ are the terms with $x_{n+s-1-m}$ of order $l_{s-1-m}+1$ and higher.

We now have to pick out the terms which contain $x_{n+s-1-m}^{l_{s-1-m}}$ from (30) i.e. we have to solve the equation

$$l_{s-1-m} = \sum_{h=1}^{j} b_{\lambda,h,l_{s-1}+1-m,m-1}. \tag{32}$$

This implies that

$$k_{j+1} = n+s-1-m. \tag{33}$$

Further

$$b_{\lambda,j,l_{s-1}+1-m,m-1} + b_{\lambda,j+1,l_{s-1}+1-m,m-1} \mapsto b_{\lambda,j,l_{s-1}-m,m}. \tag{34}$$

When $m=1$, the $b$s have the following values:

$$b_{\lambda,1,l_{s-1}-1,1} = 1, \ldots, b_{\lambda,l_{s-2},l_{s-1}-1,1} = 2, b_{\lambda,l_{s-2}+1,l_{s-1}-1,1} = 1,$$

$$, \ldots, b_{\lambda,l_{s-1}-2,l_{s-1}-1,1} = 1, b_{\lambda,l_{s-1}-1,l_{s-1}-1,1} = 2.$$

If $l_{s-2}+1 = l_{s-1}$, the last term shall be $b_{\lambda,l_{s-2},l_{s-1}-1,1} = 3$.

This is accomplished by putting $j = l_{s-2}+1$ and $k_j = n+s-2$ followed by suppression of the index $k_j$, which results in a reordering of the $k_i$s. By equating equations (30) and (31) , cancelling the factor

$(-1)^{n+s-m+l_{s-1-m}}$ and equating the coefficients of $x_{n+s-1-m}^{l_{s-1-m}}$, we obtain

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+s-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{s-m-1}}} & \cdots & \widehat{x_{n+s-2-m}^{l_{s-m-1}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_s}} & \cdots & \widehat{x_{n+s-2-m}^{l_s}} \\ \vdots & \ddots & \vdots \\ x_1^{n+s-1} & \cdots & x_{n+s-2-m}^{n+s-1} \end{vmatrix} =$$

$$= (-1)^{\binom{l_{s-1}+2-m}{2}+n+s-m+l_{s-1-m}} \sum_{k \in U_{n+s-2-m,l_{s-1}-m}}$$

$$\prod_{j=1}^{l_{s-1}-m} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_{n+s-2-m})^{b_{\lambda,j,l_{s-1}-m,m}} \times$$

$$\times e_{n-l_s+s-1}(x_1, \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_{l_{s-1}-m}}}, \ldots, x_{n+s-2-m}) \times$$

$$\times(-1)^{k_1+\ldots+k_{l_{s-1}-m}+(n+s-1-m)+I(\pi)_{l_{s-1}+1-m}}\left(\prod_{\substack{1\leq j<i\leq n+s-2-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}-m}\}}}(x_i-x_j)\right).$$

Obviously, $I(\pi)_{l_{s-1}+1-m}=I(\pi)_{l_{s-1}-m}(-1)^{l_{s-1}-m-l_{s-1}-m}$, and a computation shows that the last determinant is equal to

$$(-1)^{\binom{l_{s-1}+1-m}{2}}\sum_{k\in U_{n+s-2-m,l_{s-1}-m}}$$

$$\prod_{j=1}^{l_{s-1}-m}(x_1\ldots\prod_{i=1}^{j}\widehat{x_{k_i}}\ldots x_{n+s-2-m})^{b_{\lambda,j,l_{s-1}-m,m}}\times$$

$$\times e_{n-l_s+s-1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{s-1}-m}}},\ldots,x_{n+s-2-m})\times$$

$$\times(-1)^{k_1+\ldots+k_{l_{s-1}-m}+I(\pi)_{l_{s-1}-m}}\left(\prod_{\substack{1\leq j<i\leq n+s-2-m,\\ i,j\neq\{k_1,\ldots,k_{l_{s-1}-m}\}}}(x_i-x_j)\right).$$

Now put $m=s-2$ to obtain equation (29).

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Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden

*E-mail address*: `Thomas.Ernst@math.uu.se`