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On local computation of the electromagnetic force field in deformable bodies

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Abstract. We present a finite-element based method for the computation of force fields, using the language and concepts of differential geometry, in the familiar "eddy-currents" context (displacement currents ignored). The approach is Lagrangian (comoving mesh) and the degrees of freedom are magnetomotive forces along material lines which coincide with the edges of the mesh. The essential result is: the edge-element implementation of the h-formulation of the eddy-currents problem can serve without any modification when conductors are mobile and deformable, provided the assembly of finite-element matrices is done at each time step. A simple formula gives the virtual work of electromagnetic forces, hence a notion of "nodal forces" from which an interpolation of the force field can be derived if needed.

1. Introduction

The present state of the art as regards the computation of force fields is not completely satisfactory. Tools like Maxwell's tensor, or $j \times b$, are not fool-proof. The integration of Maxwell's tensor yields only global quantities, and there are difficulties when the surface of integration crosses a small airgap. There are problems too with the $j \times b$ formula, because numerical methods generally treat differently the two equations $\mathrm{curl}\, h = j$ and $\partial_t b + \mathrm{curl}\, e = 0$: when one of them is exactly satisfied, the other one is only enforced in weak form, and for this reason, the two fields in the $j \times b$ formula are not consistently approximated. (Cf. [15] for another form of this argument.)

There are also more deeply buried concerns, which compound these numerical difficulties: some hesitation in the eddy-currents community about the very meaning of *local* force (some argue that only integrated, global quantities make sense); the wrong belief that force is either $q\, h$ or $(j + j_a) \times b$, where q and j_a are equivalent (fictitious) charges or currents (which does not mean that methods based on such formulas are necessarily wrong! cf. [10, 11, 20]); the confusion between the computation of electromagnetic *force* (which certainly should be independent on the nature of the matter which has to sustain it!) and of the *stress* the underlying material will suffer (which of course depends on other forces applied to it). One cannot escape the uncomfortable feeling that the whole subject is still fraught with uncertainty and controversy.

Uncertainty: As programmers, we should like to find somewhere some unambiguous and general enough formula to implement, that would give local force in all reasonably imaginable circumstances. What then are we to make with so different formulas as

$$f = -\nabla p_0 + (8\pi)^{-1}\nabla[H^2\rho\,(\partial_\rho\mu)_T] - H^2/8\pi\,\nabla\mu + \mu/c\; j \times H$$

([12], p. 190—this is for a liquid medium, and p_0 is the pressure), or

$$M \cdot \nabla B + M \times (\nabla \times B) - \nabla B \cdot (v \times (M \times v)/c^2)$$

([16], p. 218—M is magnetization), or else

$$f_i = (J \times B)_i - \mu_0/2\; H_j H_k\, \partial_i \mu_{jk} - {}^1\!/_2\; \partial_j(H_i B_j - H_j B_i) \tag{1}$$

([18], when μ does not depend on strain, and neglecting Coulomb forces between electric charges)?

Controversy: Are all these formulas the same basic one in different guises (unlikely), or perhaps in different contexts, or do we have to accept that different theories will yield different expressions for the force *density*, only *global* forces making sense? Are there different possible formulations, which may differ in their predictions, but which "if complete and properly used, lead to identical predictions of all measurable quantities" [16, p. 5]? (Note that the issues raised in the last two sentences are not strictly the same.) Brown [7] thus explains the situation: "The possibility of deriving formulas that are seemingly incompatible, but actually equivalent, arises from the fact that there is no unique way of separating the force exerted by one part of a polarized body on an adjacent part into a long-range term, to be calculated by macroscopic electrostatics, and a short-range term, to be calculated by means of 'stresses'." (He is actually discussing electrostatics, which is not our present concern, but his argument carries over to magnetostatics unchanged.) Moon [14], having pointed out that "(...) until recently, the fact that for magnetizable materials the representation of internal magnetic forces was not unique led to much confusion" (p. 19), seems to take a similar stand: "(...) to uniquely determine the internal forces in a magnetized body, the constitutive relations between strain, stress, and magnetic fields must be rigorously formulated using a nonlinear energy approach (...)". On the other hand, Robinson [18] derives (1) by the following strategy (p. 94): "To calculate the magnetostatic forces and stress we consider the work that has to be done against magnetic forces (...) to assemble a system of magnetic media in which currents of density J flow." No inkling here of a possible multiplicity of local force expressions, or of the necessity to invoke material constitutive laws. Who is right?

So is the stage. We propose here a line of approach which purports to overcome these difficulties. Its main characteristics are: 1°- The use of a Lagrangian formulation (Maxwell equations in "material form"); 2°- The use of a virtual work principle: force is obtained by differentiating the magnetic coenergy with respect to the parameters which describe the configuration; 3°- A "comoving" finite elements discretization (the mesh is linked with the matter and moves with it); 4°- The interpretation of degrees of freedom (DoF) as magnetomotive forces (mmf) along edges of the mesh, i.e., along material lines. These four characteristics cannot be separated: only with the use of DoF as described can one give precise meaning to the familiar rule that "force is the derivative of coenergy, currents being held at their present values, with respect to

configuration". This rule will be given here in a slightly different form, under which it has the status of a mathematical theorem. It implies that one obtains the force *field* by choosing the displacement field as configuration parameter.

The theory is free of logical gaps, essentially because of the use of a precise mathematical notation. The toll to pay for this, however, is heavy: it consists in giving up the traditional apparatus of vector analysis, to the benefit of a differential-geometric approach. For this reason, the present paper is, for a large part, a tutorial on differential geometry, as applied to eddy-current equations. We start from basic principles, and proceed until a finite-element procedure for the computation of fields and forces has been obtained. This is a modest objective, and certainly not enough to put an end to the still raging controversy about force [2]. But we shall at least be able to propose to the programmer an unambiguous *algorithm* to code.

Let us stress however that this is a *theory* (about forces) *within eddy-currents theory*: it acknowledges from the onset the common practice which consists in dropping the displacement currents term $\varepsilon\,\partial_t e$ from Maxwell equations, as if the permittivity ε was zero. Such a theory inherits all limitations which are those of eddy-currents theory itself. Specifically, if $\varepsilon = 0$, no energy is attributable to the *electric* field, therefore no virtual work due do *electric* forces is involved in a virtual displacement: "there are no Coulomb forces in eddy-currents theory"—or rather, recourse to a complementary theory is necessary if one wants to determine these forces. This delicate point is discussed, along with a few other issues, in the last Section. (Thanks are due to the referee who insisted on this necessary clarification.) Meanwhile, Section 2 develops the needed geometrical basis, Section 3 sets eddy-current equations, Section 4 explains how to discretize them using edge-elements and how to compute force.

2. Some mathematical concepts and notations

First, a notational point: By "$f \in U \to V$", we mean that function f maps the set U (or a part of it) into the set V.

Our model of physical space will be the Euclidean three-dimensional space E_3 of ordinary geometry, with its scalar product "$\cdot$". Time is a real parameter (the same all over the place: the theory will *not* be relativistic). Space is absolute, and matter moves with respect to it.

Imagine first that matter fills all space. To describe how it moves, we say where in space E_3 the material particle x is to be found at time t: at $u_t(x) \in E_3$. So at each time t we have a mapping u_t from the set of all particles (call it X) into E_3. Material continuity is expressed by postulating a topology on X and by requiring the continuity of u_t. An injective continuous map $u \in X \to E_3$ is called a *placement*. Kinematics is thus described by giving a one-parameter family $t \to u_t$ of placements ($t \in [0, T]$), called *trajectory*, continuous in some suitable sense with respect to t. We note v the vector field (on E_3) of material velocities, such that $v(u(x), t) = {}^{d}/_{dt}[u_t(x)]$. The derivative $\partial_t u$ at time t is thus the map $x \to \{u_t(x), v(u(x), t)\}$, which legitimates the notational abuse $v = \partial_t u$.

Of course, one might view X as being the same as E_3, via the identification $x \to u_0(x)$, for instance. But there is an essential difference. On E_3 we have a *metric structure* (notion of distance, angle, etc.), induced by the dot-product, whereas no such structure exists on X, since the distance between two particles varies with time. The right structure for X is that of a *differentiable manifold*, i.e., something which looks locally like three-dimensional space, but stripped of metric information. Let's call it the *material manifold*. Note that permeability μ, conductivity σ, etc., which are material properties, are functions, independent of time, defined on X.

Now, we keep the same terminology if matter occupies only a part of space. In that case, a part of X corresponds to air or vacuum, the other part, M say, to "hard" matter. Still, the placement u_t continues to be from *all* X into E_3. This will be useful at the discretization stage, where we shall conceive X as meshed by tetrahedra, each node n being placed at $u_n(t) \in E_3$ at time t. Of course, a whole family of different placements is now compatible with a given kinematics: the latter only specifies $u_t(x)$ for $x \in M$. We shall have to worry about the invariance of force with respect to a change of compatible placement.

The notions of *tangent vector* and of *tangent space* make sense in a manifold. (See any course in differential geometry, for this and for what follows. Our favorites are [8], [19].) One notes T_xX the tangent space at x, TX the manifold of all tangent vectors, or "material" vectors.) Say (with much abuse) that material points around x can each be noted as $x + \delta x$, where δx is a "small" tangent vector. Then, a placement u sends x and $x + \delta x$ to points $u(x)$ and $u(x + \delta x)$ respectively. The vector joining these is, if one neglects high-order terms, a linear function of δx: $u(x + \delta x) - u(x) = u_*(x)\, \delta x + ...$, where $u_*(x) \in T_xX \to T_{u(x)}E_3$, the so-called *tangent mapping*. As an example, there is a field of tangent vectors V on X such that $v = u_*V$ (at time t): $x + Vdt$ is the material point that were, at time $t - dt$, where x now is. One says that v is the *push-forward* of V by u.

The next concept is *covector at* x: an element of the dual space, $T_x^* X$, to T_xX. More generally, a *p-covector at* x is a p-linear map $\Omega_x \in T_xX \times ... \times T_xX \to IR$, *alternating* (or *skew-symmetric*), i.e., $\Omega_x(V_1, V_2, ..., V_P) = -\Omega_x(V_2, V_1, ..., V_P)$, etc. We'll also consider covectors at points of E_3. If ω_y is such a covector at point $y = u(x)$, the mapping $\{v_1, ..., v_P\} \to \omega_y(u_*(x)v_1, ..., u_*(x)v_P)$ is a p-covector at x, that is noted $\Omega_x = u^*(y)\omega_y$. A *differential form* (DF) of degree p on X, or *p-form*, is a field $x \to \Omega_x$ of p-covectors, noted Ω. We'll also consider DF on E_3. If ω is one of these, we note $u^*\omega$ the DF Ω we have just defined. One calls it the *pull-back* of ω by u.

Differential forms, as one knows, are objects that can appear under a summation sign: if Ω is a p-form, it can be integrated over a submanifold S of dimension p (a line, if $p = 1$, a surface if $p = 2$, etc.). If $\Omega = u^*\omega$, one has $\int_S \Omega = \int_{u(S)} \omega$. One may define a linear operator d, called *exterior derivative*, that transforms p-forms into (p + 1)-forms, in such a way that *Stokes theorem* holds: $\int_S d\Omega = \int_{\partial S} \Omega$, where ∂S is the boundary of S. It follows from this definition that d and u^* commute:

(2) $$u^*\, d\omega = d\, u^*\omega = d\, \Omega.$$

The *contraction*, or *inner product* of a p-form Ω by a vector field W, noted $i_W\Omega$ (a (p – 1)-form) will also play a role. It's the field of (p – 1)-covectors $\{V_2, ..., V_p\} \rightarrow \Omega(W, V_2, ..., V_p)$. Again, there is commutativity:

(3) $$u^* i_w \omega = i_W \Omega = i_W u^* \omega,$$

if $w = u_* W$.

In E_3, all these concepts have their counterparts in the language of vector analysis, for if w is a vector-field on E_3, the maps $v \rightarrow w_y \cdot v$ and $\{v_1, v_2\} \rightarrow w_y \cdot v_1 \times v_2$ are respectively a 1- and a 2-covector at y, hence a 1-form and a 2-form, denoted ${}^1w$ and ${}^2w$. Similarly, one may associate with a function φ a 0-form ${}^0\varphi$ and a 3-form ${}^3\varphi$ (which is the field of covectors at y, $\{v_1, v_2, v_3\} \rightarrow v_1 \cdot v_2 \times v_3 \ \varphi_y$). It's then easy to see that relations like $h = \text{grad}\ \varphi$, $b = \text{rot}\ a$, $\partial_t \rho = -\ \text{div}\ j$, etc., translate as ${}^1h = d\,{}^0\varphi$, ${}^2b = d\,{}^1a$, $\partial_t\,{}^3\rho = -\,d\,{}^2j$, etc., in the language of differential geometry. One also has $i_v\,{}^1w = {}^0(w \cdot v)$ and $i_v\,{}^2w = {}^1(w \times v)$. But note all this correspondences are only made possible by the metric structure of E_3 (dot-product and cross-product), and thus are not available on the material manifold.

We shall now discuss Maxwell equations. To avoid cumbersome notation like ${}^1h$, ${}^2b$, etc., let us agree that h, b, j, etc., will denote differential forms in E_3, and that the corresponding vector-fields will be set in boldface: **h**, **b**, **j**, etc. (Thus, boldface will serve as a reminder that such entities are not "true" vectors, contrary to, e.g., the velocity field v.) Differential forms living on X will be noted, as we have been doing so far, with small capitals: H, B, J, etc.

3. Eddy-current equations

3.1 Eddy-current equations in material form ("Lagrangian" formulation)

Maxwell equations, when expressed on the material manifold, and when one neglects displacement currents, read as:

(4) $$\partial_t B + dE = 0, \quad dH = J,$$

where E and H are 1-forms on X, and B and J, 2-forms. To understand their meaning, begin with E. Place a unit electric charge at point $x \in X$. Consider a virtual displacement δx of this charge. Some virtual work δW is involved. We *define* the electric field at x as the linear mapping $\delta x \rightarrow \delta W$, i.e., as a covector, noted E_x, hence the 1-form E as the *mathematical* representation of the electric field, conceived as the *physical* agency which is responsible for forces acting on charges at rest. Similarly, the 2-covector J_x at x tells anything about the current flow at x, if we interpret $J_x(V_1, V_2)$ as the quantity of electric charge that crosses, each second, the parallelogram built on V_1 and V_2 (we gloss over questions of *orientation* here), and B contains all the information about the magnetic part of the electromagnetic field (it tells about the flux of induction, and also, as we'll see later, about Lorentz force). As for H, as a 1-form, its integrals along material lines make sense. Such an integral is called a *magnetomotive force* (mmf).

One recognizes Faraday's law and Ampère's theorem in eqs. (4). No mention of the placement u was made, so *these laws are always valid, whatever the movement of matter* in space. Of course, movement will affect the field, but only through the intermediation of behavior laws.

3.2 Eddy-current equations, in "Eulerian" formulation

We now want to "push-forward" the eqs. (4). For this, let h, b and j be such that $H = u^*h$, $B = u^*b$, $J = u^*j$. (The corresponding vector fields **h**, **b** and **j** are the familiar ones.) Thanks to (2), one has $dh = j$, as in eq. (4), right. Faraday's law is more of a problem, for it is certainly *not* true that $u^*\partial_t b = \partial_t(u^*b) = \partial_t B$ when u depends on time.

To find the right formula, consider a material surface S, and evaluate the time-derivative $\partial_t(\int_S u_t^* b)$. There are two contributions to this quantity, one due to the variations of b, one to the variations of the placement. We need not detail this computation, which is fairly classical (though usually performed in the vectorial framework, and assuming $\mathrm{div}\,\mathbf{b} = 0$, which we don't, for the moment). The result is

$$\partial_t u^*b = u^*[\partial_t b + (i_v d + d i_v)b], \tag{5}$$

and this is valid for forms of all degrees. One sets $L_v = i_v d + d i_v$. This operator is called the *Lie derivative* in the direction of v. Thanks to (2) and (3), one has again the commutativity relation

$$u^* L_v \omega = L_V \Omega = L_v u^* \omega. \tag{6}$$

One then *defines* e, a 1-form on E_3, *not* as $(u^*)^{-1}E$, but in such a way that $E = u^*(e - i_v b)$, all this at time t, hence $\partial_t b + de = 0$, after (5). (It will help at this stage to remark that $e - i_v b$ has $\mathbf{e} + v \times \mathbf{b}$ as its vectorial equivalent.) Thus, e is the electric field "in the laboratory frame", i.e., the field of forces acting on charges *at rest in* E_3. In contrast, the virtual work of the field $(u^*)^{-1}E$ on a unit charge at $y = u(x)$ undergoing the virtual displacement $\delta y = u_*(x)\delta x$ is $E_x(\delta x) \equiv ((u^*)^{-1}E)_{u(x)}(u_*(x)\delta x) = (e - i_v b)_y(\delta y)$, which identifies $e - i_v b$ as the *Lorentz force*. We conclude that eqs. (4) push forward as

$$\partial_t b + de = 0,\ dh = j, \tag{7}$$

i.e., eddy-current equations are fully *covariant*, or "form invariant", when one goes from the material reference system to the laboratory one. (Actually, they would also look like (7) for an observer that would move with respect to "our" E_3, and for whom the placement would be different.)

3.3 Behavior laws

Let us introduce two new concepts of differential geometry, *metric* and *Hodge operator*. A metric g on X is just a (smooth) field of scalar-products on each tangent space T_xX. We already have one on E_3. Any placement u induces one on X, noted g_u, as follows: the scalar-product of two vectors at x, V_1 and V_2, is defined as $g_u(V_1, V_2) = u_*(x)V_1 \cdot u_*(x)V_2$. This metric (for which the distance between two points x and x' is

the distance of their images $u(x)$ and $u(x')$ in E_3) is thus changing with time according to the kinematics $t \to u_t$.

Assuming a metric g, the Hodge operator $*$ turns p-forms into $(n - p)$-forms ($n = 3$, here), as follows. Let Ω_x be a p-covector at x, and $\{V_1, ..., V_n\}$ a direct frame of n vectors, orthonormal with respect to the metric g. Define $*\Omega_x$ as the map $\{V_{p+1}, ..., V_n\} \to \Omega_x(V_1, ..., V_p)$. By linearity, this defines a unique $(n - p)$-covector at x. Now, the field $x \to *\Omega_x$ is an $(n - p)$-form, that we note $*\Omega$, called the *Hodge dual* to Ω. When $g = g_u$, induced by the placement u, it will be denoted by $*_u$, and the plain $*$ will be reserved for the Hodge operator in E_3. (Remark that $*\,{}^1w = {}^2w$, $*\,{}^2w = {}^1w$, $*\,{}^0\varphi = {}^3\varphi$, $*\,{}^3\varphi = {}^0\varphi$.) The Hodge operator is well-behaved with respect to pull-back:

(8) $$u^* *\omega = *_u u^*\omega = *_u \Omega.$$

Coming back to physics, we now contend that

(9) $$B = \mu *_u H,$$

in the case of a material of permeability μ. A similar relation (Ohm's law) holds between J and E:

(10) $$J = \sigma *_u E + J^g.$$

Here, σ is a positive function on X which vanishes outside a fixed region $C \subset M \subset X$ (the "conductor") and J^g a given 2-form, time-dependent, vanishing in C, with $dJ^g = 0$. We assume for simplicity that μ and σ are scalars which depend on position x only, but generalizations are easy: they could be tensors, or depend on local strain. (In that case, they would indirectly depend on u. Magnetostriction, for instance, can be dealt with this way.)

That (9) and (10) should be correct is by no means obvious, so let us check this point. At point x, where the conductivity is σ (and, for simplicity, $J^g = 0$), carve a parallelepipedic chunk of conductor, built on the three material vectors V_1, V_2, V_3, and choose these such that, once placed in E_3 by u, their images u_*V_i, $i = 1, 2, 3$, are all of length 1 and mutually orthogonal. Let U be the electromotive force between the faces joined by V_3, i.e., by definition, $E_x(V_3)$. The resistance of this cube of matter being σ^{-1}, a current $I = \sigma U$ should cross the face built on V_1 and V_2. But this is indeed what (10) predicts, since by the very definition of $*_u$, and of J, one has $\sigma E_x(V_3) = J_x(V_1, V_2) = I$. A similar justification holds for (9).

To sum up: the full set of eddy-current equations is, in Lagrangian form,

(11) $$\partial_t B + dE = 0, \quad dH = J, \quad B = \mu *_u H, \quad J = \sigma *_u E + J^g,$$

and in Eulerian form,

(12) $$\partial_t b + de = 0, \quad dh = j, \quad b = \mu_u * h, \quad j = \sigma_u * (e - i_v b) + j^g,$$

where j^g, μ_u, σ_u are such that $J^g(t) = u_t^* j^g(t)$, $\mu_u(u(x)) = \mu(x)$ and $\sigma_u(u(x)) = \sigma(x)$. Translating (12) into traditional language yields

$$\partial_t \mathbf{b} + \mathrm{rot}\, \mathbf{e} = 0, \quad \mathrm{rot}\, \mathbf{h} = \mathbf{j},$$

$$\mathbf{b} = \mu_u \mathbf{h}, \quad \mathbf{j} = \sigma_u (\mathbf{e} + v \times \mathbf{b}) + \mathbf{j}^g,$$

which is comforting, even though we won't use it.

3.4 Weak formulation

A last and necessary step, before finite element discretization, consists in finding some weak form for eqs. (11). The tool for that is the *wedge-product* $\wedge$: given a 1- and a 2-covector H_x and B_x at x, we may build from them a 3-covector $H_x \wedge B_x$, thus defined: $H_x \wedge B_x(V_1, V_2, V_3) = H_x(V_1)\, B_x(V_2, V_3) + H_x(V_2)\, B_x(V_3, V_1) + H_x(V_3)\, B_x(V_1, V_2)$, hence a 3-form $H \wedge B$. Similar definition on E_3. Of course (as usual!), $u^*(h \wedge b) = u^*h \wedge u^*b = H \wedge B$. As a 3-form, $H \wedge B$ can be integrated over X. A simple computation will show that $\int_X H \wedge B = \int_{E_3} h \wedge b = \int_{E_3} \mathbf{h} \cdot \mathbf{b}$. From this remark, we draw a simple heuristic principle: to find weak forms for equations expressed in terms of differential forms, just use "test forms" where "test fields" would normally go, a*nd substitute wedge products for dot products.*

By applying this principle, one finds the following weak form for eqs. (11): *find* H(t) (a time-dependent 1-form on X) *such that* $dH = J^g$ *outside* C, *and*

$$(13) \qquad \int_X \partial_t(\mu *_u H) \wedge H' + \int_X \sigma^{-1} *_u dH \wedge dH' = 0$$

for all "test-forms" H' (independent of t) *such that* $dH' = 0$ *outside* C.

This may seem less palatable than the standard weak form of the eddy-currents equation, which is, in the absence of motion, $\int \partial_t(\mu \mathbf{h}) \cdot \mathbf{h}' + \int \sigma^{-1}\, \mathrm{rot}\, \mathbf{h} \cdot \mathrm{rot}\, \mathbf{h}' = 0$. Yet, we'll find the finite-element discretization of (13) very easy, thanks to our last borrowing from differential geometry: Whitney forms [21].

4. Discretization

4.1 Whitney forms, as finite-elements

Assume a tetrahedral mesh of X. (These are "soft" tetrahedra, like Dali's clocks, since the notion of straightline or of plane doesn't make sense in X. The only requirement is that the topology of this mesh be compatible with finite element practice: the intersection of two tetrahedra is a face, or an edge, or a node, or is empty.) Call $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$ the sets of its nodes, edges and faces. A node n is at a specific point x_n of X, so it will be placed at time t at $u_n(t) = u_t(x_n)$. We thus have the concept of a mesh which is linked with matter, a "comoving" mesh. An edge e also corresponds to a single and unchanging material line, a face to a material surface, etc.

To node n, we attach a positive continuous function λ_n, with $\lambda_n(x_m) = 1$ if n = m, else 0. We want the λ_ns to vanish on all tetrahedra that do not contain node n, and to form a partition of the unit: $\sum_{n \in \mathcal{N}} \lambda_n = 1$. (The usual requirement about linearity has to be treated with some care in the present context: see next remark.) Now, let e be the edge that goes from n to m. *Whitney's edge-form* W_e is, by definition,

$$W_e = \lambda_n \, d\lambda_m - \lambda_m \, d\lambda_n .$$

Under the placement u, W_e is the pull-back of a 1-form on E_3 that we shall denote by ${}^u w_e$: one has $W_e = u^* \, {}^u w_e$. The associated vector-field, ${}^u\mathbf{w}_e$, will be recognized by the reader as the standard *edge-element*, associated with the image of the mesh by the placement u. It depends on the latter, and thus changes with time.

Remark. About piecewise linearity, what we require is that the push-forward ${}^u\lambda_n$, defined by ${}^u\lambda_n(y) = \lambda_n(u^{-1}(x))$, be linear for *one* placement. Then, provided all placements u_t are linearly interpolated from the nodal placements $u_n(t)$, ${}^u\lambda_n u$ is indeed piecewise linear at all times. It may help here to carefully check that ${}^u\mathbf{w}_e = {}^u\lambda_n \, \mathrm{grad} \, {}^u\lambda_m - {}^u\lambda_m \, \mathrm{grad} \, {}^u\lambda_n$. ◊

Note that the mesh must contain a *finite* number of tetrahedra, in practice. This is not contradictory with meshing all X, since some tetrahedra may have an image under u which extends to infinity in E_3. (This point of view is probably the best from which to deal with the concept of "infinite" finite elements [1], but such is not our present concern.)

We distinguish the set $\mathcal{F}_0$ of faces that are not inside the region of X where $\sigma > 0$. If $f \in \mathcal{F}_0$, the intensity through it is known in advance (and computable from the problem's data): it's the integral over face f of the given current density $J^g(t)$. Let us denote the vector of these quantities (when f spans $\mathcal{F}_0$) by $\underline{j}^g(t)$.

Now comes the decisive move. Let us assign to each edge $e \in \mathcal{E}$ a real, time-dependent value $\underline{h}_e(t)$, and let us look for H in (13) as the linear combination

$$H(t) = \Sigma_{e \in \mathcal{E}} \, \underline{h}_e(t) \, W_e . \tag{14}$$

Call $\underline{h}$ the vector of edge-DoF: $\underline{h} = \{\underline{h}_e : e \in \mathcal{E}\}$. The constraint $dH = J^g$ outside C is approximately satisfied by imposing that intensities through faces of $\mathcal{F}_0$ be equal to their known values. This (due to properties of Whitney forms, [4]) results in the algebraic relation $R \, \underline{h}(t) = \underline{j}^g(t)$, where R, a discrete analog to rot in the non-conducting region, is a matrix whose elements are all 0, 1 or −1. To satisfy (13) approximately, in the best possible way, we demand that it hold for all test-forms $H' = \Sigma_{e \in \mathcal{E}} \, \underline{h}_e' \, W_e$, under the algebraic constraints $R\underline{h}' = 0$ on the values $\underline{h}_e'$. All this results in a system of algebraic-differential equations of the form:

$$\partial_t(A \, \underline{h}) + B \, \underline{h} = R^t \underline{a}, \tag{15}$$

$$R \, \underline{h} = \underline{j}^s(t), \tag{16}$$

where $\underline{a}$ is a Lagrange multiplier corresponding to the constraint $R\underline{h}' = 0$. The dimension of $\underline{a}$ is the number of faces in $\mathcal{F}_0$, and we shall obtain the symmetric matrices A and B in a moment.

Note at this juncture the physical interpretation of the DoF $\underline{h}_e(t)$ in (14): they are (time-dependent) *magnetomotive forces* along the edges of the mesh on X (the material mesh-edges), or as well, along the images of these edges under u_t (the moving mesh-edges in E_3). It is very pleasant to have such an *intrinsic* interpretation,

independent of any coordinate system. Less pleasant is the fact that $\underline{h}$ is not a vector of *independent* DoF. This is why users of this method won't stop at (15)(16), but will either introduce nodal (scalar) DoF $\underline{\varphi}_n$ in the region where $\sigma = 0$ (hence the "h–φ" formulation, with edge-variables in conductors, node-variables elsewhere), or look for a spanning-tree in the mesh graph, in order to get rid of $\underline{a}$ and to obtain a genuine differential system. We ignore these issues here.

One may have remarked that (15)(16) are precisely the discretized form of the eddy-current equations, in h-formulation and with edge-elements, in the case of *non-moving* conductors (see, e.g., [5]). So what's the difference, and *where is the velocity hiding?* Answer: in the dependence of matrices A and B on u_t, therefore on time. Indeed, the entries of A and B, indexed over couples of edges e and e', are

$$A_{ee'}(u) = \int_X \mu *_u W_e \wedge W_{e'}\ ,\quad B_{ee'}(u) = \int_X \sigma^{-1} *_u dW_e \wedge dW_{e'}\ .$$

To *practically* compute them, one evaluates the contributions of each tetrahedron T to these integrals, as expressed in *physical* space (E_3), that is

$$A^T_{ee'}(u) = \int_{u(T)} \mu_u\ {}^u\mathbf{w}_e \cdot {}^u\mathbf{w}_{e'}\ ,\ B^T_{ee'}(u) = \int_{u(T)}\ \sigma_u^{-1}\ \mathrm{rot}\ {}^u\mathbf{w}_e \cdot \mathrm{rot}\ {}^u\mathbf{w}_{e'}\ ,$$

where, let us recall, the ${}^u\mathbf{w}_e$ are the edge-element basis fields in configuration u. This reversal to vectorial notation is not compulsory, but it well shows that $A(u)$ and $B(u)$ are obtained by strictly *the same assembly process* as in the case of non-moving conductors. The only difference (of course, quite meaningful, in terms of computing cost) is that this assembly has to be done anew *at each time-step* of the numerical scheme by which one will solve (15)(16). The outcome of this computation is the vector of edge-mmf's, as a function of time. The physical magnetic field is obtained via the summation $\mathbf{h} = \Sigma_{e \in \mathcal{E}}\ \underline{h}_e\ {}^u\mathbf{w}_e$. Notice the two reasons why this vector field changes with time: because the edge-mmf's are changing, and because the placement of the edges is changing as well.

4.2 The computation of forces

We assume that the dynamics of the deformable bodies is well described, and that some structural dynamics code is available which can simulate this dynamics: at each time step, given the present-step u and electrodynamic forces as input, it yields the next-step value of u as output. Thus the key ingredient in coupling (15)(16) with the dynamic equations is the computation of forces at each time step. We now show how to do this.

The magnetic coenergy Φ of the system is a function of the placement u and of the DF H:

$$\Phi(u, H) = {}^1\!/_2 \int_X \mu\, H \wedge *_u H \equiv {}^1\!/_2 \int_{E_3} \mu_u\, \mathbf{h} \cdot \mathbf{h}. \tag{17}$$

Now, force in configuration u for the electromagnetic state as described by H is simply *the partial derivative of* Φ *with respect to* u: $f = \partial_u \Phi(u, H)$.

This innocent-looking statement is the key to the method, and not a trivial one to prove, because the above expression of coenergy is unconventional, being in terms of

the "material" magnetic field H, not of some more familiar entity like $\mathbf{h}$. For a proof that force is indeed this, see [6]. (The proof in [6] is valid on the premises of eddy-currents theory, i.e., $\varepsilon = 0$, no energy attributable to the electric field.) We just state here in what sense "force" should be understood: *force is a covector-valued 3-form* F *on* X, or equivalently, a covector-valued 3-form f on E_3, with $F = u^*f$. To understand this, note that a virtual variation δu of the placement can be characterized by a virtual velocity v (so that $\delta u = v\,\delta t$). For f_y to be a covector-valued 3-covector means that $f_y(v_y)\delta t$ is a 3-covector. It represents the density of virtual work about point $y \in E_3$ associated with the virtual displacement $v\delta t$. Its integral over E_3 is the virtual work, that we note $\langle f, \delta u \rangle$ for convenience.

By the very definition of matrix A above, we have the following approximation of coenergy:

$$\Phi(u, H) \sim {}^1\!/_2\, (A(u)\, \underline{h}, \underline{h}),$$

where $(\ ,\ )$ denotes the scalar product in $\mathbb{R}^{\#(\mathcal{E})}$. Therefore, an approximation of force is ${}^1\!/_2\, (\partial_{\underline{u}} A(u)\, \underline{h}, \underline{h})$, where $\underline{u}$ is the vector of nodal positions (at time t), $\underline{u} = \{u_n : n \in \mathcal{N}\}$. This derivative is an array (of length $\#(\mathcal{N})$, the number of elements in $\mathcal{N}$, like $\underline{u}$ itself) of ordinary three-dimensional vectors $\mathbf{f}_n$, one per node, and an approximation to the virtual work is

$$\langle f, \delta u \rangle \sim \Sigma_{n \in \mathcal{N}}\, \mathbf{f}_n \cdot \delta u_n.$$

While these "nodal forces" $\mathbf{f}_n$ are a satisfactory outcome, they may not be exactly what is needed by a specific structural code, so some interpolation may be called for. There are several possibilities. One of them [17] is as follows. We assign to each node a "control volume", equal to the sum of the volumes of all neighboring tetrahedra, divided by 4. The nodal force is then divided by this volume, thus giving an estimation of the nodal *density* of force (different, of course, from $\mathbf{f}_n$). These values of density may then be interpolated inside tetrahedra.

In early realizations of the virtual work idea, people would compute the field *twice* (or even more, as pointed out, e.g., in [13]), for two close values of u, and take the difference of the computed coenergies as an approximation to the virtual work. This is all right when only one resultant or torque is looked for, but otherwise there is advantage in precomputing the matrix $\partial_{\underline{u}} A$. This (as observed in [9]) is again an assembly process, which is about as costly as the one that yields A itself.

5. Discussion

So we have a practical procedure to both solve for the electromagnetic field and obtain an approximation of the force-field, worth feeding into a structural dynamics code. Our objective is thus fulfilled. A few loose ends still have to be tied up, however.

First, it is legitimate to ask for a *formula* giving the force field, if only to get an independent check on the validity of the present theory. This will appeal to a technical

tool, the *integration by parts* formula (an immediate consequence of Stokes theorem):

(18) $\int_X \Omega \wedge d\Theta = \int_X d\Omega \wedge \Theta$

where Ω and Θ are a 1-form and a 2-form vanishing at the boundary of X (if there is one).

Since the force in configuration $\{u, H\}$ does not depend on previous history, we may, in order to compute the virtual work associated with the virtual displacement $v\,\delta t$, assume any trajectory $\tau \to u_\tau$ such that $u_\tau = u$ and $\partial_\tau u = v$ at time $\tau = t$. The virtual *power* will thus be obtained by differentiating the function $\tau \to {}^1\!/_2 \int_X \mu\, H \wedge *_u H$ with respect to τ. (We denote by ∂_t the derivative at $\tau = t$.) Suppose first that $\mu = \mu_0$ all over. Since $H = u^*h$ is kept constant in the process, we have, on the model of (5),

$$0 = \partial_t(u^*h) = u^*(\partial_t h + L_v h)$$

hence $\partial_t h = -L_v h$. Now, let's proceed (note the use of (18)):

$$\begin{aligned}
&\partial_t \textstyle\int_X \mu/2\, H \wedge *_u H \equiv \partial_t \int_{E_3} \mu/2\, h \wedge * h \\
&= \textstyle\int_{E_3} \mu * h \wedge \partial_t h = -\int_{E_3} \mu * h \wedge L_v h = -\int_{E_3} b \wedge (i_v j + d\, i_v h) \\
&= -\textstyle\int_{E_3} b \wedge i_v j - \int_{E_3} db \wedge i_v h = -\int_{E_3} b \wedge i_v j \\
&\equiv \textstyle\int_{E_3} \mathbf{j} \times \mathbf{b} \cdot v,
\end{aligned}$$

so we do get the $j \times b$ formula.

When μ depends on x, we have the additional term ${}^1\!/_2 \int_{E_3} \partial_t(\mu_u)\, h \wedge * h$. To compute $\partial_t(\mu_u)$ at a *fixed* point $y \in E_3$, we use the τ-dependent x such that $y = u_\tau(x)$. Using $\mu_u(u(x)) = \mu(x)$, and some care, we find that $\partial_t(\mu_u) = -\nabla\mu_u \cdot v$. So, at last,

$$\mathbf{f} = \mathbf{j} \times \mathbf{b} - {}^1\!/_2\, |\mathbf{h}|^2\, \nabla\mu_u,$$

which is (1), in the isotropic case. Of course, $\nabla\mu_u$ is taken in the sense of distributions in this formula.

Note that the freedom in the definition of the placement outside of "hard matter" appears at the end to be of no consequence, as regards the force formula. Force is null, as it should be, where there is no current and no variation of permeability. Beware, however, that this conclusion does *not* hold at the discretized level: the nodal force $\mathbf{f}_n$ can very well be non-zero at nodes n *in the air*. Such values of $\mathbf{f}_n$ have a residual character: they are *numerical* errors, and one may argue that it is a good thing to have them computed, as an indication of how accurate the other $\mathbf{f}_n$s (those in matter) are.

Let us finally tackle the issue of Coulomb force. For the purpose of this discussion, we may assume non-moving bodies ($v = 0$), and use classical language. So the relevant equations are:

(19) $\partial_t \mathbf{b} + \mathrm{rot}\, \mathbf{e} = 0$, (20) $\mathrm{rot}\, \mathbf{h} = \mathbf{j}$,

(21) $\mathbf{b} = \mu\, \mathbf{h}$, (22) $\mathbf{j} = \sigma\, \mathbf{e} + \mathbf{j}^g$.

Call C the region where $\sigma > 0$ (the conductor), I the region where $\mathbf{j}^g$ may assume non-zero values (the inductor), and suppose C and I do not intersect. Assume also, for simplicity, that C is in one piece, simply connected (no loops), with a one-piece boundary ∂C (no holes in C). (The argument which follows has already been presented, though more shortly, in [3].)

Eddy-current theory has established that, starting from specified initial conditions, and given $\mathbf{j}^g$ as a function of time, eqs. (19) to (22) fully determine $\mathbf{h}$, $\mathbf{b}$ and $\mathbf{j}$. *But not* $\mathbf{e}$. The electric field can be computed from $\mathbf{j}$ *in the conductor,* thanks to Ohm's law, but *not outside*, because information on div $\mathbf{e}$ outside C is lacking. What is known is the curl of $\mathbf{e}$, rot $\mathbf{e} = -\partial_t \mathbf{b}$, by eq. (19), and the tangential part $n \times \mathbf{e}$ of $\mathbf{e}$ on ∂C (which makes sense, by continuity of the tangential part of $\mathbf{e}$). Thus $\mathbf{e}$ is only determined up to the gradient of a function v, with v a constant on ∂C.

This indetermination of $\mathbf{e}$, after all, is not such a surprise: not having taken into account the inner structure of the inductor (how threads are coiled, how connections to the mains are made, etc.), we can't expect a complete determination of the electromagnetic situation. Observe that *not* being compelled to describe this fine structure is a welcome comfort in eddy-current computations! But the price to pay for this comfort is not having access to the electric field outside the passive conductors.

Now, *if* $\rho = \text{div } \varepsilon\mathbf{e}$ was known outside C, this additional information *would* determine $\mathbf{e}$ outside C: Just solve the Dirichlet problem $\text{div}(\varepsilon \text{ grad } v) = \rho$ outside C, $v = \text{Cte}$ on ∂C, and adjust the constant in order to get $\int_{\partial C} \varepsilon \, \partial v/\partial n = 0$ (zero total charge in C, a natural requirement). So suppose these outside charges are known, by some means, say $\rho = 0$ outside C, for definiteness. Obtain v. One thus knows $\mathbf{e}$ in *all* E_3 after this two-step process. There are two parts in the total field $\mathbf{e}$, one which is given by the eddy-currents model (which assumes $\varepsilon = 0$), one which is obtained after taking some extra information into account and reintroducing ε at its actual physical value. The second part of field $\mathbf{e}$ (the outside part) is thus obtained through a *perturbation process,* with ε as the relevant small parameter.

In the same spirit, Coulomb force appears as a first-order correction (in ε). For, having obtained $\mathbf{e}$ over all E_3, we may compute $\text{div}(\varepsilon\mathbf{e})$, as a mathematical distribution, over all E_3 as well, hence in particular the *jump* of $\varepsilon \, n \cdot \mathbf{e}$ at the boundary ∂C, hence the surface charge on C. (If the ratio σ/ε is not constant in C, there are also charges there, since $\text{div}(\sigma\mathbf{e}) = 0$, which implies $\text{div}(\varepsilon\mathbf{e}) \neq 0$.) Now, there certainly exist Coulomb forces between these surface charges, and they may have a non-negligible mechanical effect. But these forces have the same theoretical status as the surface charges themselves: 1°- they *cannot* be determined by eddy-current equations *alone;* they can only be obtained after additional information about the electric field outside C, *in particular, in the inductor*, has been provided, 2°- they are first-order in ε, the dielectric constant, and thus appear as a corrective term to a theory whose basic assumption was that $\varepsilon = 0$.

Conclusion

We have presented a Lagrangian finite-element method to compute the magnetic field and electromagnetic forces in deformable bodies in the context of eddy-currents theory. The magnetic coenergy appears as an explicitely known quadratic form of the degrees of freedom at the discretized level, and this makes it easy to obtain force by differentiation with respect to the configuration parameters, which are, in the most general case, the positions of the mesh-nodes. The theory requires a modern mathematical apparatus (differential forms), and we have presented a few sample calculations showing that calculus in this framework is relatively easy (and gives correct formulas). We have discussed a limitation of this theory, due to its very consistency with eddy-currents theory (displacement currents ignored, as if one had $\varepsilon = 0$): Coulomb forces, like electric charges themselves, are not directly obtained, but only as a first-order corrective term, and through a process which requires more data than what is sufficient to compute eddy-currents.

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