A Note on the Castelnuovo Theory

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1 Introduction

Configurations of projective points in linearly general position (i.e. no $k+2$ of them are contained in a plane of dimension k) have been much investigated since such configurations arise by cutting a variety of dimension d with a linear variety of dimension $n-d$. The old Castelnuovo result that such a configuration of $n+3$ points in $\mathbb{P}^n$ lies in a rational normal curve has been generalized in the context of schemes [**EH**]; around the Castelnuovo Theory, there arose the question ([**CRV**]) whether a suitable configuration lies in a rational normal scroll. In [**CRV**] it was proved for each d, $1 \leq d \leq n-2$, that a configuration of $n+2+d$ points in $\mathbb{P}^n$ lies in a rational normal scroll of dimension d; this result has been generalized to a primary scheme in [**M**]. The aim of this note is to prove this result for any scheme by explicitly writing down the required scroll in function of the scheme.

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2 Set up, notation and proof outline

Let $\mathcal{P} = K[X_1, \dots, X_n]$, where K is an algebraic closed field of characteristic 0 and let $\mathbf{m} = (X_1, \dots, X_n)$.

If $M = (a_{ij})_{ij}$ is a $2 \times k$ matrix with entries $a_{ij} \in \mathcal{P}$, we denote by $\mathcal{R}(M)$ the ideal generated by the minors of M; also if $\Phi : \mathcal{P} \to \mathcal{P}'$ is a morphism, $\Phi(M)$ will denote the matrix $\Phi(M)_{ij} = (\Phi(a_{ij}))$.

For each linear form $f = \sum_{i=1}^n c_i X_i$, $\Lambda(f)$ will denote the form $\Lambda(f) = \sum_{i=1}^n c_i X_{i-1}$, where $X_0 = 1$.

Throughout the paper, we will use the following set up and notation:

Let $\mathcal{I} \subset \mathcal{P}$ be a 0-dimensional ideal, and let us denote

- $\mathcal{Z}(\mathcal{I}) = \{\mathbf{a}_1, \dots, \mathbf{a}_r\} \subset K^n$ the set of zeros of $\mathcal{I}$,
- $\mathbf{m}_i \subset \mathcal{P}$ the maximal ideal whose root is $\mathbf{a}_i$,
- $\mathbf{q}_i$ the $\mathbf{m}_i$-primary component of $\mathcal{I}$, so that $\mathcal{I} = \cap_i \mathbf{q}_i$,
- $\mu_i = \mathrm{mult}(\mathbf{q}_i)$.

Let us assume that there is $\delta, 1 \leq \delta \leq n-1$ such that $\mathrm{mult}(\mathcal{I}) = n + 2 + \delta$ and that the roots are ordered so that $\mu_i \geq \mu_j$ for $i < j$.

Let $\mathcal{I}''$ be an ideal such that, denoting $\forall i$ $\mathbf{q}''_i$ the $\mathbf{m}_i$-primary component of $\mathcal{I}''$, one has:

- $\mathcal{I}'' \supseteq \mathcal{I}$,
- $\mathrm{mult}(\mathcal{I}'') = n + 3$,
- there is $s \leq r$ such that
  - the set of roots of $\mathcal{I}''$ is $\{\mathbf{a}_1, \dots, \mathbf{a}_s\}$,
  - for $i < s$, $\mathbf{q}''_i = \mathbf{q}_i$.

It then follows that:

  - $\mathrm{mult}(\mathbf{q}_s) \neq n+1, n+2$ – since there cannot be two roots $\mathbf{a}_\alpha, \mathbf{a}_\beta$ such that $\mu_\alpha \geq \mu_\beta \geq n+1$,
  - $\mathrm{mult}(\mathbf{q}_s) \geq n+3 \iff s = 1$

To fix the notation let us assume that (up to a translation) $\mathbf{a}_s$ is the origin and let us denote

- $\mathcal{I}' = \mathbf{q}_s \cap \mathcal{I}''$, so that $\mathcal{I}'' \supseteq \mathcal{I}' \supseteq \mathcal{I}$,
- d such that $\mathrm{mult}(\mathcal{I}') = n + 2 + d$ and so $1 \leq d \leq \delta$.
- $a := \mathrm{mult}(\mathbf{q}_s), b := \mathrm{mult}(\mathbf{q}''_s)$.

With this notation, it is then possible to outline the argument by which we prove our claim that if $\mathcal{I}$ is (an ideal defining a scheme) in linearly general position, then there is a rational normal scroll (defined by the ideal) $\mathcal{S}$ of dimension δ s.t. $\mathcal{I} \supset \mathcal{S}$.

Our starting point is the result of [**EH**], on the basis of which we know that, denoting

$$M'' = \begin{pmatrix} X_1 & X_2 & \dots & X_i & \dots & X_n \\ 1 & X_1 & \dots & X_{i-1} & \dots & X_{n-1} \end{pmatrix},$$

it holds (up to a change of coordinate) $\mathcal{I}'' \supset \mathcal{R}(M'')$ [1].

[1] Remark, by the way, that, under the assumption that $\mathcal{I}$ defines a scheme in linearly general position, there is a unique ideal $\mathcal{I}''$, satisfying the assumptions above (*cf.* Rem. 8).

On the basis of that, first we show the existence of a matrix M' such that the rational normal scroll $\mathcal{S}' = \mathcal{R}(M')$ satisfies $\dim(\mathcal{S}') = d$ and $\mathcal{S}' \subset I'$; there are two cases to consider:

- $\mathrm{mult}(\mathbf{q}_s) \geq n+3$ (§4) so that $\mathcal{I}' = \mathbf{q}_s$. In this case the scroll $\mathcal{S}'$ has been already computed in [**M**]; we give here a more simple proof (Th. 9).
- $\mathrm{mult}(\mathbf{q}_s) \leq n$ (§6). The argument then depends on the multiplicity of $\mathbf{q}''_s$:
  - if $\mathrm{mult}(\mathbf{q}''_s) = 1$, the linear forms contained in $\mathbf{q}_s$ allow easily to compute a matrix M' as above, whose columns are linear combinations of those of M'' (Prop. 12).
  - Also in the case $\mathrm{mult}(\mathbf{q}''_s) > 1$, M' is obtained by computing its columns as linear combinations of those of M''. The computation is however more complex: let us denote $\mathbf{q}^{(c)}$, $b \leq c \leq a$ to be the unique[2] $\mathbf{m}$-primary such that $\mathrm{mult}(\mathbf{q}^{(c)}) = c$ and
  $$\mathbf{q}''_s = \mathbf{q}^{(b)} \supset \mathbf{q}^{(b+1)} \supset \ldots \mathbf{q}^{(a-1)} \supset \mathbf{q}^{(a)} = \mathbf{q}_s.$$
  We will produce $\forall c$ a matrix $M'^{(c)}$ such that $\mathcal{I}'' \cap \mathbf{q}^{(c)} \subset \mathcal{R}(M'^{(c)})$ by means of an iterative computation: assuming $M^{(c)}$ is already computed, let κ be the maximal value such that the minor consisting of the first and κ^{th} columns is not zero mod. $\mathbf{q}^{(c+1)}$; the matrix $M'^{(c+1)}$ is then easily obtained by removing the κ^{th} column and suitably adding it to the other columns (Lemma 13).

Once $M' = (\ell_{ij})_{ij}$ is obtained, we can then compute a matrix M such that $\mathcal{S} = \mathcal{R}(M)$ satisfies $\dim(\mathcal{S}) = \delta$ and $\mathcal{S} \subset \mathcal{I}$, thus completing our task (§5). The computation is the following: let Π be the maximal linear ideal [3] passing thru the origin and contained in all the primaries $\mathbf{q}_{s+1}, \ldots, \mathbf{q}_n$ which have not yet being considered; let $\wp$ be the linear ideal contained in Π and $(\ell_{1j}, 1 \leq j \leq n+1-d)$ and let $Z_1, \ldots Z_C$ be linear forms generating $\wp$. Then the matrix M is the one whose first row is $(Z_1, \ldots Z_C)$ and whose columns are linear combinations of those of M'.

3 Castelnuovo Theory

Definition 1 *A 0-dimensional ideal $\mathcal{I} \subset \mathcal{P}$ is said to be in* linearly general position *(l.g.p.) if for each linear ideal $\wp \subset \mathcal{P}$ it holds* $\mathrm{mult}(\mathcal{I} + \wp) \leq 1 + \dim(\wp)$.

Definition 2 A rational normal scroll *(of dimension $n+1-k$) is the variety defined by the ideal $\mathcal{R}(M) \subset \mathcal{P}$ generated by the minors of a matrix*

$$M = \begin{pmatrix} \ell_{11} & \ldots & \ell_{1j} & \ldots & \ell_{1k} \\ \ell_{21} & \ldots & \ell_{2j} & \ldots & \ell_{2k} \end{pmatrix}$$

such that

S.1 $\ell_{ij} \in \mathrm{Span}_K(\{1, X_1, \ldots, X_n\}), \forall 1 \leq i \leq 2, 1 \leq j \leq k$;

S.2 $\lambda\ell_{11} + \mu\ell_{21}, \ldots \lambda\ell_{1j} + \mu\ell_{2j}, \ldots \lambda\ell_{1k} + \mu\ell_{2k}$ *are linearly independent* $\forall(\lambda, \mu) \in \mathbb{P}^2$.

A rational normal curve *is a rational normal scroll such that $k = n$.*

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[2] Uniqueness is, again, a consequence of the assumption that $\mathcal{I}$ defines a scheme in linearly general position (*cf.* Rem. 8).

[3] a linear ideal is an ideal generated by linear forms, and so defining a linear subspace of $\mathbb{P}^n$.

Theorem 3 [EH] *Let $\mathcal{I} \subset \mathcal{P}$ be a 0-dimensional ideal in l.g.p., then:*
1) if $\text{mult}(\mathcal{I}) \geq n+3$ *then $\mathcal{I} \supset \mathcal{C}$ where $\mathcal{C}$ is the ideal of a smooth curve unramified at each zero of $\mathcal{I}$.*
2) If $\text{mult}(\mathcal{I}) = n+3$ *then there is a unique ideal $\mathcal{C}$ defining a smooth rational normal curve such that $\mathcal{I} \supset \mathcal{C}$.*

Theorem 4 [CRV] *Let $\mathcal{I} \subset \mathcal{P}$ be a 0-dimensional radical ideal in l.g.p. such that* $\text{mult}(\mathcal{I}) = n+2+d$, $1 \leq d \leq n-1$. *Then there is an ideal $\mathcal{S}$ defining a rational normal scroll such that* $\dim(\mathcal{S}) = d$ *and $\mathcal{I} \supset \mathcal{S}$.*

Since the aim of this paper is to generalize Th. 4 to any 0-dimensional ideal in l.g.p., in order to simplify the presentation let us introduce the following

Definition 5 *A 0-dimensional ideal $\mathcal{I} \subset \mathcal{P}$ is said to be a* scrolled ideal *if*

$$\text{mult}(\mathcal{I}) = n+2+d, 1 \leq d \leq n-1,$$

and there is an ideal $\mathcal{S}$ defining a rational normal scroll such that $\dim(\mathcal{S}) = d$ *and $\mathcal{I} \supset \mathcal{S}$.*

Let $\mathbf{q} \subset K[X_1, \ldots, X_n]$ be an $\mathbf{m}$-primary ideal of multiplicity μ and such that its embedding dimension is 1. There is therefore a unique chain of $\mathbf{m}$-primary ideals

$$\mathbf{m} = \mathbf{q}^{(1)} \supset \mathbf{q}^{(2)} \supset \ldots \mathbf{q}^{(\mu-1)} \supset \mathbf{q}^{(\mu)} = \mathbf{q}$$

with $\text{mult}(\mathbf{q}^{(j)}) = j$.

Theorem 6 [M] *Under the notation above, if $\mathbf{q}$ is in l.g.p. and of embedding dimension 1, then there is a coordinate frame $Y_1, \ldots, Y_n$ such that*

$$\mathbf{q} = \begin{cases} (Y_1^{\mu}, Y_2 - Y_1^2 - Y_1^{n+1} f_2(Y_1), \ldots, Y_n - Y_1^n - Y_1^{n+1} f_n(Y_1)) & \text{if } \mu > n \\ (Y_1^{\mu}, Y_2 - Y_1^2, \ldots, Y_{\mu-1} - Y_1^{\mu-1}, Y_{\mu}, \ldots, Y_n) & \text{if } \mu \leq n. \end{cases} \quad (1)$$

Corollary 7 *Under the notation above, if $\mathbf{q}$ is in l.g.p. and of embedding dimension ≤ 1, there is a coordinate frame $Y_1, \ldots, Y_n$ such that*

$$\forall j, 1 \leq j \leq \min(\mu, n), \mathbf{q}^{(j)} = (Y_1^j, Y_2 - Y_1^2, \ldots, Y_{j-1} - Y_1^{j-1}, Y_j, \ldots, Y_n)$$

Proof: By Theorem 6, there is a coordinate frame $Y_1, \ldots, Y_n$ such that (1) holds.

Denoting $\mathbf{p}^{(j)} = \mathbf{q} + \mathbf{m}^j \ \forall j, 1 \leq j \leq \min(\mu, n)$, the thesis follows from the facts that

- $\mathbf{p}^{(j)} = (Y_1^j, Y_2 - Y_1^2, \ldots, Y_{j-1} - Y_1^{j-1}, Y_j, \ldots, Y_n) \ \forall j$,
- $\mathbf{p}^{(i)} \supset \mathbf{p}^{(j)}$ if $i < j$,
- $\text{mult}(\mathbf{p}^{(j)}) = j \ \forall j$. ▮

Remark 8 As a consequence of Th. 3, the assumption that $\mathcal{I}$ is in l.g.p. implies that $\forall i$ the primary component $\mathbf{q}_i$ not only is in l.g.p., but also has embedding dimension ≤ 1. This implies that

- the results of Cor. 7 apply to $\mathbf{q} := \mathbf{q}_s$ (this will be used in §6)
- and, in particular, there is a unique $\mathbf{m}$-primary $\mathbf{q}^{(b)}$ s.t.

$$\mathbf{m} \supset \mathbf{q}^{(b)} \supset \mathbf{q} \text{ and } \operatorname{mult}(\mathbf{q}^{(b)}) = b,$$

so that

- the 0-dimensional ideal $\mathcal{I}''$ satisfying the required assumptions is unique, it being the one s.t. $\mathbf{q}''_s = \mathbf{q}^{(b)}$.

4 The rational normal scroll containing a primary in l.g.p.

Let $\mathbf{q}^{(n+2+d)}$ be a l.g.p. $\mathbf{m}$-primary ideal of multiplicity $\mu = n+2+d$, $1 \leq d \leq n-1$, and such that its embedding dimension is 1.

Then, as a consequence of Th. 6, one has

$$\mathbf{q}^{(n+2+d)} = (Y_1^{(n+2+d)}, g_2, \ldots, g_n)$$

where $\forall j$

$$g_j = Y_j - Y_1^j - \sum_{i=1}^{d+1} d_{ji} Y_1^{n+i}.$$

Let us define c_i, $2 \leq i \leq n$ to be the solution of the system

$$\left\{ d_{\rho-1\ 1} + \sum_{j=\rho+1}^{n} d_{j\ 1} c_{j+1-\rho} + d_{n\ 1} d_{\rho\ 1} - c_{n+2-\rho} - d_{\rho\ 2} = 0 \quad 2 \leq \rho \leq n \right. \tag{2}$$

and $\ell_j, 1 \leq i \leq n$ to be the linear forms (where $Y_0 = 1$)

$$\ell_j = Y_{j-1} + \sum_{i=j+1}^{n} c_{i+1-j} Y_i + d_{j1} Y_n.$$

Let also

$$M^{(d)} = \begin{pmatrix} Y_d & \ldots & Y_j & \ldots & Y_n \\ \ell_d & \ldots & \ell_j & \ldots & \ell_n \end{pmatrix}$$

and let $\mathcal{S}^{(d)} = \mathcal{R}(M^{(d)})$.

Theorem 9 [M] *With the notation above, $\forall d$, $1 \leq d \leq n-1$, one has $\mathcal{S}^{(d)} \subset \mathbf{q}^{(n+2+d)}$.*

Proof: Let $\Psi : K[Y_1, \ldots, Y_n] \mapsto K[t]$ defined by

$$\Psi(Y_1) = t$$

$$\Psi(Y_j) = t^j + \sum_{i=1}^{d+1} d_{ji} t^{n+i} \qquad j \geq 2.$$

It is then obvious that $\mathbf{q}^{(n+2+d)} = \Psi^{-1}(t^{n+2+d})$ and so the thesis is proved if it is shown that $\Psi(\mathcal{S}^{(d)}) \subset (t^{n+2+d})$.

To do that we have just to compute $\Psi(\ell_j)$ which is clearly (putting $d_{0\ 1} = 0$):

$$\Psi(\ell_j) = t^{j-1} + \sum_{i=j+1}^{n} c_{i+1-j} t^i + d_{j\ 1} t^n + \left(d_{j-1\ 1} + \sum_{i=j+1}^{n} c_{i+1-j} d_{i\ 1} + d_{j\ 1} d_{n\ 1} \right) t^{n+1} + \dots$$

Let us denote $M^{(\lambda\rho)} = Y_\lambda \ell_\rho - Y_\rho \ell_\lambda, d \leq \lambda < \rho \leq n$ which is the minor over the λ^{th} and ρ^{th} columns of the matrix $M^{(d)}$. Modulo $(t^{n+\lambda+2})$ one has

$$\begin{aligned}\Phi(Y_\lambda \ell_\rho) \equiv t^{\lambda+\rho-1} &+ \sum_{i=2}^{n-\rho+1} c_i t^{i+\rho+\lambda-1} + d_{\rho\ 1} t^{n+\lambda} + \\ &+ \left(d_{\rho-1\ 1} + \sum_{j=\rho+1}^{n} c_{j+1-\rho} d_{j\ 1} + d_{\rho\ 1} d_{n\ 1} \right) t^{n+\lambda+1} + d_{\lambda\ 1} t^{n+\rho};\end{aligned}$$

$$\Phi(Y_\rho \ell_\lambda) \equiv t^{\lambda+\rho-1} + \sum_{i=2}^{n-\lambda+1} c_i t^{i+\rho+\lambda-1} + d_{\lambda\ 1} t^{n+\rho} + d_{\rho\ 1} t^{n+\lambda} + d_{\rho\ 2} t^{n+\lambda+1}.$$

As a consequence one has

$$\Phi(M^{(\lambda\rho)}) \equiv$$

$$\begin{aligned}\Phi(M^{(\lambda\rho)}) \equiv & t^{\lambda+\rho-1} + \sum_{i=2}^{n-\rho+1} c_i t^{i+\rho+\lambda-1} + d_{\rho\ 1} t^{n+\lambda} + \\ &+ \left(d_{\rho-1\ 1} + \sum_{j=\rho+1}^{n} c_{j+1-\rho} d_{j\ 1} + d_{\rho\ 1} d_{n\ 1} \right) t^{n+\lambda+1} + d_{\lambda\ 1} t^{n+\rho} - \\ &- t^{\lambda+\rho-1} - \sum_{i=2}^{n-\lambda+1} c_i t^{i+\rho+\lambda-1} - d_{\lambda\ 1} t^{n+\rho} - d_{\rho\ 1} t^{n+\lambda} - d_{\rho\ 2} t^{n+\lambda+1} \equiv \\ \equiv & \left(d_{\rho-1\ 1} + \sum_{j=\rho+1}^{n} c_{j+1-\rho} d_{j\ 1} + d_{\rho\ 1} d_{n\ 1} - d_{\rho\ 2} \right) t^{n+\lambda+1} - \\ &- \sum_{i=n-\rho+2}^{n-\lambda+1} c_i t^{i+\rho+\lambda-1} \equiv \\ \equiv & c_{n+2-\rho} t^{n+\lambda+1} - c_{n-\rho+2} t^{n+\lambda+1} \equiv 0 \bmod (t^{n+\lambda+2}),\end{aligned}$$

– where

$$\left(d_{\rho-1\ 1} + \sum_{j=\rho+1}^{n} c_{j+1-\rho} d_{j\ 1} + d_{\rho\ 1} d_{n\ 1} - d_{\rho\ 2} \right) = c_{n+2-\rho}$$

follows from (2) – so that $\Phi(M^{(\lambda\rho)}) \equiv 0 \bmod (t^{n+\lambda+2})$ $\forall\lambda, \rho$, $d \leq \lambda < \rho \leq n$ and therefore $\Psi(\mathcal{S}^{(d)}) \subset (t^{n+2+d})$. ▮

5 Reducing the problem

The aim of this section is to prove that, if $\mathcal{I}$ is in l.g.p. and $\mathcal{I}'$ is scrolled, such is $\mathcal{I}$. To prove that, we need some further notation: let

- $\gamma := \mathrm{mult}(\mathcal{I}) - \mathrm{mult}(\mathcal{I}') = \delta - d = \sum_{i=s+1}^{r} \mu_i$;
- ℓ_{ij} $\forall 1 \leq i \leq 2, 1 \leq j \leq n+1-d = k$ be linear forms satisfying S.1, S.2 such that denoting

$$M' = \begin{pmatrix} \ell_{11} & \dots & \ell_{1j} & \dots & \ell_{1k} \\ \ell_{21} & \dots & \ell_{2j} & \dots & \ell_{2k} \end{pmatrix}$$

  it holds $\mathcal{R}(M') \subset \mathcal{I}'$; moreover, since $\mathbf{a}_s$ is the origin, so $\mathcal{R}(M) \subset \mathbf{m}$, we will assume, up to row and column operations, that $\ell_{1j}(0) = 0$ $\forall j$;
- $\forall j > s$
  - a_{ij} such that $\mathbf{a}_j = (a_{1j}, \dots, a_{nj})$,
  - $\rho_j : \mathcal{P} \mapsto \mathcal{P}$ the translation such that $\rho(X_i) = X_i + a_{ij}$ so that $\rho_j(\mathbf{m}_j) = \mathbf{m}$,
  - $Y_1, \dots, Y_n$ the frame of coordinates such that

$$\rho_j(\mathbf{q}_j) = (Y_1^{\mu_j}, Y_2 - Y_1^2, \dots, Y_{\mu_j-1} - Y_1^{\mu_j-1}, Y_{\mu_j}, \dots, Y_n),$$

  - $\Phi_j : \mathcal{P} \mapsto K[s]$, the morphisms such that $\Phi_j(Y_i) = \begin{cases} s^i & i < \mu_j \\ 0 & i \geq \mu_j, \end{cases}$.

Let us consider the vector space V of all the linear forms Z thru the origin such that $\Phi_j\rho_j(Z) = 0, \forall j > s$; denoting $c := \dim(V)$, so that $c \geq n - \gamma$, let $Z_1, \dots, Z_c$ be linear forms such that

- $V = \mathrm{Span}_K(Z_1, \dots, Z_c)$,
- there is $C \leq c$ such that $\mathrm{Span}_K(Z_1, \dots, Z_C) = V \cap \mathrm{Span}_K(\ell_{11}, \dots \ell_{1k})$.

Let $\wp := (Z_1, \dots, Z_C)$ and let us compute c_{ij}, $1 \leq i \leq C$, $1 \leq j \leq k$ such that $Z_i = \sum_j c_{ij}\ell_{1j}$ and let us define $Z'_i := \sum_j c_{ij}\ell_{2j}$, $1 \leq i \leq C$ and

$$M = \begin{pmatrix} Z_1 & \dots & Z_C \\ Z'_1 & \dots & Z'_C \end{pmatrix}.$$

It is then easy to verify that $\mathcal{R}(M) \subset \mathcal{I}$, so that $\mathcal{I}$ is scrolled:

Proposition 10 *If $\mathcal{I}$ is in l.g.p., $n+3 \leq \mathrm{mult}(I) \leq 2n+1$, and, using the notation above, $\mathcal{I}'$ is a scrolled ideal, then such is $\mathcal{I}$.*

Proof: Using the notation above, $\Phi_j\rho_j(Z) = 0$ clearly implies $Z \in \mathbf{q}_j$ and, therefore, $\mathcal{R}(M) \subset \wp \subset \mathbf{q}_j, \forall j > s$. Since $\mathcal{R}(M) \subset \mathcal{I}'$ too, we conclude $\mathcal{R}(M) \subset \mathcal{I}$. The thesis then follows since the Grassman formula gives

$$C = \dim(V \cap W) = \dim(V) + \dim(W) - \dim(V \cup W) \geq k + c - n \geq n + 1 - d - \gamma = n + 1 - \delta,$$

where $W := \mathrm{Span}_K(\ell_{11}, \ldots \ell_{1k})$. ∎

Example 11 Let us assume $\mathcal{I} = \mathcal{I}' \cap \mathbf{q}_1 \cap \mathbf{q}_2 \subset \mathcal{P} = K[X_1, \ldots, X_9]$ be an ideal in l.g.p., where

- $\mathrm{mult}(\mathcal{I}) = 18, \mathrm{mult}(\mathcal{I}') = 14, \mathrm{mult}(\mathbf{q}_1) = 1, \mathrm{mult}(\mathbf{q}_2) = 3$,
- $\mathcal{I}'$ is contained in the scroll

$$M' = \begin{pmatrix} X_1 & \ldots & X_4 & X_7 & \ldots & X_9 \\ 1 & \ldots & X_3 & X_6 & \ldots & X_8 \end{pmatrix};$$

- $\mathbf{q}_1 = \mathbf{m}_1 = (X_1 + 1, \ldots X_i + 1, \ldots, X_9 + 1)$ is the maximal ideal of $\mathbf{a}_1 = (-1, \ldots, -1)$;
- $\mathbf{m}_2 = (X_1, X_2 - 1, X_3, \ldots, X_9)$ is the maximal ideal of $\mathbf{a}_2 = (0, 1, 0, \ldots, 0)$ and $\mathbf{q}_2$ is the $\mathbf{m}_2$-primary ideal defined by $\mathbf{q}_2 = (X_4^3, X_1 - X_4^2, X_2 - 1, X_3, X_5, \ldots, X_9)$.

With the notation above we have

- $\Phi_2(X_i) = \begin{cases} s & i = 4 \\ s^2 & i = 1 \\ 0 & i \neq 1, 4 \end{cases}$
- $Z = \sum_{i=1}^{9} a_i X_i \in V \iff \begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 0 \\ \Phi_2\rho_2(Z) = a_2 + a_4 s + a_1 s^2 = 0 \end{cases}$
- $c = 5, C = 3$
- $Z_1 = X_3 - X_7, Z_2 = X_7 - X_8, Z_3 = X_8 - X_9, Z_4 = X_9 - X_5, Z_5 = X_5 - X_6$
- $M = \begin{pmatrix} X_3 - X_7 & X_7 - X_8 & X_8 - X_9 \\ X_2 - X_6 & X_6 - X_7 & X_8 - X_9 \end{pmatrix}$. ∎

6 A further reduction

On the basis of Prop. 10, it is clear that, to prove that each 0-dimensional ideal $\mathcal{I}$ in l.g.p. is scrolled, we have to prove the same for a 0-dimensional ideal $\mathcal{I}' = \mathcal{I}'' \cap \mathbf{q}_s$, where (up to change of coordinates)

- $\mathcal{I}'$ is in l.g.p.,
- $\mathbf{q} := \mathbf{q}_s$ is an $\mathbf{m}$-primary ideal,
- $\mathrm{mult}(\mathcal{I}'') = n + 3$, $\mathrm{mult}(\mathcal{I}') = n + 2 + d$
- $\mathcal{I}'' \supset \mathcal{R}(M'')$ where

$$M'' = \begin{pmatrix} X_1 & X_2 & \ldots & X_i & \ldots & X_n \\ 1 & X_1 & \ldots & X_{i-1} & \ldots & X_{n-1} \end{pmatrix},$$

- $\text{mult}(\mathbf{q}) \leq n$.
- $\mathbf{q}''$, the $\mathbf{m}$-primary component of I'', is such that $\mathbf{q} \subset \mathbf{q}'' \subset \mathbf{m}$, $\mathbf{q} \neq \mathbf{q}''$, so that $d > 1$,

$$\text{mult}(\mathbf{q}'') = b < a = \text{mult}(\mathbf{q})$$

and $a - b = d - 1 > 0$;

Before discussing this case, let us introduce further notation. Let:
- $Y_1, \ldots, Y_n$ be linearly independent forms such that (*cf.* Cor. 7),

$$\mathbf{q} = (Y_1^a, Y_2 - Y_1^2, \ldots, Y_{a-1} - Y_1^{a-1}, Y_a, \ldots, Y_n),$$

$$\mathbf{q}'' = (Y_1^b, Y_2 - Y_1^2, \ldots, Y_{b-1} - Y_1^{b-1}, Y_b, \ldots, Y_n);$$

- $\forall \sigma \leq a, \mathbf{q}^{(\sigma)} = \mathbf{q} + \mathbf{m}^\sigma = (Y_1^\sigma, Y_2 - Y_1^2, \ldots, Y_{\sigma-1} - Y_1^{\sigma-1}, Y_\sigma, \ldots, Y_n)$ so that

$$\mathbf{q} = \mathbf{q}^{(a)}, \mathbf{q}'' = \mathbf{q}^{(b)};$$

- $\forall \sigma \leq a, \Psi_\sigma : K[X_1, \ldots, X_n] \mapsto K[s]$ be defined by $\Psi_\sigma(Y_i) = \begin{cases} s^i & \text{if } i < \sigma \\ 0 & \text{if } i \geq \sigma \end{cases}$ so that $\Psi_\sigma^{-1}(s^\sigma) = \mathbf{q}^{(\sigma)}$.

Let us first consider the case $b = 1$ for which it holds:

Proposition 12 *If $b = 1$ let*

$$M' = \begin{pmatrix} Y_a & \ldots & Y_n \\ \Lambda(Y_a) & \ldots & \Lambda(Y_n) \end{pmatrix}.$$

It holds $\mathcal{R}(M') \subset \mathcal{I}'$.

Proof: Clearly $\mathcal{R}(M') \subset \mathcal{I}''$, and $\mathcal{R}(M') \subset (Y_a, \ldots, Y_n) \subset \mathbf{q}$. ∎

In the case $b > 1$, in order to prove that $\mathcal{I}'$ is scrolled, we will directly prove that $\mathcal{I}'' \cap \mathbf{q}^{(c)}$ is such, $\forall c, b \leq c \leq a$, by proving $\forall c$ the property

P(c) There are linearly independent linear forms $Z_2^{(c)}, \ldots, Z_{n-c+b}^{(c)}$ such that setting

$$M'^{(c)} := \begin{pmatrix} X_1 & Z_2^{(c)} & \ldots & Z_j^{(c)} & \ldots & Z_{n-c+b}^{(c)} \\ 1 & \Lambda(Z_2^{(c)}) & \ldots & \Lambda(Z_j^{(c)}) & \ldots & \Lambda(Z_{n-c+b}^{(c)}) \end{pmatrix}$$

it holds $\mathcal{R}(M'^{(c)}) \subset \mathcal{I}'' \cap \mathbf{q}^{(c)}$

by inductive argument:

Lemma 13 *Let c, $a \geq c \geq b > 1$; if* **P(c)** *holds, then* **P(c+1)** *holds.*

Proof: Since $\mathcal{R}(M'^{(c)}) \subset \mathbf{q}^{(c)} = \Psi_c^{-1}(s^c)$, and so $\Psi_c(\mathcal{R}(M'^{(c)})) \subset (s^c)$, there are suitable $\gamma_k^{(c+1)} \in K$ for $k \geq 2$ such that $\Psi_{c+1}\left(X_1\Lambda(Z_k^{(c)}) - Z_k^{(c)}\right) \equiv \gamma_k^{(c+1)} s^c \bmod s^{c+1}$. Let κ be the maximal index such that $\gamma_\kappa^{(c+1)} \neq 0$, if any, and let us define

$$Z_i^{(c+1)} = \begin{cases} \gamma_\kappa^{(c+1)} Z_i^{(c)} - \gamma_i^{(c+1)} Z_\kappa^{(c)} & i < \kappa \\ Z_{i+1}^{(c)} & \kappa \leq i \leq n-c-1+b \end{cases}$$

so that $\Psi_{c+1}\left(X_1\Lambda(Z_k^{(c+1)}) - Z_k^{(c+1)}\right) \in (s^{c+1}), \forall k \geq 2$.

As a consequence it also holds, $\forall i, j$:

$$\Psi_{c+1}(Z_i^{(c)}\Lambda(Z_j^{(c+1)})) \equiv \Psi_{c+1}(X_1\Lambda(Z_i^{(c+1)})\Lambda(Z_j^{(c+1)})) \equiv \Psi_{c+1}(Z_j^{(c+1)}\Lambda(Z_i^{(c+1)})) \bmod s^{c+1}$$

and therefore we can conclude

$$\Psi_{c+1}(\mathcal{R}(M'^{(c+1)})) \subset (s^{c+1}), \mathcal{R}(M'^{(c+1)}) \subset \Psi_{c+1}^{-1}(s^{c+1}) = \mathbf{q}^{(c+1)}.$$

Since also $\mathcal{R}(M'^{(c+1)}) \subset \mathcal{R}(M'^{(c)}) \subset \mathcal{I}''$, $\mathbf{P(c+1)}$ holds. ▮

Corollary 14 *If $b > 1$, $\forall c$, $b \leq c \leq a$, $\mathcal{I}'' \cap \mathbf{q}^{(c)}$ is scrolled, and in particular $\mathcal{I}'$ is such.*

Proof: Obviously $\mathbf{P(b)}$ holds choosing $Z_i^{(b)} = X_i, \forall i \geq 2$ since $\mathcal{R}(M'') \subset \mathbf{q}''$. Therefore, by Lemma 13, we know that $\mathbf{P(c)}$ holds $\forall c$. ▮

Example 15 Let us assume $\mathcal{I}' = \mathcal{I}'' \cap \mathbf{q} \subset \mathcal{P} = K[X_1, \ldots, X_8]$ be an ideal in l.g.p., where

- $\mathrm{mult}(\mathcal{I}') = 15, \mathrm{mult}(\mathcal{I}'') = 11, \mathrm{mult}(\mathbf{q}) = 8, \mathrm{mult}(\mathbf{q}'') = 4$;
- $\mathcal{I}'' \supset \mathcal{R}(M'')$;
- $\begin{cases} Y_1 = X_1 + X_3 + X_5 \\ Y_2 = X_2 + X_6 \\ Y_3 = X_3 \\ Y_4 = X_8 \\ Y_5 = X_4 \\ Y_6 = X_5 + X_7 \\ Y_7 = X_5 - X_7 \\ Y_8 = X_6 \end{cases}$ so that $\Phi_8(X_i) = \begin{cases} s - s^3 - \frac{1}{2}s^6 - \frac{1}{2}s^7 & i = 1 \\ s^2 & i = 2 \\ s^3 & i = 3 \\ s^5 & i = 4 \\ \frac{1}{2}s^6 + \frac{1}{2}s^7 & i = 5 \\ 0 & i = 6 \\ \frac{1}{2}s^6 - \frac{1}{2}s^7 & i = 7 \\ s^4 & i = 8 \end{cases}$

Denoting $\Gamma^{(c+1)} = \left(\gamma_2^{(c+1)}, \ldots, \gamma_k^{(c+1)}\right)$ we have:

$$M'^{(4)} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ 1 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \end{pmatrix}$$

$$\Psi_5(M'^{(4)}) = \begin{pmatrix} s - s^3 & s^2 & s^3 & 0 & 0 & 0 & 0 & s^4 \\ 1 & s - s^3 & s^2 & s^3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{(5)} = (-2, 0, 1, 0, 0, 0, -1)$$

$$M'^{(5)} = \begin{pmatrix} X_1 & X_2 - 2X_8 & X_3 & X_4 + X_8 & X_5 & X_6 & X_7 \\ 1 & X_1 - 2X_7 & X_2 & X_3 + X_7 & X_4 & X_5 & X_6 \end{pmatrix}$$

$$\Psi_6(M'^{(5)}) = \begin{pmatrix} s - s^3 & s^2 - 2s^4 & s^3 & s^4 + s^5 & 0 & 0 & 0 \\ 1 & s - s^3 & s^2 & s^3 & s^5 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{(6)} = (0, -1, -1, 0, 0, 0)$$

$$M'^{(6)} = \begin{pmatrix} X_1 & X_2 - 2X_8 & X_3 - X_4 - X_8 & X_5 & X_6 & X_7 \\ 1 & X_1 - 2X_7 & X_2 - X_3 - X_7 & X_4 & X_5 & X_6 \end{pmatrix}$$

$$\Psi_7(M'^{(6)}) = \begin{pmatrix} s - s^3 - \frac{1}{2}s^6 & s^2 - 2s^4 & s^3 - s^4 - s^5 & \frac{1}{2}s^6 & 0 & \frac{1}{2}s^6 \\ 1 & s - s^3 - \frac{3}{2}s^6 & s^2 - s^3 - \frac{1}{2}s^6 & s^5 & \frac{1}{2}s^6 & 0 \end{pmatrix}$$

$$\Gamma^{(6)} = \left(1, 1, \frac{1}{2}, 0, -\frac{1}{2}\right)$$

$$M'^{(7)} = \begin{pmatrix} X_1 & X_2 + 2X_7 - 2X_8 & X_3 - X_4 + 2X_7 - X_8 & X_5 + X_7 & X_6 \\ 1 & X_1 + 2X_6 - 2X_7 & X_2 - X_3 + 2X_6 - X_7 & X_4 + X_6 & X_5 \end{pmatrix}$$

$$\Psi_8(M'^{(7)}) =$$

$$= \begin{pmatrix} s - s^3 - \frac{1}{2}s^6 - \frac{1}{2}s^7 & s^2 - 2s^4 + s^6 - s^7 & s^3 - s^4 - s^5 + s^6 - s^7 & s^6 & 0 \\ 1 & s - s^3 - \frac{3}{2}s^6 + \frac{1}{2}s^7 & s^2 - s^3 - \frac{1}{2}s^6 + \frac{1}{2}s^7 & s^5 & \frac{1}{2}s^6 + \frac{1}{2}s^7 \end{pmatrix}$$

$$\Gamma^{(8)} = \left(-1, \frac{1}{2}, 0, \frac{1}{2}\right)$$

$$M'^{(8)} = \begin{pmatrix} X_1 & X_2 + 2X_6 + 2X_7 - 2X_8 & X_3 - X_4 - X_6 + 2X_7 - X_8 & X_5 + X_7 \\ 1 & X_1 + 2X_5 + 2X_6 - 2X_7 & X_2 - X_3 - X_5 + 2X_6 - X_7 & X_4 + X_6 \end{pmatrix}$$

$$\Psi_8(M'^{(8)}) = \begin{pmatrix} s - s^3 - \frac{1}{2}s^6 - \frac{1}{2}s^7 & s^2 - 2s^4 + s^6 - s^7 & s^3 - s^4 - s^5 + s^6 - s^7 & s^6 \\ 1 & s - s^3 - \frac{1}{2}s^6 - \frac{3}{2}s^7 & s^2 - s^3 - s^6 & s^5 \end{pmatrix}$$

7 Conclusion

Theorem 16 *If $\mathcal{I} \subset \mathcal{P}$ is a 0-dimensional ideal in l.g.p., then it is scrolled.*

Proof: Using the notation of §2, let $\mathcal{I}', \mathcal{I}'', \mathbf{q}_s$ be as defined there.

If $\mathrm{mult}(\mathbf{q}_s) \leq n$, then $\mathcal{I}'$ is scrolled, as a consequence of Prop. 12 and Cor. 14. If, instead, $\mathrm{mult}(\mathbf{q}_s) \geq n+3$, then $s = 1$ and so $\mathcal{I}' = \mathbf{q}_s$ which is therefore $\mathbf{m}$-primary and so it is scrolled as a consequence of Th. 9. $\mathcal{I}'$ being then scrolled, so also is $\mathcal{I}$ by Prop. 10. ∎

Remark 17 It is quite evident that the assumption that $\mathcal{I}$ is in l.g.p. is used to assume that

- $\mathcal{I}''$ is scrolled as a consequence of Th. 3,
- each primary $\mathbf{q}_i$, $i \geq s$ in l.g.p. and of embedded dimension ≤ 1, so allowing to apply Cor. 7 both in §4 and §5. As a consequence it holds also:

Corollary 18 *Let $\mathcal{I} \subset \mathcal{P}$ be a 0-dimensional ideal such that*
- $\mathcal{I} = \mathcal{I}' \cap_{j=1}^{s} \mathbf{q}_j$,
- $\mathcal{I}'$ *is in l.g.p. and such that* $\text{mult}(\mathcal{I}') = n + 3$,
- $\mathbf{q}_j$ *is a primary ideal in l.g.p. and of embedded dimension* $1\ \forall j$,
- $\mathbf{q}_j + \mathcal{I}' = (1)$ *if* $j \neq 1$,
- $\text{mult}(\mathcal{I}') = n + 2 + \delta \leq 2n + 1$.

Then $\mathcal{I}$ is scrolled.

Proof: In fact, the argument of §5 can be applied in this setting to compute a $C \times 2$ matrix M $(C \leq n + 1 - \delta)$ whose minors generate an ideal $\mathcal{S}$ defining a rational normal scroll such that $\dim(\mathcal{S}) = n + 1 - C \geq \delta$ and $\mathcal{I} \supset \mathcal{S}$. A suitable submatrix can then be extracted from M. ∎

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