Bells, Motels and Permutation Groups

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Abstract

This article is about the mathematics of ringing the changes. We describe the mathematics which arises from a real-world activity, that of ringing the changes on bells. We present Rankin's solution of one of the famous old problems in the subject.

1 Introduction: Motels and Bells

In chapter 6 of his book "Time Travel and other Mathematical Bewilderments" [3], Martin Gardner discusses the following problem, a special case of D. H. Lehmer's "motel problem" from [7].

Mr. Smith manages a motel. It consists of n rooms in a straight row. There is no vacancy. Smith is a psychologist who plans to study the effects of rearranging his guests in all possible ways. Every morning he gives them a permutation. The weather is miserable, raining almost daily. To minimize his guests' discomfort, each daily rearrangement is made by exchanging only the occupants of two adjoining rooms. Is there a simple algorithm that will run through all possible rearrangements by switching only one pair of adjacent occupants at each step?

For the purposes of this article, we refer to this problem as the *motel problem*. Before discussing a solution we consider another problem, which at first might seem unrelated. This problem concerns the change-ringing of bells, so we first provide a brief introduction to this topic and an explanation of how permutations arise in the ringing of bells.

Around the year 1600 in England it was discovered that by altering the fittings around each bell in a bell tower, it was possible for each ringer to maintain precise control of when his (there were no female ringers then) bell sounded. This enabled the ringers to ring the bells in any particular order, and either maintain that order or change the order in a precise way.

Suppose there are n bells being rung, numbered $1, 2, 3, \ldots, n$ in order of pitch, number 1 being the highest. When the bells are rung in order of descending pitch $1, 2, 3, \ldots, n$, we say they are being rung in *rounds*. A change in the order of the bells, such as rounds 1 2 3 4 5 being changed to 2 1 4 3 5, can be considered as a permutation in the symmetric group on five objects. In the years 1600–1650 a new craze emerged where the ringers would continuously change the order of the bells for as long as possible, while not repeating any particular order, and return to rounds at the end. This game evolved into a challenge to ring the bells in every possible order, without any repeats, and return to rounds. We will give a precise statement of this challenge shortly. However, the reader can now see why bell-ringers were working with permutations. It has been pointed out before [20] that permutations were first studied in the 1600's in the context of the ringing of bells in a certain order, and not in the 1770's by Lagrange, in the context of roots of polynomials. And, by the way, the craze continues to this day.

A solution to the motel problem is a sequence of all the arrangements (orderings) of n objects, with the property that each arrangement is obtained from its predecessor by a single interchange of two adjacent objects. Algorithms for generating the $n!$ arrangements subject to this condition can be found in S. M. Johnson [5] and H. F. Trotter [17]; see also the papers by D. H. Lehmer [7] and M. Hall Jr.–D. Knuth [4]. Johnson and Trotter discovered the same algorithm, and we describe their algorithm below. Gardner [3] credits Steinhaus [14] as being the first to discover this method. Such algorithms are similar to what bell-ringers are trying to find, as we shall now explain. In fact, the Johnson-Steinhaus-Trotter algorithm was discovered in the 17th century by bell-ringers. We elaborate on this in section 2.

Let us refer to the n bells listed in a particular order or arrangement as a *row*. For example, when $n = 5$, rounds is the row 1 2 3 4 5. We will use cycle notation for permutations. Permutations act on the positions of the bells, and not on the bells themselves; for example, the permutation (1 2)(3 4) changes the row 5 4 3 2 1 into the row 4 5 2 3 1. It is important not to confuse the numbers standing for the bells with the numbers in the cycles, which refer to the positions of the bells. Permutations act on the right, i.e., XY means first do the permutation X, then do Y. As usual, S_n and A_n denote the symmetric group and alternating group on n objects.

The task enjoyed by bell-ringers, which we refer to as the *ringers problem*, is to make a list of rows subject to the following rules:

(1) The first and last rows must be rounds.

(2) No row may be repeated (apart from rounds which appears twice, first and last).

(3) Each bell may only change place by one position when moving from one row to the subsequent row.

(4R) No bell occupies the same position for more than two successive rows.

A *method* is any set of rows that obeys these rules. The origins of rules (1) and (2) have been explained in the introduction. Rule (3) is there for physical reasons. The purpose of rule (4R) is mainly to make ringing interesting for the ringers. For more details see [2] or [19]. Clearly, $n! + 1$ is the longest possible length that a list of rows satisfying rules (1), (2), (3), and (4R) could have, and a list of this length is called an *extent*. The ringers problem is essentially to construct methods of various lengths, and of particular interest is the construction of extents.

The motel problem is not identical to the ringers problem, for if we replace rule (4R) by the following:

(4M) Only two bells may change place when moving from one row to the subsequent row

then a list of $n! + 1$ rows satisfying rules (1), (2), (3), and (4M) is precisely a solution to the motel problem.

We can now see that there is certainly a similarity between the motel problem and the ringers problem. Some previous *Monthly* articles have considered the motel problem [4], [7], and other *Monthly* articles have considered the ringers problem, [1], [2], [19], [20].

We should mention that in practice ringers do have some other rules, but rules (1), (2), (3), (4R) are the most important. We ignore the other rules, and some other ringing matters, for the purposes of this article. For more details see [2] or [19].

In the following sections we discuss these two problems and some solutions. In section 2 we present the first solution to the motel problem, which was discovered by ringers in the 1600's. Section 3 discusses the simplest solution to the ringers problem, and section 4 gives a more complicated solution. We shall see how ringers were manipulating cosets of a dihedral subgroup of the symmetric group in an effort to meet their challenge. Section 5 proposes a new method. In section 6 we present a general group-theoretic framework for the problems. We discuss approaches to finding extents in section 7, and give some proofs of impossibility. In section 8 we also present a proof of a remarkable 1948 result of R. A. Rankin which provides a beautiful solution to a particular problem dating from 1741. The first solution to this problem was given in 1886 by W. H. Thompson. This solution is more subtle than the proofs in section 7. In section 9 we present a small result concerning S_4.

No knowledge of change-ringing is assumed for this article. We keep the bell-ringing terminology to a minimum, although a little is necessary. The mathematics involved is elementary group theory.

2 The Johnson-Steinhaus-Trotter solution

The Johnson-Steinhaus-Trotter solution of the motel problem described below satisfies rules (1), (2), (3), (4M), but not rule (4R) for $n > 3$, and is therefore not a solution to the ringers problem for $n > 3$.

The solution to the motel problem in Johnson [5], Steinhaus [14] and Trotter [17] (see also [3], [4], [7]) is a beautifully simple idea. We summarise the idea as follows. Construct all arrangements of $n+1$ objects inductively, using all arrangements of n objects. The induction begins because 1 is a solution for $n = 1$, or $12, 21$ is a solution for $n = 2$. Expand the list of all arrangements of n objects by replacing each arrangement by $n+1$ copies of itself. Place the new $(n+1)$th object M on the left of the first arrangement. To obtain the next arrangement, we interchange M with the object to the right of M. After doing this n times, we have M on the extreme right. Then we leave M here for one step, as the old n objects undergo their first rearrangement. Next M moves to the left, one place at a time, and then M stays at the extreme left for one step after arriving there, as the old n objects undergo their second rearrangement. We continue this process until we reach the end of the list.

For example, to obtain the six arrangements of three objects from the two arrangements of two objects, we first expand and obtain 23, 23, 23, 32, 32, 32. Then we weave 1 through these as instructed in the algorithm: 123, 213, 231, 321, 312, 132.

To get the solution for four objects, we write the solution for three objects as 234, 324, 342, 432, 423, 243. Then, with 1 as the new object M, we obtain the list of rows shown below. The reader should check the list, and ignore the A's, B's and C's for the moment.

1 2 3 4					
	A		A		A
2 1 3 4		3 1 4 2		4 1 2 3	
	B		B		B
2 3 1 4		3 4 1 2		4 2 1 3	
	C		C		C
2 3 4 1		3 4 2 1		4 2 3 1	
	A		A		A
3 2 4 1		4 3 2 1		2 4 3 1	
	C		C		C
3 2 1 4		4 3 1 2		2 4 1 3	
	B		B		B
3 1 2 4		4 1 3 2		2 1 4 3	
	A		A		A
1 3 2 4		1 4 3 2		1 2 4 3	
	C		C		C
1 3 4 2		1 4 2 3		1 2 3 4	

Now we explain the A, B, C notation. The A stands for the permutation (1 2), B for (2 3) and C for (3 4), acting on the positions. Between two rows we have listed the permutation used to get from one to the other. A shorter way to write this is simply to list the sequence of A's, B's and C's,

$$A, B, C, A, C, B, A, C, A, B, C, A, C, B, A, C, A, B, C, A, C, B, A, C,$$

and we will sometimes do this in the sequel.

We may claim that the motel problem was actually solved in the 1600's by bell-ringers. In the early days ringers did not have rule (4R), and all early methods of changing the order of the bells involved only two adjacent bells switching place at any one time. Such changes were called "plain changes." And so, in fact, rule (4M) was being used instead of rule (4R). Thus, the ringers problem in those days was identical with the motel problem. An exact copy of the solution given above to the motel problem for $n = 4$ can be seen in a book dating from c. 1621. The full 120 plain changes on five bells were being rung by the mid seventeenth century.

Fabian Stedman's Campanalogia [13] published in 1677 gives the Johnson-Steinhaus-Trotter solution of the motel problem for $n = 3, 4, 5$ and 6.

Moreover, the general pattern of the algorithm has been noted, as is evidenced by the following (paraphrased) quote concerning the 6! plain changes on 6 bells.

The method of the seven hundred and twenty, has an absolute dependency upon the method of the sixscore changes on five bells; for five of the notes are to make the sixscore changes, and the sixth note hunts continually through them, and every time it leads or lies behind them, one of the sixscore changes must then be made. The method of the seven hundred and twenty is in effect the same as that of the sixscore; for as the sixscore comprehended the twenty-four changes on four, and the six on three; so likewise the seven hundred and twenty comprehend the sixscore changes on five, the twenty-four changes on four, and the six changes on three. – F. Stedman, 1677

Although the general pattern had been observed, plain changes on more than 6 bells are difficult to ring and bell-ringers progressed to other ways of ringing. There is an earlier book than Stedman's, entitled *Tinntinnalogia*, written by Richard Duckworth and published in 1668, which also discusses plain changes on 3, 4, 5 and 6 bells. This book presumably gives the Johnson-Steinhaus-Trotter solution to the motel problem, but we have not been able to confirm this. Nevertheless, the evidence from Stedman's book would appear to show that their algorithm had been discovered three hundred years before. Knuth [6] states that there is a document written by Mundy dating from 1653 which gives the algorithm.

The following observation will be important in the next sections. The solution to the motel problem given above for $n = 4$ is equivalent to a way of writing down the elements of S_4 in an ordered list, with the property that each element of the list is obtained from the preceding element by multiplication on the right by one of A, B or C. The list would begin $A, AB, ABC, ABCA, ABCAC, \ldots$.

3 Plain Hunt

The simplest method for bell-ringers is called Plain Hunt. We will describe Plain Hunt on 5 bells, and then the generalisation to n bells. But note that not all methods generalise easily to n bells.

Plain Hunt on 5 bells uses two permutations applied alternately to rounds, until rounds comes back again. The permutations are $X = (1\ 2)(3\ 4)$ and $Y = (2\ 3)(4\ 5)$. We write X or Y between two rows to indicate which permutation has been used to get from one to the other. After each element X or Y we list the product of all the elements so far.

1 2 3 4 5	
	X, $(1\ 2)(3\ 4)$
2 1 4 3 5	
	Y, $XY = (1\ 3\ 5\ 4\ 2)$
2 4 1 5 3	
	X, $XYX = (1\ 4)(3\ 5)$
4 2 5 1 3	
	Y, $(1\ 5\ 2\ 3\ 4)$
4 5 2 3 1	
	$X, (1\ 5)(2\ 4)$
5 4 3 2 1	
	$Y, (1\ 4\ 3\ 2\ 5)$
5 3 4 1 2	
	$X, (1\ 3)(2\ 5)$
3 5 1 4 2	
	$Y, (1\ 2\ 4\ 5\ 3)$
3 1 5 2 4	
	$X, (2\ 3)(4\ 5)$
1 3 2 5 4	
	Y,identity
1 2 3 4 5	

Using the above shorthand, we could write this as

$$X, Y, X, Y, X, Y, X, Y, X, Y.$$

After 10 permutations we return to rounds. In other words, there is a total of 11 rows in Plain Hunt on 5, if we include both rounds at beginning and end. We could have predicted this; since X and Y generate a group of order 10 it follows that there can be no more than 10 permutations in Plain Hunt on 5, and indeed in any method only using X and Y. However, we should note that given any two permutations A and B, if they generate a group of order m it does not follow that we can ring a method with $m + 1$ rows using A and B. This is because of Rule (3).

In Plain Hunt on 6 bells, the permutations used are $X = (1\ 2)(3\ 4)(5\ 6)$ and $Y = (2\ 3)(4\ 5)$. On 7 bells we use $X = (1\ 2)(3\ 4)(5\ 6)$ and $Y = (2\ 3)(4\ 5)(6\ 7)$. On 8 bells we use $X = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$ and $Y = (2\ 3)(4\ 5)(6\ 7)$. The generalisation to n bells is now clear.

Note above that X and Y are products of disjoint transpositions of consecutive numbers. This is demanded by rule (3).

The permutations X and Y generate a subgroup H_{2n} of S_n of order $2n$, which we will call the *hunting subgroup* (even though it is better known as the dihedral group D_{2n}). Note that if we take a list of the elements after the comma above, we obtain a list of the elements of the subgroup H_{10}.

Remark 1. Here is the general idea for solving the ringers problem: to devise a method with more than $2n$ rows, we must throw another permutation into the mix. We wish to use as few permutations as possible in order to keep the method as simple as possible, while obtaining a method as long as possible, hopefully with all $n! + 1$ rows.

The first solutions employed by ringers to the ringers problem involved cosets of the hunting/dihedral subgroup H_{2n} of S_n generated by $X = (1\,2)(3\,4)\dots(n-1,\ n)$ and $Y = (2\,3)(4\,5)\dots(n-2,\ n-1)$ when n is even, and $X = (1\,2)(3\,4)\dots(n-2,\ n-1), Y = (2\ 3)(4\ 5)\dots(n-1,\ n)$ when n is odd. These particular involutions are used for the transitions between successive rows because rules (3) and (4R) will be obeyed. As long as we do not apply X or Y twice in succession, rule (2) will be obeyed.

Remark 2. Let us mention here another rule, which roughly states that each bell follows the same path. We will not go into any further detail on this. One can see that this is indeed the case in Plain Hunt, and that it will also hold in a coset of the hunting subgroup. This is why we shall assume for this article that all methods are a union of cosets of the hunting subgroup. Using cosets keeps the method simple, one of the goals from Remark 1. How to choose the cosets such as to obey rules (1)-(3) and (4R) is the real question.

4 Plain Bob

Probably the next simplest method after Plain Hunt is called Plain Bob. This method dates from about 1650. We will describe Plain Bob on 4 bells, and then

6 bells. The idea of Plain Bob is to combine Plain Hunt with some particular cosets.

We do this because Plain Hunt is not yet a solution to the ringers problem of finding an extent of all $n! + 1$ rows. We obtain a solution to the ringers problem for $n = 4$ using the hunting subgroup H_8, a group of order 8, and two left cosets, which are $(2\ 4\ 3)H_8$ and $(2\ 3\ 4)H_8$. The advantage of using cosets of a subgroup is that distinct cosets are disjoint and therefore rule (2) is automatically satisfied. This fact was surely known to, and utilised by, the early composers. Here is the full solution, which uses one other permutation, namely $Z = (3\ 4)$, apart from X and Y. This Z is used in order to switch into the cosets.

We spell this out in detail. First consider Plain Hunt on 4 bells:

1 2 3 4	
	$X, (1\ 2)(3\ 4)$
2 1 4 3	
	$Y, (1\ 3\ 4\ 2)$
2 4 1 3	
	$X, (1\ 4)$
4 2 3 1	
	$Y, (1\ 4)(2\ 3)$
4 3 2 1	
	$X, (1\ 3)(2\ 4)$
3 4 1 2	
	$Y, (1\ 2\ 4\ 3)$
3 1 4 2	
	$X, (2\ 3)$
1 3 2 4	
	Y, identity
1 2 3 4	

where $X = (1\ 2)(3\ 4)$ and $Y = (2\ 3)$. The sequence of elements after the commas consists of the elements of H_8.

In Plain Bob on 4 bells, instead of doing the final Y which takes us back to rounds, we do $Z = (3\ 4)$ instead. We then continue with X and Y alternately. This has the effect of taking us into the left coset $(Y^{-1}Z)H_8$ of the hunting subgroup H_8, i.e., after the comma we will be listing the elements of $(Y^{-1}Z)H_8$. Here is what happens.

3 1 4 2	
	$X,\ XYXYXYX = (2\ 3) = Y^{-1}$
1 3 2 4	
	$Z,\ Y^{-1}Z = (2\ 3)(3\ 4) = (2\ 4\ 3)$
1 3 4 2	
	$X,\ Y^{-1}ZX{=}(1\ 2\ 3)$
3 1 2 4	
	$Y,\ Y^{-1}ZXY{=}(1\ 3)$
3 2 1 4	
⋮	⋮ ⋮

At the same point in this coset when we have reached $Y^{-1}ZXYXYXYX = Y^{-1}ZY^{-1}$, instead of doing the final Y next (which would cause us to repeat 1 3 4 2, disobeying rule (2)) we do Z again, which takes us into the coset $(Y^{-1}Z)^2H_8$. Again we alternate between X and Y to take us through this coset, and at the same point again we do Z, which takes us back to rounds. Here is the full set of rows.

1 2 3 4					
	$X, (1\ 2)(3\ 4)$		$X, (1\ 2\ 3)$		$X, (1\ 2\ 4)$
2 1 4 3		3 1 2 4		4 1 3 2	
	$Y, (1\ 3\ 4\ 2)$		$Y, (1\ 3)$		$Y, (1\ 3\ 2\ 4)$
2 4 1 3		3 2 1 4		4 3 1 2	
	$X, (1\ 4)$		$X, (1\ 4\ 3\ 2)$		$X, (1\ 4\ 2\ 3)$
4 2 3 1		2 3 4 1		3 4 2 1	
	$Y, (1\ 4)(2\ 3)$		$Y, (1\ 4\ 2)$		$Y, (1\ 4\ 3)$
4 3 2 1		2 4 3 1		3 2 4 1	
	$X, (1\ 3)(2\ 4)$		$X, (1\ 3\ 4)$		$X, (1\ 3\ 2)$
3 4 1 2		4 2 1 3		2 3 1 4	
	$Y, (1\ 2\ 4\ 3)$		$Y, (1\ 2\ 3\ 4)$		$Y, (1\ 2)$
3 1 4 2		4 1 2 3		2 1 3 4	
	$X, (2\ 3)$		$X, (2\ 4)$		$X, (3\ 4)$
1 3 2 4		1 4 3 2		1 2 4 3	
	$Z, (2\ 4\ 3)$		$Z, (2\ 3\ 4)$		Z,identity
1 3 4 2		1 4 2 3		1 2 3 4	

This solution could be represented by the sequence of permutations

$$X, Y, X, Y, X, Y, X, Z, X, Y, X, Y, X, Y, X, Z, X, Y, X, Y, X, Y, X, Z.$$

There are a few important observations to make here. Firstly, by adding cosets of H_8 to Plain Hunt, we have increased the number of rows in our set of changes. This is the basic idea of all the methods considered in this paper, as we said in Remark 2.

Secondly, we have listed (after the comma) the permutations in the order

$$H_8\backslash\{\text{identity}\},\ (Y^{-1}Z)H_8,\ (Y^{-1}Z)^2H_8,\ \text{identity}$$

where within each coset we use the order from Plain Hunt.

Thirdly, we note that we obtained 3 cosets in total because $Y^{-1}Z = (2\ 4\ 3)$ has order 3. We also note that the union of the 3 cosets is all of S_4, so in this case we obtained the maximum number of permutations possible, an extent. This does not happen in general.

Next we summarise Plain Bob on 6 bells. Recall that the idea is to add cosets of H_{12} to Plain Hunt on 6, so that we obtain more rows. Of course, this must be done without disobeying any of rules (1), (2), (3), (4R). We recall that the generators of the hunting subgroup are $X = (1\ 2)(3\ 4)(5\ 6)$ and $Y = (2\ 3)(4\ 5)$. In addition we use $Z = (3\ 4)(5\ 6)$, in the same way as in Plain Bob on 4 bells. Since $Y^{-1}Z = (2\ 3)(4\ 5)(3\ 4)(5\ 6) = (2\ 4\ 6\ 5\ 3)$ has order 5, we obtain 5 cosets of H_{12} for a total of 60 permutations (61 rows including both rounds).

Let us check that rules (1)–(3) and (4R) are satisfied. Because X, Y, Z are all products of disjoint transpositions of consecutive numbers and they are the only permutations used to get from one row to the next, rule (3) is satisfied. Rule (1) is also satisfied because of the construction, and rule (2) is satisfied because the cosets are distinct, and therefore disjoint. Rule (4R) is satisfied because it is satisfied for Plain Hunt, and Plain Bob is a union of cosets of Plain Hunt. Here we see that group theory provides us with a construction of a method, and a proof that it obeys the rules, without writing out all the rows.

One can also ring Plain Bob on an odd number of bells. For any n, Plain Bob on n bells uses $n-1$ cosets of H_{2n} and so has $2n(n-1)$ permutations. Only when $n = 4$ does $2n(n-1) = n!$.

This solution to the ringers problem dates from the 17th century, and can be found in Stedman's 1677 book [13].

Note that bell 1 here behaves in the same way as in the section 2 solution to the motel problem. The other bells behave differently however. Unfortunately, this solution to the ringers problem does not generalise to arbitrary n in an obvious way, unlike the motel problem. The task of combining cosets of H_{2n} to obtain a solution to the ringers problem for $n > 4$ is highly nontrivial, and solutions involve many clever ideas. See [19] for more details.

5 A New Method

We now introduce a "new" method on 5 bells which is a union of cosets of H_{10}. We mimic the construction of Plain Bob on 5 bells except that instead of using $Z = (3\ 4)$ we will use $Z = (1\ 2)$. This results in 6 cosets of H_{10} because $Y^{-1}Z = (2\ 3)(4\ 5)(1\ 2) = (1\ 2\ 3)(4\ 5)$ has order 6. Thus we obtain a method with 61 rows (including both rounds).

This method is not listed in the collection of known methods, so we propose to call it Christ Church Dublin Differential Doubles. This is now recognised, see the web site [12] under differentials. We outline a bob for this method in section 7.

6 A Group-Theoretic Formulation

As explained at the end of section 2, the solution to the ringers problem given above is equivalent to a way of writing down the elements of S_4 in a list, with the property that each element of the list is obtained from the preceding element by multiplying by one of X, Y or Z. This idea motivates the following definition.

6.1 Unicursal Generation

Definition. Let G be a finite group of order n, and let T be a subset of G. We say that T generates G *unicursally* if the elements of G can be ordered $g_1, g_2, \ldots, g_n$ so that for each integer i, there exists $t_i \in T$ such that $g_{i+1} = g_i t_i$. (Here subscripts are considered modulo n.)

In this framework, the motel problem can be restated as follows: is S_n generated unicursally by $T = \{(1\ 2), (2\ 3), (3\ 4), \ldots, (n-1,\ n)\}$?

The ringers problem can be restated as follows: is S_n generated unicursally by a subset T satisfying the following conditions:

1. each element of T is a product of disjoint transpositions of consecutive numbers (this is rule 3)
2. t_i and t_{i+1} have no common fixed point for any i (this is rule 4R).

Example 1. Let $G = S_3$, and let $T = \{(1\ 2), (2\ 3)\}$. Then T generates G unicursally. The reader will find it useful to verify this small example.

Example 2. Let G be the D_4 from section 3, and let $T = \{(1\ 2)(3\ 4), (2\ 3)\}$. Then T generates G unicursally as shown in section 3.

Example 3. Let $G = S_4$, and let $T = \{(1\ 2)(3\ 4), (2\ 3), (3\ 4)\}$ as in section 4 in the solution to the ringers problem for $n = 4$.

Example 4. Let $G = S_n$. The solution to the the motel problem described in section 2 shows that S_n is generated unicursally by the $n-1$ elements of $T = \{(1\ 2), (2\ 3), (3\ 4), \ldots, (n-1,\ n)\}$.

Example 5. Let G be any finite group and let $T = G$. Then G is generated unicursally by T, and any ordering of the elements of G can be used, since $g(g^{-1}h) = h$ for any $g, h \in G$.

Example 6. Let $G = S_n$ and let

$$T = \{(1\ 2), (1\ 2)(3\ 4)(5\ 6)\ldots, (2\ 3)(4\ 5)(6\ 7)\ldots\}.$$

It is shown in [10] (see also [19]) that T generates G unicursally. Knuth [6] states that Rapaport's result has been generalised by Savage.

Remarks. In order for a subset T to generate G unicursally it is necessary that T generates G. This condition is not sufficient, as examples 7, 8 and 9 below show.

We are usually interested in the case when T is a small, and often minimal, set of generators. The general question of whether a given G is generated unicursally by a given T seems very difficult. This problem may be related to word problems in the group G.

Given a generating set T for a finite group G, the Cayley colour graph $C_T(G)$ is the graph with the elements of G for vertices, and all directed edges (x, xt) where $t \in T$. Each directed edge is coloured by the generator t. If every element of T has order 2, then the graph may be considered undirected. Usually assumptions are made to ensure that $C_T(G)$ has no loops or multiple edges. The group G acts regularly and transitively on the vertices of $C_T(G)$, and is the automorphism group of $C_T(G)$. The following theorem is clear.

Theorem 1 *A group G is generated unicursally by T if and only if the Cayley colour graph defined by G and T is Hamiltonian.*

This is the point of view of Rapaport [10] and White [18], [19], [20]. White has written several papers on bells and topological graph theory.

The motel problem and the ringers problem are concerned with specific types of subset T of the symmetric group S_n. For the motel problem, as we have said above, T is the set $\{(1\ 2), (2\ 3), (3\ 4), \ldots (n-1,\ n)\}$. For the ringers problem, elements of T will be products of disjoint transpositions of consecutive numbers (because of rule 3), and one must ensure that t_i and t_{i+1} have no common fixed point (because of rule 4R). Which particular T is chosen depends on the method. Two methods will be discussed in this paper, Plain Bob and Grandsire.

Let us generalise and consider the unicursal generation of S_n by elements other than products of disjoint transpositions. First we make a few simple observations about arbitrary groups. The classification of groups generated by T of size 1 is straightforward.

Theorem 2 *A group G is generated unicursally by a subset of size 1 if and only if G is cyclic.*

Proof. Suppose G is generated unicursally by $T = \{x\}$. Assuming $g_1 = 1$ (w.l.o.g.) then $g_2 = x$, and then $g_3 = x^2$, and $g_{i+1} = g_i x = x^i$ for any i. The other implication is clear.

The classification of groups generated by T of size 2 is nontrivial. Clearly if G is isomorphic to a direct product of two cyclic groups then G is unicursally generated by a subset of size 2. The following theorem is a remarkable result of R. A. Rankin on this case. The result in Rankin's paper [8] is more general than the version we state here, and is somewhat based on ideas of Thompson [16]. The end result is very simply stated in group theoretic language, even though the problem and the proof are somewhat combinatorial. We give a proof in section 8.

Let $\langle g\rangle$ denote the subgroup generated by g.

Theorem 3 *(Rankin, 1948) Let G be a finite group. Suppose that G is generated by $T = \{x, y\}$, and that $\langle x^{-1}y\rangle$ has odd order. If G is generated unicursally by T, then $\langle x\rangle$ and $\langle y\rangle$ have odd index in G.*

Example 7. It is easily checked that $G = A_4$ is generated by $T = \{A, B\}$ where $A = (1\ 2\ 3), B = (1\ 2\ 4)$. However, A_4 is not generated unicursally by T by Rankin's theorem, because $A^{-1}B = (1\ 3\ 4)$ has odd order but $\langle A\rangle$ has index 4 in A_4.

Example 8. It is easily checked that A_5 is generated by $T = \{A, B\}$, where $A = (1\ 3\ 5\ 4\ 2)$ and $B = (3\ 5\ 4)$. Here $A^{-1}B$ has order 3, so Rankin's theorem applies. Since $\langle B\rangle$ has even index in A_5 we conclude that A_5 is not generated unicursally by T.

Example 9. It is not hard to show that S_n is generated by the transposition $A = (n-1,\ n)$ and the $(n-1)$-cycle $B = (1\ 2\ 3\ \ldots n-1)$. We may well ask whether S_n is generated unicursally by A and B. Suppose n is odd and $n > 3$. Then S_n is not generated unicursally by $T = \{A, B\}$ by Rankin's theorem, because $A^{-1}B = (1\ 2\ \ldots n)$ has odd order but $\langle A\rangle$ has even index in S_n for $n > 3$.

Example 10. It is not hard to show that S_n is generated by $\sigma = (1\ 2\ \ldots n)$ and $\tau = (1\ 2)$. We (of course) ask if S_n is generated unicursally by these elements. Rankin's theorem shows that the answer is negative if $n \geq 4$ is even, since $\tau^{-1}\sigma$ is an $(n-1)$-cycle. We will mention this example again soon.

From the discussion in the previous sections, the following is now obvious (and has been observed before, see [4] for example). Let t_i be as above.

Theorem 4 *1. The existence of an extent on n bells satisfying rules (1)-(3), where the allowed permutations between rows are $X_1, \ldots, X_k$, is equivalent to $T = \{X_1, \ldots, X_k\}$ generating S_n unicursally.*

2. *The existence of an extent on n bells satisfying rules (1),(2),(3),(4R), where the allowed permutations between rows are $X_1, \ldots, X_k$, is equivalent to $T = \{X_1, \ldots, X_k\}$ generating S_n unicursally with the additional property that t_i and t_{i+1} have no common fixed point for any i.*

A permutation X_i being an allowed transition between rows is equivalent to X_i being a product of disjoint transpositions of consecutive numbers, by rule (3). No bell staying in the same place for more than two rows (rule (4R)) is equivalent to no two consecutive transitions having a common fixed point.

As we mentioned in Remark 2 at the end of section 3, we assume that the methods in this article are cosets of H_{2n}. Therefore, included in T will be X and Y, the generators for the hunting subgroup H_{2n}. But note that even if S_n is generated unicursally by T, it does not follow that it is generated unicursally as cosets of H_{2n}.

Remark 3. In the ringing methods we discuss in this paper, one can divide up an extent on n bells into groups of $2n$ rows called *leads*, roughly (but not exactly) corresponding to cosets of H_{2n}. A method composed of cosets of H_{2n} is also a method made up of a succession of leads. In this paper we use only two types of leads, and we consider methods and extents made from a sequence of these leads. In these cases then, one can show that the existence of an extent on n bells of this type is equivalent to the alternating group A_{n-1} being generated unicursally by T', where T' is a set of generators related to T. For more details on this, see section 6.

The general question of whether a given G is generated unicursally by a given T seems very difficult.

6.2 The famous old question

The famous old question mentioned in the introduction concerns the three permutations $X = (1\ 2)(3\ 4)(5\ 6), Y = (2\ 3)(4\ 5)(6\ 7)$ and $Z = (1\ 2)(4\ 5)(6\ 7)$ in S_7, and whether S_7 is generated unicursally by X, Y, Z in a particular way. We will explain this in detail in section 7.

This was asked in 1741 by a bell-ringer John Holt, who was able to construct a method of 4998 permutations, but could not obtain a method of $7! = 5040$ permutations. He then (naturally!) queried the existence of such an extent. As in Remark 3, it can be shown (see [8], or section 7) that this question is equivalent to:

Question A. Is A_6 (acting on $\{2, 3, 4, 5, 6, 7\}$) generated unicursally by the two permutations $(3\ 4\ 6\ 7\ 5)$ and $(2\ 4\ 7)(3\ 6\ 5)$?

The first proof that the answer is no is due to Thompson [1] (1886) [16], with some case-by-case analysis. An insightful proof was given by Rankin (1948) [8],

[1] Thompson was a civil servant in India at the time, and a Cambridge mathematics gradute.

where he came up with theorem 3 based somewhat on Thompson's ideas. See also [9] and a proof by Swan [15]. We present Rankin's proof in section 8, in our special case only. Rankin's result is more general. Most of the ideas of the proof can be found in our proof of the special case in section 8. We also show in section 8 that 4998 is best possible.

6.3 Open Questions

The first concerns example 9. The argument there works when n is odd, so it is natural to inquire as to what happens when n is even. Thus, we wonder whether S_n is generated unicursally by $A = (n-1,\ n)$ and $B = (1\,2\,3\ \ldots n-1)$. Rankin's theorem does not apply directly. However, in the $n = 4$ case, by modifying the argument in the proof of Rankin's theorem, we will show (see section 9) that S_4 is not generated unicursally by $A = (1\,2\,3)$ and $B = (3\,4)$. The question for even $n \geq 6$ remains open, as far as we are aware.

Problem 1: Let $n \geq 6$ be even. Is S_n generated unicursally by $T = \{A, B\}$ where $A = (n-1,\ n)$ and $B = (1\ 2\ 3\ \ldots n-1)$?

According to Knuth [6] a similar question was asked in 1975 by Nijenhuis and Wilf in their book *Combinatorial Algorithms*. They asked if S_n is generated unicursally by $\sigma = (1\ 2\ \ldots n)$ and $\tau = (1\ 2)$. Rankin's theorem shows that the answer is negative if $n \geq 4$ is even (see example 10). Recently, Ruskey-Jiang-Weston [11] did a computer search for $n = 5$ and did find that S_5 IS unicursally generated by σ and τ. Thus this question is different to problem 1.

This example is not relevant to bells because the generators are not products of disjoint transpositions. However, as the answer to problem 1 may well be negative, the discussion raises the natural question of whether S_n is generated unicursally by any two of its elements. (Example 4 shows that S_n is generated unicursally by three of its elements.) We have found that S_4 IS generated unicursally by $A = (1\ 2\ 3)$ and $B = (1\ 2\ 3\ 4)$. Here is one listing which does the trick:

$$B, A, B, A, A, B, B, B, A, B, B, A, A, B, B, B, A, A, B, A, B, B, B, A.$$

The question for $n \geq 5$ remains open.

Problem 2: Is S_n generated unicursally by some two elements for all n?

By the above comments, the $n = 5$ case has been done.

Remarks. A similar eighteenth-century problem to our famous old question, concerning another method called Stedman, remained unsolved until 1995. We may discuss this in a future article. Right transversals of $PSL(2, 5)$ in S_7 are used to construct extents in this problem, see [19].

Readers interested only in the proof of the answer to question A should skip ahead to section 8.

7 Leads

In this section we shall explain in detail the two types of leads mentioned in Remark 3. We only deal in this section with the methods of Plain Bob on 4 and 6 bells, and Grandsire on 5 and 7 bells, although our remarks have wider application. Then we shall explain the origin of the famous old question.

7.1 Plain Bob

Consider the 25 rows in Plain Bob on 4 bells in section 4; the first one is rounds, and then the rows can be divided into three sets of eight. Each of these sets of eight is called a *lead*. In each of these leads, note that each bell is twice in the first position. Also note that bell number 1 is always in the first position in the last two rows of each lead. The second of these two rows, which is the last row of the lead, is called the *lead head*. This holds in general, for Plain Bob on n bells, where leads have $2n$ rows. In Plain Bob on 4 bells, the lead heads are 1 3 4 2, 1 4 2 3, and 1 2 3 4. The following are simple observations:

1. The first, second and third lead heads are the result of $P = Y^{-1}Z = (2\ 4\ 3)$, P^2 and P^3 respectively acting on rounds.
2. When considering only lead heads, we may drop the 1 in the first position.
3. Plain Bob can then be described by elements of S_{n-1} acting on lead heads.

Now we can fully describe Plain Bob on 6 bells by its lead heads. Here is the first lead (with initial rounds included as well):

1 2 3 4 5 6

$X, (1\ 2)(3\ 4)(5\ 6)$

2 1 4 3 6 5

$Y, (1\ 3\ 5\ 6\ 4\ 2)$

2 4 1 6 3 5

$X, (1\ 4)(3\ 6)$

4 2 6 1 5 3

$Y, (1\ 5\ 4)(2\ 3\ 6)$

4 6 2 5 1 3

$X, (1\ 6)(2\ 4)(3\ 5)$

6 4 5 2 3 1

$Y, (1\ 6)(2\ 5)(3\ 4)$

6 5 4 3 2 1

$X, (1\ 5)(2\ 6)(3\ 4)$

5 6 3 4 1 2

$Y, (1\ 4\ 5)(2\ 6\ 3)$

5 3 6 1 4 2

$X, (1\ 3)(2\ 5)(4\ 6)$

3 5 1 6 2 4

$Y, (1\ 2\ 4\ 6\ 5\ 3)$

3 1 5 2 6 4

$X, (2\ 3)(4\ 5)$

1 3 2 5 4 6

$Z, (2\ 4\ 6\ 5\ 3)$

1 3 5 2 6 4

In this case $P = Y^{-1}Z = (2\ 4\ 6\ 5\ 3)$ and the lead heads are 3 5 2 6 4, 5 6 3 4 2, 6 4 5 2 3, 4 2 6 3 5 and 2 3 4 5 6, corresponding to P, P^2, P^3, P^4 and P^5 respectively, acting on 2 3 4 5 6. Each of these leads is called a *plain* lead. Each lead head is obtained from the previous lead head by applying P. There are five leads because P has order 5. This sequence of five plain leads is called a *plain course*.

As we said in Remark 3, there are only two types of lead considered in this paper. Let us now describe the other type of lead, at least as far as Plain Bob on 6 bells goes. Mathematically there is no reason to have a method made of only two or three types of lead, but this is usually what is done in practice for simplicity and historical reasons (recall Remark 1). The complete method is made up of a succession of leads.

Any plain lead may be described by the sequence of permutations

$$X, Y, X, Y, X, Y, X, Y, X, Y, X, Z.$$

Alternatively, considering only lead heads, we describe a plain lead by P, and the plain course (which has 60 permutations) by the sequence of five plain leads

$$P, P, P, P, P.$$

The other type of lead is called a *bob* lead and may be described by the sequence of permutations

$$X, Y, X, Y, X, Y, X, Y, X, Y, X, W$$

where $W = (2\ 3)(5\ 6)$. If this were done from rounds we would get

$$\begin{array}{ll}
1\ 2\ 3\ 4\ 5\ 6 & \\
 & X,\ (1\ 2)(3\ 4)(5\ 6) \\
2\ 1\ 4\ 3\ 6\ 5 & \\
 & Y, (1\ 3\ 5\ 6\ 4\ 2) \\
2\ 4\ 1\ 6\ 3\ 5 & \\
 & X, (1\ 4)(3\ 6) \\
4\ 2\ 6\ 1\ 5\ 3 & \\
\vdots & \vdots \\
3\ 5\ 1\ 6\ 2\ 4 & \\
 & Y, (1\ 2\ 4\ 6\ 5\ 3) \\
3\ 1\ 5\ 2\ 6\ 4 & \\
 & X, (2\ 3)(4\ 5) \\
1\ 3\ 2\ 5\ 4\ 6 & \\
 & W, (4\ 6\ 5) \\
1\ 2\ 3\ 5\ 6\ 4 &
\end{array}$$

This is a bob lead.

As we used P to denote a plain lead we shall use B to denote a bob lead. We can now construct longer methods using a combination of plain and bob leads. Here is one such method:

$$P, P, P, P, B, P, P, P, P, B, P, P, P, P, B.$$

This corresponds to doing the first 59 of the 60 permutations in the plain course. The 60th permutation that would be performed in a plain course is Z, which would bring us back to rounds. Instead of this last Z, we do $W = (2\ 3)(5\ 6)$. This has the effect of putting us into another coset, namely BH_{12} where $B = Z^{-1}W = (3\ 4)(5\ 6)(2\ 3)(5\ 6) = (2\ 3\ 4)$. We then repeat the same 59 permutations, then do W again, then the 59 and then W again, which returns us to rounds since B has order 3. We finish up with a method of 180 permutations. By a similar argument as in section 4, rules (1),(2),(3),(4R), are obeyed.

We have still not succeeded in getting an extent of $6! = 720$ permutations. It is possible that some other sequence of plain and bob leads will give us an extent. The following result ends all hope of this.

Theorem 5 *There does not exist an extent of Plain Bob on 6 bells using plain and bob leads. The longest possible method using plain and bob leads has* 360 *permutations, and there does exist such a method.*

Proof: The key to the proof is to observe that P, B, Z and W are all even permutations. The fact that P and B are even implies that any lead head will be an even permutation of 2 3 4 5 6. Also, the row before a lead head is the result of applying either Z^{-1} or W^{-1} to the lead head. Since Z and W are even, we see that in any method of plain and bob leads all rows with bell 1 in the first position are followed by an even permutation of 2 3 4 5 6. The result follows, because if we did obtain an extent we would get all possible permutations of 2 3 4 5 6 following 1.

This argument also shows that any method using plain and bob leads has at most $5!/2 = 30$ leads, since each lead has two rows with 1 in the first position, and these rows must be followed by an even permutation of 2 3 4 5 6. Each lead has 12 rows, so a method with plain and bob leads has at most $12 \times 30 = 360$ permutations.

To show that 360 is possible we give an ordering:

$$B, P, P, P, B, B, P, P, P, P, B, P, P, P, B, B, P, P, P, P, B, P, P, P, B, B, P, P, P, P.$$

The reader may check that this sequence of plain and bob leads obeys rules (1),(2),(3),(4R).

Remarks. The use of plain and bob leads applies to Plain Bob on n bells (and other methods). The number of leads in an extent on n bells is $n!/(2n) = (n-1)!/2$, which is the cardinality of A_{n-1}. If P and B generate A_{n-1} unicursally, then we can construct an extent made up of plain and bob leads.

On 6 bells it is true that P and B generate A_5, but example 4 and theorem 5 both show that they do not generate A_5 unicursally. Example 4 used Rankin's theorem, but theorem 5 gives a different and shorter proof. The argument in theorem 5 is shorter because parity can be used to answer the question. The famous old question is an analogous question about A_6 requiring a more delicate argument since it is nearly possible to acheive an extent.

In the language of section 8, the longest chain generated by P and B has length 30.

This proof gives the idea of how to construct an extent: use odd permutations for Z and W but even permutations for P and B. Any method with these properties has a chance of working. This idea leads to results of Saddleton (see theorems 4.8 and 4.11 of [19]).

In practice another type of lead, called a single lead, is used to obtain an extent.

7.2 Grandsire

We now consider another method, the last of this article. The method named Grandsire (pronounced grand-sir) is rung on an odd number of bells. It was developed in the 1650's by Robert Roan on 5 bells, and extensions to 7 and more bells took place in the late 1600's or later. The problem we referred to in the abstract is on 7 bells, but first we explain Grandsire on 5 bells.

The hunting subgroup H_{10} is generated by $X = (1\ 2)(3\ 4)$ and $Y = (2\ 3)(4\ 5)$ as usual. We introduce $Z = (1\ 2)(4\ 5)$, but the first difference in Grandsire from Plain Bob is that we do Z at the very start. This is irrelevant from a mathematical point of view. Then we do Y, and then alternate X and Y until we have run through the coset ZH_{10}. The last permutation done will be Y, and in total we will have done $ZYXYXYXYXY = ZX$. Then we repeat the permutations, i.e., do $Z, Y, X, Y, \ldots, X, Y$ until we have run through the coset $(ZXZ)H_{10}$, and then we repeat the permutations again, running through $(ZXZXZ)H_{10}$, and then we are back at rounds. Here are the rows of 3 plain leads, a plain course. Neglecting the first row which is the initial rounds, each column is a plain lead.

1 2 3 4 5					
	Z		Z		Z
2 1 3 5 4		2 1 5 4 3		2 1 4 3 5	
	Y		Y		Y
2 3 1 4 5		2 5 1 3 4		2 4 1 5 3	
	X		X		X
3 2 4 1 5		5 2 3 1 4		4 2 5 1 3	
	Y		Y		Y
3 4 2 5 1		5 3 2 4 1		4 5 2 3 1	
	X		X		X
4 3 5 2 1		3 5 4 2 1		5 4 3 2 1	
	Y		Y		Y
4 5 3 1 2		3 4 5 1 2		5 3 4 1 2	
	X		X		X
5 4 1 3 2		4 3 1 5 2		3 5 1 4 2	
	Y		Y		Y
5 1 4 2 3		4 1 3 2 5		3 1 5 2 4	
	X		X		X
1 5 2 4 3		1 4 2 3 5		1 3 2 5 4	
	Y		Y		Y
1 2 5 3 4		1 2 4 5 3		1 2 3 4 5	

It is because the first lead head is the result of $ZX = (1\ 2)(4\ 5)(1\ 2)(3\ 4) = (3\ 4\ 5)$ which has order 3, that we get back to rounds after $3 \times 10 = 30$ permutations, and a plain course has 3 plain leads.

As with the Plain Bob method, we sometimes add bob leads to the above plain course to obtain a longer method. The bob lead uses the permutation Z applied instead of the last X in a plain lead, which would be *two places before* the next appearance of Z in a plain course.

1 2 3 4 5
Z
2 1 3 5 4
Y
2 3 1 4 5
X
3 2 4 1 5
Y
3 4 2 5 1
X
4 3 5 2 1
Y
4 5 3 1 2
X
5 4 1 3 2
Y
5 1 4 2 3
Z
1 5 4 3 2
Y
1 4 5 2 3
Z

If we use the bob before the first lead head, as shown above, the first lead head will be 1 4 5 2 3 instead of 1 2 5 3 4. This is the result of (2 4)(3 5) applied to rounds. Since this permutation has order 2, we will return to rounds after using the bob twice in that place. In other words, the method consisting of leads B, P, P, B, P, P increases the number of permutations from 30 to 60.

It is reasonable to ask, as usual, if we could obtain a larger set of permutations by using plain and bob leads in a different arrangement. The answer is no. To see this, simply note that each of X, Y, Z is even. Therefore the largest possible number of permutations they can generate is 60 (the order of A_5), which in fact is the case as we have shown.

Theorem 6 *There does not exist an extent of Grandsire on 5 bells using plain and bob leads. The longest possible method using plain and bob leads has* 60 *permutations, and there does exist such a method.*

To obtain the maximum number of permutations on 5 bells we would need to use an odd permutation, which involves another type of lead called a single lead. We do not discuss single leads in this article.

Next we consider Grandsire on 7 bells, or Grandsire Triples as it is known to ringers, which is more interesting mathematically than Grandsire on 5 bells. In this case $X = (1\ 2)(3\ 4)(5\ 6)$ and $Y = (2\ 3)(4\ 5)(6\ 7)$ generate the hunting subgroup H_{14}, and Grandsire Triples uses $Z = (1\ 2)(4\ 5)(6\ 7)$. As on 5 bells, we do Z first and then alternate Y and X, and repeat. The first lead head is the result of $ZX = (1\ 2)(4\ 5)(6\ 7)(1\ 2)(3\ 4)(5\ 6) = (3\ 4\ 6\ 7\ 5)$ on rounds, which is 1 2 5 3 7 4 6. Since ZX has order 5 there are 5 plain leads and 70 permutations in a plain course of Grandsire on 7 bells.

To extend the method we use bob leads. The bob lead uses the permutation Z applied instead of the last X in a plain lead, as in Grandsire on 5 bells. If we do a bob lead on the earliest possible occasion, the first lead head would become 1 7 5 2 6 3 4. This is the result of $B = (2\ 4\ 7)(3\ 6\ 5)$ acting on rounds. Since B has order 3, we obtain a total of 210 permutations if we use $B, P, P, P, P, B, P, P, P, P, B, P, P, P, P$.

It is possible to obtain larger sets of permutations by using plain and bob leads in different sequences. We may then reasonably wonder as to the largest set we can get. On 5 bells we used that fact that X, Y, Z are even to obtain an upper bound (which was 60). This argument will not work here, since X and Y are odd. It is, in fact, conceivable that we could achieve an extent of all $7! = 5040$ permutations. As before, it is enough to consider the action of P and B on lead heads. In 1741 John Holt came very close to an extent and obtained 4998 permutations. This gave rise to the famous old question mentioned in the introduction and section 6:

Famous Old Question. Is it possible to ring all the 5040 permutations on seven bells using the Grandsire method and plain and bob leads only? In other words, does there exist an extent of Grandsire on 7 bells using plain and bob leads?

Considering only lead heads, first note that there would be $5040/14 = 360$ lead heads. Next check that $P = (3\ 4\ 6\ 7\ 5)$ and $B = (2\ 4\ 7)(3\ 6\ 5)$ generate A_6. If P and B generate A_6 unicursally then we would obtain the extent we are looking for. We therefore arrive at the following question in order to answer the Famous Old Question:

Question A. Is A_6 (acting on $\{2, 3, 4, 5, 6, 7\}$) generated unicursally by $P = (3\ 4\ 6\ 7\ 5)$ and $B = (2\ 4\ 7)(3\ 6\ 5)$?

7.3 A bob for the new method

We must define a bob lead for the "new" method proposed in section 5. We propose the permutation $W = (3\ 4)$ instead of the last Z in a lead. Thus a plain lead has the permutations

$$X, Y, X, Y, X, Y, X, Y, X, Z$$

and a bob lead has the permutations

$$X, Y, X, Y, X, Y, X, Y, X, W.$$

The sequence of leads $P, P, P, B, P, P, P, B, P, P, P, B$ yields an extent of all 120 permutations.

8 Proofs

We now proceed to give Rankin's proof that question A has a negative answer. As we explained in section 7, this implies that the answer to the Famous Old Question is no. The proof is by contradiction.

We begin by supposing that A_6 is generated unicursally by $P = (3\ 4\ 6\ 7\ 5)$ and $B = (2\ 4\ 7)(3\ 6\ 5)$. Assume there exists an ordering $g_1, g_2, \ldots, g_{360}$ of the elements of A_6, with the property that for each i, $g_{i+1} = g_i t_i$ for some $t_i \in T = \{P, B\}$ (with subscripts modulo 360). We will call any sequence of elements $g_1, \ldots, g_m$ a *chain* of length m if it has the property that for each i, $g_{i+1} = g_i t_i$ for some $t_i \in T = \{P, B\}$. With subscripts modulo m, we consider a chain to be an infinite cyclic sequence. If $g_{i+1} = g_i P$ then we say that g_i is *acted on* by P, and similarly for B. Each element of A_6 is acted on by exactly one of P and B, by assumption.

The key is to consider left cosets of the cyclic subgroup C of order 5 generated by

$$\gamma = BP^{-1} = (2\ 4\ 7)(3\ 6\ 5)(3\ 5\ 7\ 6\ 4) = (2\ 3\ 4\ 6\ 7).$$

This was noted by Thompson, who called these cosets "Q-sets". We present Rankin's argument in a series of observations, each one a lemma. The idea of the proof is to show that, under a certain transformation, the parity of the number of chains remains constant.

Lemma 1 *Every element of a coset xC of C is acted on by the same element.*

Proof: Suppose $x\gamma^i$ is acted on by P (a similar argument holds for B). Then the next element in the chain is $x\gamma^i P$. But $x\gamma^i P = x\gamma^{i-1}(BP^{-1})P = x\gamma^{i-1}B$. To avoid repetition therefore, B cannot act on $x\gamma^{i-1}$, so P must act on $x\gamma^{i-1}$. This argument is valid for any i.

We now consider where the elements of a coset xC appear in the chain. For each i between 1 and 5, we define a positive integer k_i between 1 and 5, by letting $x\gamma^{k_i}$ be the next element of xC in the chain after $x\gamma^i$. This defines a permutation in S_5, which in two-line notation is

$$\sigma(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ k_1 & k_2 & k_3 & k_4 & k_5 \end{pmatrix}$$

for each coset xC.

Lemma 2 *The permutation $\sigma(x)$ is a 5-cycle.*

Proof: Our assumption on the existence of a chain of length 360 implies that $\sigma(x)$ is a 5-cycle.

We will now rearrange the chain. It is possible that the 'length 360 chain' property may be destroyed during the rearrangement, i.e., the single chain of length 360 may become several (disjoint) chains of smaller length.

Here is how the rearrangement is done. Let xC be a coset acted on by B (if there is no such coset then every element of A_6 is acted on by P, which is impossible if there is only one chain). Then the next element in the chain after $x\gamma^i$ is $x\gamma^i B$. The chain can be divided up into 5 segments with respect to xC, each segment being a sequence beginning with $x\gamma^i B$ and ending with $x\gamma^{k_i}$. By definition of k_i, there are no elements of xC in the chain from $x\gamma^i B$ to the element immediately preceding $x\gamma^{k_i}$. In other words, in these segments with respect to xC, the only element in a segment that is in xC is the last element.

$$\cdots x\gamma^i][x\gamma^i B \cdots x\gamma^{k_i}][x\gamma^{k_i} B \cdots$$

We permute these segments, so that the segment after $x\gamma^i$ now begins with $x\gamma^{i-1}B$ (and ends with $x\gamma^{k_{i-1}}$).

$$\cdots x\gamma^i][x\gamma^{i-1} B \cdots x\gamma^{k_{i-1}}][x\gamma^{k_{i-1}-1} B \cdots$$

Lemma 3 *After the rearrangement, the coset xC is acted on by P. All other cosets are unaffected, in terms of whether they are acted on by P or B.*

Proof: Note that $x\gamma^{i-1}B = x\gamma^{i-1}BP^{-1}P = x\gamma^i P$, so the element after $x\gamma^i$ is $x\gamma^i P$. This implies that $x\gamma^i$, and therefore the coset xC, is now acted on by P in the rearrangement. The proof of the second part is clear from the construction.

Lemma 4 *After the rearrangement there may be more than one chain, but every chain contains an element of xC.*

Proof: The rearrangement may alter the number of chains because, for example, we may have $k_1 = 3$ and $k_2 = 2$. In this case, after rearranging, we would have

$$[x\gamma B \cdots x\gamma^3][x\gamma^2 B \cdots x\gamma^2]$$

which is a chain, and the other segments would form at least one other chain.

If there were a chain not containing any element of xC after the rearrangement, this chain would not be affected by the rearranging, and would therefore have existed before the rearrangement. But there was only one chain before the rearrangement!

The next element of xC after $x\gamma^i$ in the new arrangement is $x\gamma^{k_{i-1}}$, so we define a permutation $\tau(x)$ in a similar manner to $\sigma(x)$:

$$\tau(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ k_5 & k_1 & k_2 & k_3 & k_4 \end{pmatrix}.$$

In the following, "cycles" means disjoint cycles including 1-cycles, as usual.

Lemma 5 *The number of cycles in $\tau(x)$ is equal to the number of chains after the rearrangement.*

Proof: This follows from Lemma 4.

Lemma 6 *$(1\ 2\ 3\ 4\ 5)\tau(x) = \sigma(x)$.*

Proof: This is straightforward.

Lemma 7 *The number of cycles in $\tau(x)$ is odd.*

Proof: Suppose $\tau(x)$ has k cycles. The number of cycles in $(1\ 5)\tau(x)$ is $k+1$ if 1 and 5 are in the same cycle in $\tau(x)$, and $k-1$ otherwise. Hence the number of cycles in $(1\ 4)(1\ 5)\tau(x)$ has the same parity as the number of cycles in $\tau(x)$. Similarly the number of cycles in $(1\ 2)(1\ 3)(1\ 4)(1\ 5)\tau(x)$ has the same parity as the number of cycles in $\tau(x)$. Lemmas 2 and 6 now give the result.

We point out that this proof works since 5 is odd, so any odd number could be used. If an even number is used instead of 5 then the proof shows that the parity changes. This will be used in the proof of theorem 8.

Let r_x denote the number of chains after the rearrangement with respect to xC. The following fact is all-important.

Lemma 8 *r_x is odd.*

Proof: Combine Lemmas 5 and 7.

We now rearrange again with respect to another coset yC that is acted on by B. (If there is no such coset, skip to the Famous Old Theorem below.) Let r_y denote the number of chains after the rearrangement with respect to yC. Again we define $k_i = k_i(y)$ and $\sigma(y)$ in a similar manner. Lemma 2 becomes

Lemma 9 *The number of chains (before rearrangement with respect to yC) containing elements of yC is equal to the number of cycles in $\sigma(y)$.*

Lemma 3 shows that after rearranging, yC is acted on by P. Lemma 4 becomes

Lemma 10 *The number of chains not containing elements of yC remains constant.*

Proof: Shown in proof of Lemma 4.

We define $\tau(y)$ similarly to $\tau(x)$. Lemma 5 becomes

Lemma 11 $r_x - (\textit{number of cycles in } \sigma(y)) = r_y - (\textit{number of cycles in } \tau(y))$.

Proof: The lefthand side is the number of chains before rearrangement not containing elements of yC, and the righthand side is the number of such chains after rearrangement.

Lemma 6 remains the same, with y in place of x, and Lemma 7 becomes

Lemma 12 $(\textit{number of cycles in } \tau(y)) \equiv (\textit{number of cycles in } \tau(x)) \mod 2$.

Proof: Same as Lemma 7.
Again, the following will be important.

Lemma 13 *r_y is odd.*

Proof: By Lemma 8 and Lemmas 11 and 12.

The series of lemmas shows that after this second rearrangement with respect to yC, the number of chains is still odd. Repeat the rearrangement with respect to every coset acted on by B until all cosets are acted on by P. The point is that after each rearrangement with respect to a coset acted on by B, the coset is now acted on by P, and also we can apply the lemmas and conclude that the number of chains remains odd.

Theorem 7 (*Thompson*)

1. *A_6 (acting on $\{2,3,4,5,6,7\}$) is not generated unicursally by $P = (3\ 4\ 6\ 7\ 5)$ and $B = (2\ 4\ 7)(3\ 6\ 5)$.*
2. *It is not possible to ring all the 5040 permutations on seven bells using the Grandsire method and plain and bob leads. In other words, there does not exist an extent of Grandsire on 7 bells using plain and bob leads.*

Proof: (Rankin) We showed in section 7 that (2) is implied by (1). To prove (1), by the above lemmas we may assume that all cosets are acted on by P. Then every element of A_6 is acted on by P. The chains we have must therefore be the cosets of the subgroup M generated by P. By Lemma 13, the number of chains is odd. However, since P has order 5, there are $|A_6 : M| = 360/5 = 72$ cosets, which is an even number. This contradiction proves the theorem.

We remark that the roles of P and B could be interchanged in the above proof, since the subgroup generated by B also has even index.
The following is shown in [1].

Corollary 1 *The largest number of permutations that can be rung on seven bells, using the Grandsire method and plain and bob leads, is 4998.*

Proof: We know that one chain of length 360 (in A_6) is not possible. The shortest possible chain has length 3 since B has order 3, so the longest possible chain has length ≤ 357. A chain with 357 leads has $357 \times 14 = 4998$ permutations.
As we said earlier, there does exist a method with 4998 permutations due to Holt, so 4998 is best possible.

9 A New Result

We next prove a small result concerning problem 1 of section 6. Observe that the result is not trivial since there are 2^{24} possible orderings of A and B.

Theorem 8 *S_4 is not unicursally generated by $A = (3\ 4)$ and $B = (1\ 2\ 3)$.*

Proof: Suppose to the contrary that S_4 is unicursally generated by A and B. As in the sequence of lemmas, we rearrange with respect to left cosets of $C = \langle A^{-1}B \rangle$ that are acted on by A. After the rearrangement, such a coset is acted on by B.

Since C has order 4 which is even, the proof of Lemma 7 shows that after the rearrangement with respect to xC, the number of cycles in $\tau(x)$ is even, and in general that the parity of the number of cycles changes after each rearrangement with respect to a coset of C.

Suppose we perform a total of m rearrangements to get all cosets acted on by B. By Lemma 3 second sentence, the number of cosets acted on initially by A is m. Note that $m \leq 6$ as $|C| = 4$. After all m rearrangements, the number of chains has the opposite parity to m, by the previous paragraph.

By assumption, we start with one chain of length 24. If m is even then the number of chains (after all rearrangements) is odd. The proof of Rankin's theorem above shows that $\langle B \rangle$ has odd index, a contradiction.

The only alternative is that m is odd, and we have 1, 3, or 5 cosets of C acted on by A initially. If 1 coset was acted on by A, then 5 cosets were acted on by B. Therefore 20 elements of S_4 were acted on by B, which implies there are three consecutive B's. This is not possible as B has order 3. If 5 cosets were acted on by A, then 1 coset was acted on by B, and a similar argument gives a contradiction.

The final possibility is that 3 cosets were acted on by A and 3 by B. Then 12 elements of S_4 are acted on by each of A and B. Since A has order 2, we must have A and B alternating in the chain. But AB and BA have order 4, so neither of these work.

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