ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Making use of familiar Salagean derivative [7], we have introduced a new subclass of analytic functions with negative coefficients in unit disc $U$. Coefficient estimate, distortion theorem, closure properties and radii of convexity for functions belonging to the class $S_{n,\alpha}(\beta, \mu, \gamma)$ are determined.

1. Introduction and Definitions

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let $S$ denote the subclass of functions that are univalent in $U$. A function $f \in S$, is said to be starlike of order $\alpha$ in $U$, if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in U,$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$, the class of functions in $S$, which are starlike of order $\alpha$ in $U$. Further, $f \in S$ is said to be convex of order $\alpha$ in $U$, if and only if

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U.$$
for some $\alpha$ ($0 \leq \alpha < 1$), we denote by $K(\alpha)$, the class of functions in $S$, which are convex of order $\alpha$ in $U$. It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$, ($0 \leq \alpha < 1$).

Let $T$ denote subclass of $S$, consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \quad (1.2)$$

The function,

$$S_\alpha(z) = z(1-z)^{-2(1-\alpha)}, \quad 0 \leq \alpha \leq 1, \quad (1.3)$$

is the familiar extremal function for the class $S^*(\alpha)$, setting

$$C(\alpha, k) = \prod_{i=2}^{k} \frac{(i-2\alpha)}{(k-1)}, \quad k \geq 2 \quad (1.4)$$

then

$$S_\alpha(z) = z + \sum_{k=2}^{\infty} a_k C(\alpha, k) z^k, \quad a_k \geq 0. \quad (1.5)$$

We note that $C(\alpha, k)$ is a decreasing function in $\alpha$, and that

$$\lim_{k \to \infty} C(\alpha, k) = \begin{cases} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2 \end{cases} \quad (1.6)$$

The Hadamard product (convolution) of two analytic functions given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is denoted by $(f \ast g)(z)$, and is defined by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (1.7)$$
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For a function \( f(z) \in A \), we define

\[
\begin{align*}
D^0 f(z) & = f(z), \\
D^1 f(z) & = Df(z) = 2f'(z), \\
D^n f(z) & = D(D^{n-1} f(z)).
\end{align*}
\]

(1.8)

The differential operator \( D^n \) was introduced by Salagean [7]. It can be easily seen that

\[
D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k
\]

(1.9)

**Definition 1.1.** A function \( f(z) \in T \), be defined by (1.2) is in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \) if it satisfies

\[
\left| \frac{(\phi_{n,\alpha}(z))' - 1}{\mu(\phi_{n,\alpha}(z))' + 1 - (1 + \mu)\gamma} \right| < \beta
\]

(1.10)

where \( 0 \leq \mu \leq 1, 0 \leq \gamma < 1, 0 < \beta \leq 1 \) and

\[
\phi_{n,\alpha}(z) = D^n(f \ast S_{\alpha}(z)).
\]

Easily, we can deduce that

\[
\phi_{n,\alpha}(z) = z - \sum_{k=2}^{\infty} k^n C(\alpha, k) a_k z^k, \quad a_k \geq 0, \quad z \in U.
\]

(1.11)

We note that, by specifying the parameters \( \beta, \mu \) and \( \gamma \), the class \( S_{n,\alpha}(\beta, \mu, \gamma) \) generalize and extends classes studied and introduced by several research workers, for e.g. Aouf and Cho [1], Silverman [12], Aouf et al [2,3], and others {[5],[7],[8]}.

Among several definitions of fractional calculus (that is, fractional derivative and fractional integral) have been extensively studied by many researchers. We find it to be convenient to restrict ourselves to the following definitions given by Srivastava et al [11] of fractional integral operator

**Definition 1.2** (11,p 413 definition 3). For real numbers \( \lambda > 0, \delta \) and \( \eta \), the fractional integral operator \( I_{0,z}^{\lambda,\delta,\eta} \) is defined by

\[
I_{0,z}^{\lambda,\delta,\eta} f(z) = \frac{z^{\lambda-\delta}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} 2F_1 \left( \lambda + \delta, -\eta, 1 - \frac{t}{z} \right) f(t) dt
\]

(1.12)
where \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin with order

\[
f(z) = o(|z|^\varepsilon), \quad z \to 0
\]

where \( \varepsilon > \max(0, \delta - \eta) - 1 \), and the multiplicity of \((z - t)^{\lambda - 1}\) is removed by requiring \( \log(z - t) \) to be real when \((z - t) > 0\).

It is easy to observe that,

\[
I_{\lambda, \delta, \eta}^{0, z} f(z) = D_{z}^{-\lambda} f(z), \quad (\lambda > 0)
\]

where \( D_{z}^{-\lambda} f(z) \) is the fractional integral operator considered by Owa [4],

**Lemma 1.1** (11,p15, Lemma 3). *If \( \lambda > 0 \), and \( k > \delta - \eta - 1 \), then*

\[
I_{0, z}^{\lambda, \delta, \eta} f(z) = \frac{\Gamma(k + 1)\Gamma(k - \delta + \eta + 1)}{\Gamma(k - \delta + 1)\Gamma(k + \lambda + \eta + 1)} z^{k-\delta}.
\]

**2. COEFFICIENT ESTIMATE**

**Theorem 2.1.** A function \( f(z) \) defined by (1.2) belongs to the class \( S_{n, \alpha}(\beta, \mu, \gamma) \) if and only if

\[
\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k)(1 + \mu \beta) a_k \leq \beta(1 + \mu)(1 - \gamma)
\]

(2.1)

The result (2.1) is sharp.

**Proof.** Assume that the inequality (2.1) holds true and let \(|z| = 1\), then from (1.10), we have

\[
|(\phi_{n, \alpha}(z))' - 1| - \beta |(\phi_{n, \alpha}(z))' + 1 - (1 + \mu)\gamma| =
\]

(2.2)

\[
= \left| -\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^k \right| - \beta(1 + \mu)(1 - \gamma) + \mu \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^k \leq 0.
\]

Hence, by the maximum modulus principle, \( f(z) \in S_{n, \alpha}(\beta, mu, \gamma) \).
Conversely assume that $f(z)$ defined by (1.2) is in the class $S_{n,\alpha}(\beta, \mu, \gamma)$, then we have
\[
\left| \frac{(\phi_{n,\alpha}(z))' - 1}{\mu(\phi_{n,\alpha}(z))' + 1 - (1 + \mu)\gamma} \right| = \left| \frac{-\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^k}{(1 + \mu)(1 - \gamma) - \mu \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_z z^k} \right| < \beta
\]
Since $|\Re(z)| \leq |z|$ for all $z$, we have
\[
\Re \left\{ \frac{-\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k z^k}{(1 + \mu)(1 - \gamma) - \mu \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_z z^k} \right\} < \beta
\]
Choose the values of $z$ on the real axis so that $[\phi_{n,\alpha}(z)]'$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1$, through real values, we get
\[
\sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_k \leq \beta(1 + \mu)(1 - \gamma) - \mu \sum_{k=2}^{\infty} k^{n+1} C(\alpha, k) a_z,
\]
which implies that the inequality (2.1). The result (2.1) is sharp for the function
\[
f(z) = z - \frac{\beta(1 + \mu)(1 - \gamma)}{k^{n+1} C(\alpha, k)(1 + \mu)^{\frac{1}{2}}} z^k, \quad k \geq 2.
\]
□

Now making use of Lemma 1.1, we state and prove following Theorem.

### 3. Distortion and Growth Theorem.

**Theorem 3.1.** Let $\lambda > 0$, $\delta < 2$, $\lambda + \eta > -2$, $\delta(\lambda + \eta) \leq 3\lambda$. If $f(z)$ defined by (1.2) is in the class $S_{n,\alpha}(\beta, \mu, \gamma)$, then
\[
|I_{0,\alpha}^{\delta, \eta} f(z)| \geq \frac{\Gamma(2 - \delta + \eta)}{\Gamma(2 - \delta)} \left( 1 - \frac{\beta(1 + \mu)(1 - \gamma)(2 - \delta + \eta)}{2^n(2 - \delta)(2 + \lambda + \eta)(1 + \mu\beta)(1 - \alpha)} \right)
\]
and
\[ |I_{0,z}^{\lambda,\delta,\eta} f(z)| \leq \frac{\Gamma(2 - \delta + \eta)|z|^{1-\delta}}{\Gamma(2 - \delta)\Gamma(2 + \lambda + \eta)} \left( 1 + \frac{\beta(1 + \mu)(1 - \gamma)(2 - \delta + \eta)}{2^n(2 - \delta)(2 + \lambda + \eta)(1 + \mu\beta)(1 - \alpha)} \right) \] (3.2)

for \( z \in U_0 \),

\[ U_0 = \begin{cases} U, & \delta \leq 1 \\ U - 0, & \delta > 1 \end{cases} \]

The result is sharp and is given by

\[ f(z) = z - \frac{\beta(1 + \mu)(1 - \gamma)}{2^n(1 + \mu\beta)(1 - \alpha)} z^2 \] (3.3)

Proof. Since the function \( f(z) \) defined by (1.2) is in the class \( S_{n,a}(\beta, \mu, \gamma) \), we have from Theorem 2.1,

\[ \sum_{k=2}^{\infty} a_k \leq \frac{\beta(1 + \mu)(1 - \gamma)}{2^{n+1}(1 + \mu\beta)(1 - \alpha)} \] (3.4)

By virtue of Lemma 1.1, we have

\[ |I_{0,z}^{\lambda,\delta,\eta}| = \frac{\Gamma(2 - \delta + \eta)|z|^{1-\delta}}{\Gamma(2 - \delta)\Gamma(2 + \lambda + \eta)} z^{1-\delta} - \sum_{k=2}^{\infty} \frac{\Gamma(k + 1)\Gamma(k - \delta + \eta + 1)}{\Gamma(k - \delta + 1)\Gamma(k + \lambda + \eta + 1)} a_k z^{k-\delta}. \] (3.5)

Setting

\[ H(z) = \frac{\Gamma(2 - \delta)\Gamma(2 + \lambda + \eta)}{\Gamma(2 - \delta + \eta)} z^\delta I_{0,z}^{\lambda,\delta,\eta} f(z) = z - \sum_{k=2}^{\infty} h(k) a_k z^k \] (3.6)

where

\[ h(k) = \frac{(2 - \delta + \eta)_{k-1}(1)_k}{(2 - \delta)_{k-1}(2 + \lambda + \eta)_{k-1}}, \quad k \geq 2. \] (3.7)

It can be easily verified that \( h(k) \) is non-decreasing for \( k \geq 2 \), and thus we have

\[ 0 \leq h(k) \leq h(2) = \frac{(2 - \delta + \eta)^2}{(2 - \delta)(2 + \lambda + \eta)}. \] (3.8)
Hence, using (3.8) and (3.5), we have
\[
|H(z)| \geq |z|-|h(z)||z|^2 \sum_{k=2}^{\infty} a_k \geq |z|-\frac{(1+\mu)(1-\gamma)(2-\delta+\eta)}{(2-\delta)(2+\lambda+\eta)2^n(1+\mu\beta)(1-\alpha)}
\]
which proves (3.1), and other part (3.2) can be proved on similar lines, details are omitted. □

Taking \( \delta = -\lambda \) in Theorem 3.1, we get following corollary:

**Corollary 3.2.** Let the function \( f(z) \) defined by (1.2) be in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \). Then
\[
|D^{-\lambda}f(z)| \geq |z|^{1+\lambda} \left( 1 - \frac{\beta(1+\mu)(1-\gamma)}{2^n(2+\lambda)(1+\mu\beta)(1-\alpha)} \right)
\]
and
\[
|D^{-\lambda}f(z)| \leq |z|^{1+\lambda} \left( 1 + \frac{\beta(1+\mu)(1-\gamma)}{2^n(2+\lambda)(1+\mu\beta)(1-\alpha)} \right)
\]
for \( \lambda > 0, \ z \in U \). The result is sharp for the function
\[
D^{-\lambda}f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 - \frac{\beta(1+\mu)(1-\gamma)}{2^n(2+\lambda)(1+\mu\beta)(1-\alpha)} \right).
\]

4. **Integral Operator**

**Theorem 4.1.** Let \( c \) be a real number such that \( c > -1 \). If \( f(z) \in S_{n,\alpha}(\beta, \mu, \gamma) \), then the function \( F(z) \) defined by
\[
F(z) = \frac{c+1}{z^c} \int_0^\pi t^{c-1} f(t)dt
\]
also belongs to \( S_{n,\alpha}(\beta, \mu, \gamma) \).

**Proof.** Let \( f(z) \) defined by (1.2) be in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \) then \( f(z) \) can be written as
\[
F(z) = z - \sum_{k=2}^{\infty} b_k z^k
\]
where
\[
b_k = \frac{c+1}{c+n} a_k.
\]
Therefore
\[ \sum_{k=2}^{\infty} (1 + \mu \beta) b_k C(\alpha, k) k^{n+1} < \sum_{k=2}^{\infty} k^{n+1} (1 + \mu \beta) C(\alpha, k) b_k \leq \beta (1 + \mu) (1 - \gamma) \]

since \( f(z) \in S_{n, \alpha}(\beta, \mu, \gamma) \). Hence by Theorem 2.1, \( F(z) \in S_{n, \alpha}(\beta, \mu, \gamma) \).

**Theorem 4.2.** Let the function
\[ F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \]
be in the class \( S_{n, \alpha}(\beta, \mu, \gamma) \), and \( c \) be real number such that \( c > -1 \). Then \( F(z) \) defined by (4.1) is univalent in \( |z| < R^* \), where
\[ R^* = \inf \left\{ \frac{(c + 1) C(\alpha, k) (1 + \mu \beta)}{\beta (1 + \mu) (1 - \gamma) (c + k)} \right\}^{\frac{1}{c k}}, \quad k \geq 2. \quad (4.4) \]

The result (4.4) is sharp.

**Proof.** From (4.1) we have
\[ f(z) = \frac{z^{1-c} (z^c F(z))'}{c + 1} = z - \sum_{k=2}^{\infty} \frac{c + k}{c + 1} a_k z^k. \]

In order to obtain the required result it suffices to show that \( |f'(z) - 1| \) is in \( |z| < R^* \). Now
\[ |f'(z) - 1| = \left| - \sum_{k=2}^{\infty} k \frac{1 + k}{c + k} a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c + k}{c + 1} a_k |z|^{k-1}. \]

Then \( |f'(z) - 1| < 1 \) if
\[ \sum_{k=2}^{\infty} k \frac{c + k}{c + 1} a_k |z|^{k-1} \leq 1 \quad (4.5) \]

But Theorem 2.1 confirm that
\[ \sum_{k=2}^{\infty} \frac{C(\alpha, k) (1 + \mu \beta) k^{n+1}}{\beta (1 + \mu) (1 - \gamma)} a_k \leq 1. \]
Hence (4.5) will be satisfied if
\[ k \frac{c + k}{c + 1} a_k |z|^{k-1} \leq \frac{k^{n+1} C(\alpha, k)(1 + \mu \beta)}{\beta(1 + \mu)(1 - \gamma)}, \quad n \geq 2. \]
or if
\[ |z| \leq \left\{ \frac{(c + 1) C(\alpha, k)(1 + \mu \beta)k^n}{\beta(1 + \mu)(1 - \gamma)(c + k)} \right\}^{\frac{1}{k-1}}, \quad k \geq 2. \] (4.6)
The required result follows from (4.6). The result is sharp for the function
\[ f(z) = z - \frac{\beta(1 + \mu)(1 - \gamma)(c + k)}{k^{n+1}(c + 1) C(\alpha, k)(1 + \mu \beta)} z^k, \quad k \geq 2. \]
□

5. Radius of convexity

Theorem 5.1. If \( f(z) \) given by (1.2) is in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \) then \( f(z) \) is convex in \( |z| < R^* \) where
\[ R^* = \inf \left\{ \frac{C(\alpha, k)(1 + \mu \beta)k^{n+1}}{\beta(1 + \mu)(1 - \gamma)} \right\}^{\frac{1}{k-1}}. \] (5.1)
The result (5.1) is sharp.

Proof. In order to establish the required result it is sufficient to show that
\[ \frac{|zf''(z)|}{f'(z)} < 1, \quad \text{in} \ |z| < R^* \] (5.2)
In view of (1.2), we have
\[ \frac{|zf''(z)|}{f'(z)} \leq \frac{\sum_{k=2}^{\infty} k(k - 1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^{k-1}}. \] (5.3)
Hence we have
\[ \sum_{k=2}^{\infty} k^2 a_k |z|^{k-1} \leq 1. \] (5.4)
In view of Theorem 2.1, (5.4) is satisfied if
\[ k^2 |z|^{k-1} \leq \frac{C(\alpha, k)(1 + \mu \beta)k^{n+1}}{\beta(1 + \mu)(1 - \gamma)} \] (5.5)
or if
\[ |z| \leq \left\{ \frac{C(\alpha, k)(1 + \mu \beta)k^{n+1}}{\beta(1 + \mu)(1 - \gamma)} \right\}^{\frac{1}{k-1}}. \]

Hence, \( f(z) \) is convex in \( |z| < R^* \). The result is sharp and is given by (5.1) \( \square \)

6. Closure Theorems

We shall prove the following result for the closure of function in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \).

**Theorem 6.1.** Let the function \( f_j(z) \) \((j = 1, 2, \ldots, m)\) defined by
\[
f_j(z) = z - \sum_{k=2}^{\infty} a_{kj}z^k, \quad (a_{kj}) \geq 0 \quad (6.1)
\]
for \( z \in U \) be in the class \( S_{n,\alpha}(\beta, \mu, \gamma) \), then the function \( h(z) \) defined by
\[
h(z) = z - \sum_{k=2}^{\infty} b_kz^k \quad (6.2)
\]
also belongs to the class \( S_{n,\alpha}(\beta, \mu, \gamma) \), where
\[ b_k = \frac{1}{m} \sum_{j=1}^{m} a_{kj}. \]

**Proof.** Since \( f_j(z) \in S_{n,\alpha}(\beta, \mu, \gamma) \), it follows from Theorem 2.1, that
\[
\sum_{k=2}^{\infty} (1 + \mu \beta)C(\alpha, k)k^{n+1}a_{kj} \leq (\mu + 1)\beta(1 - \gamma) \quad (l = 1, 2, \ldots, m).
\]

Therefore
\[
\sum_{k=2}^{\infty} (1 + \mu \beta)C(\alpha, k)b_k = \sum_{k=2}^{\infty} \left( 1 + \mu \beta \right)C(\alpha, k) \left( \frac{1}{m} \sum_{j=1}^{m} a_{kj} \right) = \frac{1}{m} \left( \sum_{k=2}^{\infty} (1 + \mu \beta)C(\alpha, k) \right) \leq \beta(1 + \mu + 1)(1 - \gamma).
\]

Hence by Theorem 2.1, \( h(z) \in S_{n,\alpha}(\beta, \mu, \gamma) \). \( \square \)
Employing same techniques used by Silverman [12], and with the aid of Theorem 2.1, we can prove the following proofs are straightforward, hence omitted.

**Theorem 6.2.** The class $S_{n,\alpha}(\beta, \mu, \gamma)$ is closed under linear combination.

**Theorem 6.3.** Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{(1 - \gamma)\beta(1 + \mu)z^k}{k^{n+1}(1 + \mu\beta)C(\alpha, k)}, \quad k \geq 2$$

(6.3)

for $0 \leq \mu \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma < 1$.

Then $f(z)$ is in the class $S_{n,\alpha}(\beta, \mu, \gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z), \quad \Psi_n \geq 0, \quad n \geq 1$$

and

$$\sum_{k=1}^{\infty} \Psi_k = 1.$$

**Corollary 6.4.** The extreme points of the class $S_{n,\alpha}(\beta, \mu, \gamma)$ are the functions $f_k(z)$, $k \geq 1$, given by Theorem 6.3.

**References**


Received 12 July 2006

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