NECESSARY AND SUFFICIENT CONDITIONS FOR DELAY-DEPENDENT ASYMPTOTIC STABILITY OF LINEAR CONTINUOUS LARGE SCALE TIME DELAY AUTONOMOUS SYSTEMS

Sreten B. Stojanovic and Dragutin Lj. Debeljković

ABSTRACT

This paper offers new necessary and sufficient conditions for delay-dependent asymptotic stability of the linear continuous large scale time delay systems. The obtained conditions of stability are expressed by nonlinear system of matrix equations and the Lyapunov matrix equation for an ordinary linear continuous system without delay.

KeyWords: Large scale time delay systems, delay dependent stability, Lyapunov matrix equation, necessary and sufficient conditions of stability.

I. INTRODUCTION

In the past two decades, a considerable interest has been permanently shown in the problem of asymptotic stability of continuous large scale time delay systems. The stabilization problem for large scale time delay systems with or without perturbations is studied in [1-4]. Wang et al. in [5] extended the results of [2] to the problems of stabilization, estimation and robustness. Moreover, [6] derived a much more concise and less conservative result other than [5]. [7-8] have synthesized some decentralized controllers to stabilize the whole system. Xu [9] provides a new criterion for delay-independent stability of linear large scale time delay systems by employing an improved Razumikhin-type theorem and M-matrix properties. In [10], by employing a Razumikhin-type theorem, a robust stability criterion for a class of linear system subject to delayed time-varying nonlinear perturbations is given. New sufficient conditions for delay-independent asymptotic stability of large scale systems are presented by [11] using the properties of matrix norm and measure. It is shown that the presented approach simplifies the stability problem.

The basic aim of the above mentioned works is to obtain only sufficient (S) conditions for stability of large scale time delay systems. It is notorious that those conditions of stability are more or less conservative. In contrast, the major results of the paper are necessary and sufficient (NS) conditions of asymptotic stability of continuous large scale time delay autonomous systems (see [12] for similarly results for time delay systems). The obtained NS conditions are expressed by nonlinear system of matrix equations and the Lyapunov matrix equation for an ordinary linear continuous system without delay. Those conditions of stability are delay-dependent and do not possess conservatism. Unfortunately, viewed mathematically, they require somewhat more complex numerical computations.

Notation. Let us denote: $\mathbb{R}$-set of real numbers, $\mathbb{C}$-set of complex numbers, $\text{Re}$- real part of complex number $s$, $F^T$-transpose of matrix $F$, $F^*$-conjugate transpose of matrix $F$, $F > 0$-positive definite matrix $F$, det $F$-determinant of matrix $F$, $\lambda(F)$ -eigenvalue of matrix $F$, and $\sigma(F)$-spectrum.
of matrix $F$.

II. MAIN RESULTS

Consider a linear continuous large scale time delay autonomous systems composed of $N$ interconnected subsystems. Each subsystem is described as:

$$
\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j(t - \tau_{ij}), \quad 1 \leq i \leq N,
$$

(1)

with an associated function of initial state $x_i(0) = \psi_i(0)$, $0, i \leq N$. $x_i(t) \in \mathbb{R}^{n_i}$ is state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denote the system matrix, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ represents the interconnection matrix between the $i$-th and the $j$-th subsystems, and $\tau_{ij}$ is constant delay.

For the sake of brevity, we first observe system (1) made up of two subsystems ($N = 2$). For this system, we derive new necessary and sufficient delay-dependent conditions for stability, by Lyapunov’s direct method. The derived results are then extended to the linear continuous large scale time delay systems with multiple subsystems.

2.1 Large scale systems with two subsystems

Theorem 1. Given the following system of matrix equations (SME)

$$
R_1 - A_1 - e^{-\tau_{11}} A_{11} - e^{-\tau_{12}} S_2 A_{21} = 0
$$

(2)

$$
R_1 S_2 - S_2 A_2 - e^{-\tau_{21}} A_{12} - e^{-\tau_{22}} S_2 A_{22} = 0
$$

(3)

where $A_1$, $A_2$, $A_{12}$, and $A_{22}$ are matrices of system (1) for $N = 2$, $n_i$ subsystem orders and $\tau_{ij}$ time delays of the system.

If there exists solution of SME (2)-(3) upon unknown matrices $R_1 \in \mathbb{C}^{n_1 \times n_1}$ and $S_2 \in \mathbb{C}^{n_2 \times n_2}$, then the eigenvalues of matrix $R_1$ belong to a set of roots of the characteristic equation of system (1) for $N = 2$.

Proof. By introducing the time delay operator $e^{-ts}$, the system (1) can be expressed in the form

$$
\dot{x}(t) = \begin{bmatrix} A_1 + A_1 e^{-\tau_{11}} & A_{12} e^{-\tau_{21}} \\ A_{21} e^{-\tau_{12}} & A_2 + A_{22} e^{-\tau_{22}} \end{bmatrix} x(t)
$$

$$
= A_e(s) x(t), \quad x(t) = [x_1^T(t) \quad x_2^T(t)]^T
$$

(4)

Let us form the following matrix

$$
F(s) = \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = sI_{n_1 \times n_1} - A_e(s)
$$

Its determinant is

$$
\det F(s) = \det \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = \det G(s, S_2)
$$

(6)

where $G_1(s, S_2) = sI_{n_1} - A_1 - A_1 e^{-\tau_{11}} - S_2 A_{21} e^{-\tau_{21}}$

(7)

$G_2(s, S_2) = sS_2 - S_2 A_2 - A_{12} e^{-\tau_{22}} - S_2 A_{22} e^{-\tau_{22}}$

(8)

Relations (6)-(8) was obtained by applying a finite sequence of elementary row operations of type 3 over matrix $F(s)$ [14]. Transformational matrix $S_2$ is unknown for the time being, but condition determining this matrix will be derived in a further text.

The characteristic polynomial of system (1) for $N = 2$, defined by

$$
f(s) = \det (sI_{n_1} - A_e(s)) = \det G(s, S_2)
$$

(9)

is independent of the choice of matrix $S_2$ because the determinant of matrix $G(s, S_2)$ is invariant with respect to elementary row operation of type 3 [14].

Let us designate a set of roots of the characteristic equation of system (1) by $\Sigma \triangleq \{s \mid f(s) = 0\}$.

Substituting scalar variable $s$ by matrix $X$ in $G(s, S_2)$ we obtain

$$
G(X, S_2) = \begin{bmatrix} G_{11}(X, S_2) & G_{12}(X, S_2) \\ G_{21}(X) & G_{22}(X) \end{bmatrix}
$$

(10)

If there exist transformational matrix $S_2$ and matrix $R_1 \in \mathbb{C}^{n_1 \times n_1}$ such that $G_1(R_1, S_2) = 0$ and $G_{22}(R_1, S_2) = 0$ is satisfied, i.e. if (2)-(3) hold, then

$$
\det (R_1) = \det (R_1, S_2) \cdot \det G_{22}(R_1) = 0
$$

(11)

So, the characteristic polynomial (9) of system (1) is annihilating polynomial [14] for the square matrix $R_1$, defined by (2)-(3). In other words, $\sigma(R_1) \subset \Sigma$.

Theorem 2. Given the following SME

$$
R_2 - A_2 - e^{-\tau_{21}} S_1 A_{21} - e^{-\tau_{22}} A_{22} = 0
$$

(12)

$$
R_2 S_1 - S_1 A_1 - e^{-\tau_{11}} S_1 A_{21} - e^{-\tau_{12}} A_{22} = 0
$$

(13)

where $A_1$, $A_2$, $A_{12}$, and $A_{22}$ are matrices of system (1)
for \( N = 2, n_i \) subsystem orders and \( \tau_i \) time delays of the system.

If there exists solution of SME (12)-(13) upon unknown matrices \( R \in \mathbb{C}^{m \times m} \) and \( S \in \mathbb{C}^{m \times m} \), then the eigenvalues of matrix \( R \) belong to a set of roots of the characteristic equation of system (1) for \( N = 2 \).

**Proof.** Proof is similarly with the proof of Theorem 1. \( \blacksquare \)

**Corollary 1.** If system (1) is asymptotically stable, then matrices \( R \) and \( S \), defined by SME (2)-(3) and (12)-(13) respectively, are stable (i.e. \( \text{Re} R(R_i) < 0, 1 \leq i \leq 2 \)).

**Proof.** If system (1) is asymptotically stable, then \( \forall s \in \Sigma, \text{Res} < 0 \). Since \( \sigma(R_i) \subset \Sigma, 1 \leq i \leq 2 \), it follows that \( \forall \lambda \in \sigma(R_i), \text{Re} \lambda < 0 \). Therefore matrices \( R \) and \( S \) are stable. \( \blacksquare \)

**Definition 1.** The matrix \( R \) \((R_2) \) is referred to as solvent of SME (2)-(3) ((12)-(13)).

**Definition 2.** Each root \( \lambda_m \) of the characteristic equation (9) of the system (1) which satisfies the following condition: \( \text{Re} \lambda_m = \text{max} \text{Res}, s \in \Sigma \) will be referred to as maximal root (eigenvalue) of system (1).

**Definition 3.** Each solvent \( R \) \((R_2) \) of SME (2)-(3) ((12)-(13)), whose spectrum contains maximal eigenvalue \( \lambda_m \) of system (1), is referred to as maximal solvent of SME (2)-(3) ((12)-(13)).

**Theorem 3.** Suppose that there exists at least one maximal solvent of SME (2)-(3) and let \( R \) denote one of them. Then, for \( N = 2 \), is asymptotically stable if and only if for any matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) such that
\[
R^*P + PR = -Q
\]  
(14)

**Proof.** (Sufficient condition) Similarly [12], define the following vector continuous functions
\[
x_i = x_i(t + 0), 0 \in [-\tau_i, 0],
\]  
(15)

\[
z(x_1, x_2) = \sum_{i=1}^{2} S_{j} \left[ x_i(t) + \sum_{j=1}^{2} \int_{0}^{t} T_{ji}(\eta) x_j(t - \eta) d\eta \right]
\]  
(16)

where \( T_{ji} \in \mathbb{C}^{m \times m}, j = 1, 2 \), are some time varying continuous matrix functions and \( S_{j} = I_{m_i}, S_{2} \in \mathbb{C}^{m \times m} \).

The proof of the theorem follows immediately by defining Lyapunov functional for system (1) as
\[
V(x_1, x_2) = z^T(x_1, x_2) P z(x_1, x_2), P = P^T > 0
\]  
(17)

Derivative of (17), along the solutions of system (1) is
\[
\dot{V}(x_1, x_2) = \dot{z}^T(x_1, x_2) P z(x_1, x_2) + z^T(x_1, x_2) P \dot{z}(x_1, x_2)
\]  
(18)

From
\[
\frac{d}{d\eta} \left[ T_{ji}(\eta) x_j(t - \eta) \right] = T_{ji}'(\eta) x_j(t - \eta) - \frac{d}{d\eta} T_{ji}(\eta) x_j(t - \eta)
\]  
(19)

follows
\[
\dot{z}(x_1, x_2) = \sum_{i=1}^{2} S_{j} \left[ A_{ji} + \sum_{j=1}^{2} \int_{0}^{t} T_{ji}(\eta) x_j(t - \eta) d\eta \right] x_i(t - \tau_{ji})
\]  
(20)

Therefore
\[
\dot{z}(x_1, x_2) = \sum_{i=1}^{2} S_{j} \left[ A_{ji} - \sum_{j=1}^{2} \int_{0}^{t} T_{ji}(\eta) x_j(t - \eta) d\eta \right] x_i(t - \tau_{ji})
\]  
(21)

If we define new matrices
\[
R_{1i} = A_{i} + \sum_{j=1}^{2} T_{ji}(0), i = 1, 2
\]  
(22)

and if one adopts
\[
S_{j} T_{ji}(\tau_{ji}) = S_{j} A_{ji}, i, j = 1, 2
\]  
(23)

\[
S_{j} T_{ji}(\eta) = R_{1j} S_{j} T_{ji}(\eta), S_{j} R_{j} = R_{1j} S_{j}, i, j = 1, 2
\]  
(24)

then
\[
\dot{z}(x_1, x_2) = R_{1i} z(x_1, x_2)
\]  
(25)

\[
\dot{V}(x_1, x_2) = z^T(x_1, x_2) (R_{1i}^* P + P R_{1i}) z(x_1, x_2)
\]  
(26)

It is obvious that if the following equation is satisfied
\[
R_{1i}^* P + P R_{1i} = -Q < 0,
\]  
(27)

then \( \dot{V}(x_1, x_2) < 0, \forall x_1 \neq 0 \).

In the Lyapunov matrix Eq. (14), of all possible solvents \( R \) only one of maximal solvents \( R_{1i} \) is of importance, because it is containing maximal eigenvalue \( \lambda_m \) \( \in \Sigma \), which has dominant influence on the stability of the system. If a solvent, which is not maximal, is integrated into Lyapunov Eq. (14), it may happen that there will exist
positive definite solution of this equation, although the system is not stable.

(Necessary condition) Let us assume that system (1) for \( N = 2 \) is asymptotically stable, i.e. \( \forall s \in \Sigma, \text{Re} < 0 \) hold. Since \( \sigma(R_{1m}) \subset \Sigma \) follows \( \text{Re} \lambda(R_{1m}) < 0 \) (see Corollary 1) and the positive definite solution of Lyapunov matrix Eq. (14) exists.

From (23)-(24) follows
\[
S_j A_j = e^{R_{1m} T_j} S_j T_j(0), S_j = \mathbb{I}_m, i = 1, 2, j = 1, 2
\]  
(28)

Using (22) and (28), for \( i = 1 \), we obtain Eq. (2). Multiplying (22) (for \( i = 2 \)) from the left by matrix \( S_2 \) and using (24) and (28) we obtain equation (3). Taking a solvent with eigenvalue \( \lambda_m \in \Sigma \) (if it exists) as a solution of the system of Eq. (2)-(3), we arrive at a maximal solvent \( R_{1m} \).

**Theorem 4.** Suppose that there exists at least one maximal solvent of SME (12)-(13) and let \( R_{2m} \) denote one of them. Then, system (1), for \( N = 2 \), is asymptotically stable if and only if for any matrix \( Q = Q' > 0 \) there exists matrix \( P = P' > 0 \) such that
\[
R_{2m}^* P + P R_{2m} = -Q
\]  
(29)

**Proof.** Proof is almost identical to that exposed for Theorem 3.

**Conclusion 1.** Consider a following linear continuous system without time delay
\[
\dot{x}(t) = R_{1m} x(t)
\]  
(30)

where matrix \( R_{1m} \) is defined by SME (2)-(3), for \( i = 1 \), or by SME (12)-(13), for \( i = 2 \), respectively. Applying Theorems 3 or 4, the investigation of the stability of large scale time delay system (1) reduces to investigating the stability of corresponding system (30) without delay. The dimension of system (1) is infinite, while dimension of corresponding system (30) is finite and equals \( m \).

**Conclusion 2.** The proposed criteria of stability are expressed in the form of necessary and sufficient conditions and as such do not possess conservatism unlike the existing sufficient criteria of stability.

**Conclusion 3.** To the authors’ knowledge, in the literature available, there are no adequate numerical methods for direct computations of maximal solvents \( R_{1m} \) or \( R_{2m} \). Instead, using various initial values for solvents \( R_i \), we determine \( R_{1m} \) by applying minimization methods based on nonlinear least squares algorithms (see Example 1).

### 2.2 Large scale system with multiple subsystems

**Theorem 5.** Given the following system of matrix equations
\[
R_k s_j - s_j A_j - \sum_{j=1}^{N} e^{R_{1m} T_j} s_j A_j = 0
\]
\[
s_j \in \mathbb{C}^{n \times n}, \quad s_j = \mathbb{I}_n, \quad 1 \leq i \leq N
\]  
(31)

for a given \( k, 1 \leq k \leq N \), where \( A_i \) and \( A_j \), \( 1 \leq i \leq N, 1 \leq j \leq N \) are matrices of system (1) and \( \tau_j \) is time delay in the system.

If there is a solvent of (31) upon unknown matrices \( R_k \in \mathbb{C}^{n \times n} \) and \( S_j, 1 \leq i \leq N, i \neq k \), then the eigenvalues of matrix \( R_k \) belong to a set of roots of the characteristic equation of system (1).

**Proof.** Proof of this theorem is a generalization of proof of Theorem 1 or 2.

**Theorem 6.** Suppose that there exists at least one maximal solvent of (31) for given \( k, 1 \leq k \leq N \) and let \( R_{1m} \) denote one of them. Then, linear discrete large scale time delay system (1) is asymptotically stable if and only if for any matrix \( Q = Q' > 0 \) there exists matrix \( P = P' > 0 \) such that
\[
R_{1m}^* P + P R_{1m} = -Q
\]  
(32)

**Proof.** Proof is based on generalization of proof for Theorems 3 and 4. It is sufficient to take arbitrary \( N \) instead of \( N = 2 \).

### III. NUMERICAL EXAMPLE

**Example 1.** Consider following continuous large scale time delay system with delay interconnections
\[
\dot{x}_1(t) = A_1 x_1(t) + A_{12} x_2(t - \tau_{12})
\]
\[
\dot{x}_2(t) = A_2 x_2(t) + A_{21} x_1(t - \tau_{21}) + A_{23} x_3(t - \tau_{23})
\]
\[
\dot{x}_3(t) = A_3 x_3(t) + A_{31} x_1(t - \tau_{31}) + A_{32} x_2(t - \tau_{32})
\]  
(33)

\[
A_1 = \begin{bmatrix}
-6 & 2 & 0 \\
0 & -7 & 0 \\
0 & 0 & -10.9
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
3 & -2 & 0 \\
0 & 0 & 3 \\
-2 & 1 & 2
\end{bmatrix}, \quad A_{21} = \begin{bmatrix}
-1 & 0 & -1 \\
1 & 0 & 2
\end{bmatrix}
\]
\[
A_2 = \begin{bmatrix}
-1.87 & 4.91 & 10.30 \\
-2.23 & -16.51 & -24.11 \\
1.87 & -3.91 & -10.30
\end{bmatrix}, \quad A_{23} = \begin{bmatrix}
3 & 0 & 5 \\
0 & 2 & 0
\end{bmatrix}
\]
\[
A_3 = \begin{bmatrix}
-18.5 & -17.5 \\
-13.5 & -18.5
\end{bmatrix}, \quad A_{31} = \begin{bmatrix}
4 & -2 & 1 \\
2 & 0 & 1
\end{bmatrix}, \quad A_{32} = \begin{bmatrix}
1 & 2 & -1 \\
3 & 2 & 0
\end{bmatrix}
\]

Applying Theorem 5 to a given system, for \( k = 1 \), the following SME is obtained:
\[ R_1 - A_1 - e^{-R_1 \tau_{21}} S_2 A_{21} - e^{-R_1 \tau_{31}} S_3 A_{31} = 0 \]
\[ R_1 S_2 - S_2 A_2 - e^{-R_1 \tau_{21}} A_{22} - e^{-R_1 \tau_{31}} S_3 A_{32} = 0 \]  
(34)
\[ R_1 S_3 - S_3 A_3 - e^{-R_1 \tau_{21}} S_2 A_{23} = 0 \]

If for pure system time delays we adopt the following values: \( \tau_{12} = 1, \tau_{21} = 1, \tau_{23} = 1, \tau_{31} = 1, \) and \( \tau_{32} = 1 \), by applying the nonlinear least squares algorithms, we obtain a great number of solutions upon \( R_1 \) which satisfy SME (34). Among those solutions is a maximal solution:

\[ R_{1m} = \begin{bmatrix} -1.6105 & -3.3299 & 8.7623 \\ 2.8542 & 8.4472 & 21.1500 \end{bmatrix} \]

and its belonging transformational matrix:

\[ S_2 = \begin{bmatrix} 18.42 & 2.33 & 14.44 \\ -3.99 & 1.76 & -6.13 \\ -0.17 & -1.20 & 0.45 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0.44 & -0.78 \\ -0.80 & -0.97 \\ 0.58 & 0.39 \end{bmatrix} \]

The eigenvalues of matrix \( R_{1m} \) amount to: \( \lambda_1 = -0.5059, \lambda_{2,3} = -1.5785 \pm j 8.8824 \), wherefrom it follows that maximal eigenvalue of a given system is \( \lambda_m = \lambda_1 \).

To check the obtained value for \( \lambda_m \), from the characteristic equation of the system (33), by applying minimization methods, we arrived at the identical value for \( \lambda_m \). For initial guesses \( \lambda_m \) values were taken from a set of complex numbers with a large real part in order to detect maximal eigenvalue \( \lambda_m \) of a given system. Since \( \Re \lambda_m < 0 \), based on Theorem 6, a considered large scale time delay system is asymptotically stable.

If now for pure time delay we adopt the following values: \( \tau_{12} = 5, \tau_{21} = 2, \tau_{23} = 4, \tau_{31} = 5, \) and \( \tau_{32} = 3 \), by using identical procedure as in the previous case, we arrive at the following value for maximal solvent:

\[ R_{1m} = \begin{bmatrix} -0.0484 & -0.0996 & 0.0934 \\ 0.2789 & -0.3123 & 0.2104 \\ 1.1798 & -1.1970 & -0.3798 \end{bmatrix} \]

The eigenvalues of matrix \( R_{1m} \) amount to: \( \lambda_1 = -0.2517, \lambda_{2,3} = -0.2444 \pm j 0.3726 \). Therefore, for a maximal eigenvalue \( \lambda_m \) one of the values from the set \( \{ \lambda_2, \lambda_3 \} \) can be adopted. Based on Theorem 6, it follows that the large scale time delay system is asymptotically stable.

**IV. CONCLUSION**

In this paper we have established new necessary and sufficient conditions for the asymptotic stability of a linear continuous large scale time delay systems. The time-dependent criteria are derived by Lyapunov’s direct method and are based on the solution of a nonlinear system of matrix equations. Those conditions of stability do not possess conservatism, however, they require in turn somewhat more complex numerical computations.

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