Consensus of linear multi-agent systems with fully distributed control gains under a general directed graph

Jie Mei, Wei Ren, Jie Chen, and Brian D. O. Anderson

Abstract— **In this paper, we study the consensus problem for general linear multi-agent systems with gain adaption under a general directed graph. Both the cases using the relative state and output measurements with respect to the neighboring agents are considered. In the case using relative state measurements, a distributed consensus algorithm with gain adaption is proposed. In the case using relative output measurements, a distributed observer-type consensus algorithm is proposed. Novel integral-type Lyapunov functions are proposed to study the consensus convergence. In both cases, the control gains are varying and updated adaptively by distributed adaptive laws. As a result, the proposed algorithms require no global information and thus can be implemented in a fully distributed manner.**

I. Introduction

The consensus problem of multi-agent systems has received a lot of attention in the last decade due to its board applications in many scenarios, including distributed computation [1], parameter estimation [2], optimization [3], and robotic networks [4]. The objective is to design distributed algorithms for the agents to achieve a common final state by interacting with their local neighbors. One important foci of the consensus problem is the agent dynamics, including single and double integrators, general linear systems, and nonlinear systems. Most of the existing results have been mainly focused on agents with first-order or second-order integrators (see [5]–[8] and reference therein).

The present paper focuses on the consensus problem of multi-agent systems with general linear dynamics (A, B, C), which includes the single and double integrators as special cases. And this problem has also been studied extensively by using the relative state or output measurements with respect to the neighboring agents [9]–[19]. Specifically, the works in [9]–[13] rely on the assumption that the state matrix A has no eigenvalue with positive real part, including the cases with A being Hurwitz, neutrally stable, and marginally stable. And the works in [14]–[20] consider the more general case that (A, B, C) is stabilizable and detectable. As stated in [15], this condition is actually necessary for the agents to achieve consensus.

It is worth emphasizing that the extension of consensus algorithms from single integrators to general linear multi-agent systems is nontrivial. Under the good condition on the communication graph (the graph contains a directed spanning tree), the agents with single integrators will achieve consensus. But it is not true for agents with general linear multi-agent systems. Even for the special case with A having no eigenvalue with positive real part, the agents may not achieve consensus (see Theorem 3 in [10] for a proof). In [14] and [16], distributed observer-type consensus algorithms are proposed by introducing a common scalar control gain. Besides the condition on the topology and the agents' dynamics, if the common control gain is above some certain bound, the agents will achieve consensus. However, the certain bound is determined by some global information which cannot obtained in a distributed manner.

One approach to overcome such a limitation is to introduce adaptive gain updating laws based on local information. Examples include the distributed adaptive coordination algorithm design for multiple nonlinear systems [21]–[23] and general linear multi-agent systems [17]. However, the above works require a symmetric framework with an undirected graph. Recently, there are some attempts on the gain adaption under a directed graph [24]–[27]. In [24], the coordinated tracking problem is studied for multi-agent systems with first-order nonlinear dynamics. The authors utilize the property that the associated matrix (Laplacian matrix plus a diagonal matrix) for the coordinated tracking problem is positive stable. In contrast, the associated matrix (Laplacian matrix) is only semi-positive stable. Therefore, the results in [24] cannot be used for the leaderless consensus problem. The idea of [24] is extended in [25] to the coordinated tracking problem with a leader for general linear multi-agent systems. In [26], the consensus algorithm with gain adaption is proposed for second-order Lagrange systems. However, the final consensus state is stationary with a zero velocity. By establishing a connection between an undirected graph and a directed graph, we solve the consensus problem for second-order multi-agent systems with gain adaption in [27].

In this paper, we aim to propose fully distributed consensus algorithms for general linear multi-agent systems which can be implemented in a fully distributed manner under a general directed graph, by extending our previous work on the second-order systems [27]. Both the algorithm using relative state and output measurements are proposed. And we consider the general case where (A, B, C) is stabilizable and detectable[1]. Specifically, we begin with the algorithm using relative state measurements and heterogeneous constant gains, where we show that the agents achieve consensus if all the heterogeneous control gains are chosen large enough.

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[1]For the case with relative state measurements, this condition is change to that (A, B) be stabilizable.

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We then proposed a distributed consensus algorithm with gain adaption, in which the gains for the agents will always increase if they do not achieve consensus. For the case with relative output measurements, a distributed observer-type consensus algorithm with gain adaption is proposed. The control gains in the proposed algorithms are varying and updated adaptively using only local information. Novel integral-type Lyapunov functions are presented to study the consensus convergence.

Notations: Let $\mathbf{1}_m$ and $\mathbf{0}_m$ denote, respectively, the $m \times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{m\times n}$ denote the $m \times n$ matrix with all zeros and I_m denote the $m \times m$ identity matrix. Let $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote, respectively, the maximal and minimum eigenvalue of a square real matrix with real eigenvalues. Let $\sigma_{\max}(\cdot)$ denote the maximal singular value of a matrix. Let $\text{diag}(z_1, \cdots, z_p)$ be the diagonal matrix with diagonal entries z_1 to z_p. Let $\text{col}(z_1, \cdots, z_p)$ be the stacked vector of all vectors z_1 to z_p. For a complex number μ, let $\mathscr{R}(\mu)$ be its real part and $\mathscr{I}(\mu)$ be its imaginary part. For a vector function $f(t) : \mathbb{R} \mapsto \mathbb{R}^n$, it is said that $f(t) \in \mathbb{L}_2$ if $\int_0^\infty f(\tau)^T f(\tau) \mathrm{d}\tau < \infty$ and $f(t) \in \mathbb{L}_\infty$ if for each element of $f(t)$, noted as $f_i(t)$, $\sup_t |f_i(t)| < \infty$, $i = 1, \dots, n$. Throughout the paper, we use $\|\cdot\|$ to denote the Euclidean norm.

II. Background and Problem Statement

We use a directed graph to describe the network topology between the n agents. Let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be a directed graph with the node set $\mathcal{V} \triangleq \{1, ..., n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge $(i, j) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not vice versa. Here, node i is the parent node while node j is the child node. Equivalently, node i is a neighbor of node j. The set of all neighbors of node i is denoted as $\mathcal{N}_i$. A directed path from node i to node j is a sequence of edges of the form (i, i_2), (i_2, i_3), ..., (i_k, j), in a directed graph. A directed graph is strongly connected if there exists a directed path from every node to every other node. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In this paper, self edges are not allowed, *i.e.,* $a_{ii} = 0$. The (nonsymmetric) Laplacian matrix $\mathcal{L}_A = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ and hence $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1, j\neq i}^n a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

Lemma 2.1: [28], [29] Suppose that $\mathcal{G}$ is a directed graph of order n and is strongly connected. There exists a vector $\xi \triangleq [\xi_1, \dots, \xi_n]^T \in \mathbb{R}^n$ with $\sum_{i=1}^n \xi_i = 1$ and $\xi_i > 0$, $\forall i = 1, \dots, n$, such that $\xi^T \mathcal{L}_A = 0$.

The following lemma establishes a connection between a strongly connected directed graph and an undirected graph, which was first proposed in [27].

Lemma 2.2: Suppose that $\mathcal{G}$ is a directed graph of order n and is strongly connected. Define the matrix $\widehat{L} \triangleq \Xi\mathcal{L}_A + \mathcal{L}_A^T \Xi$, where $\Xi \triangleq \text{diag}(\xi_1, \dots, \xi_n)$ with ξ_i defined as in Lemma 2.1. Then $\widehat{L}$ is the symmetric Laplacian matrix associated with an undirected graph. In addition, let $\varsigma \in \mathbb{R}^n$ be any positive vector. The following inequality holds

$$\min_{\substack{\vartheta^T\varsigma=0 \\ \vartheta^T\vartheta=1}} \vartheta^T \widehat{L} \vartheta > \frac{\lambda_2(\widehat{L})}{n}, \qquad (1)$$

where $\lambda_2(\widehat{L})$ is the second smallest eigenvalue of $\widehat{L}$.

Using the properties of Kronecker product, we have the following result which will be used subsequently.

Lemma 2.3: Suppose that $U = [u_{ij}] \in \mathbb{R}^{n\times n}$ and $V = V^T \in \mathbb{R}^{p\times p}$. Let $S \in \mathbb{R}^{p\times p}$ be the unitary matrix such that $SVS^T \triangleq \text{diag}(\lambda_1(V), \lambda_2(V), \cdots, \lambda_p(V))$ and let $x_i = [x_{i1}, x_{i2}, \dots, x_{ip}]^T \in \mathbb{R}^p$, $i = 1, \dots, n$. The following equality holds

$$x^T (U \otimes V) x = \sum_{k=1}^p \lambda_k(V) v_k^T U v_k,$$

where $x = \text{col}(x_1, \dots, x_n) \in \mathbb{R}^{np}$ and $v_k = [v_{k1}, \dots, v_{kn}] \in \mathbb{R}^n$ is the stacked vector of the kth elements of all Sx_i, $i = 1, \dots, n$.

III. Consensus Algorithm with Relative State Measurements

In this section, we aim to design a fully distributed consensus algorithm for linear multi-agent systems where each agent simply chooses or updates its own control gain via the local relative state measurements with respective to its neighbors. We consider a group of n agents where the dynamics of the agents is described by the following identical general linear equations

$$\dot{x}_i(t) = Ax_i + Bu_i, \qquad i = 1, \dots, n. \qquad (2)$$

where $x_i \in \mathbb{R}^p$ is the state of agent i, $u_i \in \mathbb{R}^m$ is the control input of agent i which can only use local information from its neighbor agents, $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times m}$ are constant matrices.

For general linear multi-agent systems, the following consensus algorithms is proposed [11], [14]–[16]

$$u_i = \alpha K \sum_{j=1}^N a_{ij}(x_i - x_j), \qquad (3)$$

where α is a positive constant representing the control gain and $K = -B^T P$ where $P > 0$ is the unique solution of the following control algebraic Riccati equation (ARE)

$$A^T P + PA - PBB^T P + I_p = 0. \qquad (4)$$

Note that the above ARE has a unique solution if and only if (A, B) is stabilizable [30]. Therefore, we have the following assumptions on the dynamics of (2).

Assumption 3.1: The pair (A, B) is stabilizable.

Using (3) for (2), the agents achieve consensus if the underlying directed graph contains a directed spanning tree and the control gain is chosen such that

$$\alpha \geq \frac{1}{\min_{\mu_i \neq 0} \mathscr{R}(\mu_i)}, \tag{5}$$

where μ_i is the eigenvalue of the Laplaican matrix $\mathcal{L}_A$.

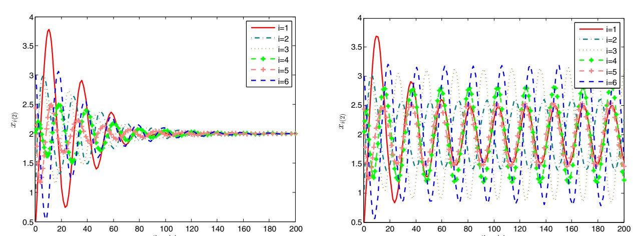
Note that in (3) all agents share the common control gain α. However, such a design is not fully distributed because in a fully distributed context each agent simply chooses its own gains and these gains are generally not identical. Moreover, the common control gain α must be above a certain lower bound (see (5)), which is determined by the (nonsymmetric) Laplacian matrix. Such a requirement is also not fully distributed as global information is needed to determine the lower bound. One possible way is to assign each agent a constant gain and tune the gains according to each agent's local information and performance, i.e., increase the control gains of the agents that do not converge to their neighbors or even move far away from their neighbors during a period of time. The above principle works when the agents share a common control gain. But for the case with heterogeneous gains is much different as shown in the following example.

A. Example

The dynamics of the agents are modeled as double integrators, and we have $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and thus $P = \begin{pmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{pmatrix}$. We consider the consensus problem for six agents with the following (nonsymmetric) Laplacian matrix

$$\begin{pmatrix} 0.15 & -0.1 & 0 & -0.05 & 0 & 0 \\ 0 & 0.3 & -0.15 & 0 & -0.15 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0.15 & -0.15 & 0 \\ 0 & -0.15 & 0 & -0.15 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.2 \end{pmatrix}.$$

Clearly, the associated graph is directed and contains a directed spanning tree. The initial states are chosen as $x_i(1) = 2i$ and $x_i(2) = 0.5i$, where $i = 1, \ldots, 6$.



(a) Velocities with a common gain $\alpha = 0.4$.

(b) Velocities with heterogeneous gains $\alpha_4 = 1.2$ and $\alpha_i = 0.4$, $i \neq 4$.

Fig. 1. The agents' velocities using a common control gain and heterogeneous control gain.

Due to the space limitation, we only show the second state of x_i, *i.e.,* the velocity of agent i. By simulation, we get that the lower bound using a common gain is nearly 0.176. Therefore, if we choose $\alpha = 0.4 > 0.176$, we can see from Fig. 1(a) that the agents achieve consensus. But if the control gains are heterogeneous, it is observed from Fig. 1(b) that the agents cannot achieve consensus even by increasing one of the control gains. Therefore, the case with heterogeneous control gains needs further investigation.

B. Consensus algorithm with heterogeneous constant control gains

We begin with the problem by investigating the consensus for agents with heterogeneous constant control gains. Here we assume that the directed graph $\mathcal{G}$ is strongly connected and Assumption 3.1 holds. The consensus algorithm for the linear multi-agent systems is proposed as

$$u_i = \alpha_i K \sum_{j=1}^{N} a_{ij}(x_i - x_j), \tag{6}$$

where α_i is a positive constant and $K \in \mathbb{R}^{m \times p}$ is a constant matrix defined as in (3). In contrast to (3), here we allow that the agents have heterogeneous control gains. Actually, the heterogeneous gains make the consensus convergence more challenging since it is not clear how to use the eigenvalue analysis as in [11], [14]–[16] since there exist n unknown variables (α_i). On the other hand, since the underlying graph $\mathcal{G}$ is directed, due to the loss of symmetry, the Lyapunov analysis as in [17] which is only valid for undirected graphs cannot be directly used in our problem. Instead, we introduce the following novel Lyapunov function candidate

$$\begin{aligned} V =& \sum_{i=1}^{n} \alpha_i \xi_i \Big[\sum_{j=1}^{n} a_{ij}(x_i - x_j) \Big]^T P \sum_{j=1}^{n} a_{ij}(x_i - x_j) \\ =& [(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda \otimes P)(\mathcal{L}_A \otimes I_p)x, \end{aligned} \tag{7}$$

where ξ_i is well defined as in Lemma 2.1 since $\mathcal{G}$ is strongly connected, $\Xi = \text{diag}(\xi_1, \ldots, \xi_n)$, $\Lambda = \text{diag}(\alpha_1, \ldots, \alpha_n)$, $x = \text{col}(x_1, \ldots, x_n)$, and P is defined as in (4). Here we integrate the graph information (ξ_i) and the control gains (α_i) into the Lyapunov function candidate. Using (6), the closed-loop system of (2) can be written as

$$\dot{x} = (I_n \otimes A + \Lambda\mathcal{L}_A \otimes BK)x. \tag{8}$$

Then the derivative of $V(t)$ is given as

$$\begin{aligned} \dot{V}(t) &= 2[(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda\mathcal{L}_A \otimes P)\dot{x} \\ &= 2s^T (\Xi\Lambda^{-1} \otimes PA + \Xi\mathcal{L}_A \otimes PBK)s, \end{aligned} \tag{9}$$

where $s \triangleq (\Lambda\mathcal{L}_A \otimes I_p)x$. Note that $K = -B^T P$. From (9), we can obtain

$$\begin{aligned} \dot{V}(t) =& s^T \Big[\Xi\Lambda^{-1} \otimes PBB^T P \Big] s - s^T (\widehat{L} \otimes PBB^T P)s \\ & - s^T (\Xi\Lambda^{-1} \otimes I_p)s, \end{aligned} \tag{10}$$

where $\widehat{L}$ is defined as in Lemma 2.2 and we have used (4) to obtain the last equality. In (10), the first term is nonnegative and the second and third terms are nonpositive. We aim to derive conditions on α_i such that $\dot{V}(t)$ is negative definite. Since $\Xi\Lambda^{-1}$ is positive definite and $\widehat{L}$ is positive semidefinite, from the first glance, it seems impossible to make $\dot{V}(t)$ negative definite. However, note that s is not arbitrary but associated with the Laplacian matrix $\mathcal{L}_A$. This fact leaves some hope for us. We next show a rigorous analysis on how to choose the control gains α_i such that $\dot{V}(t)$ is negative definite, where we will use the results in Lemma 2.2 and Lemma 2.3.

Note that PBB^TP is symmetric positive semidefinite. There exists an unitary matrix S such that $SPBB^TPS^T = \text{diag}(\lambda_1, \cdots, \lambda_p)$, where $\lambda_i \geq 0$, $i = 1, \ldots, p$, are p eigenvalues of PBB^TP. Write s as $s = [s_1, \ldots, s_n] \in \mathbb{R}^{np}$ with $s_i \in \mathbb{R}^p$. From the definition of s, we have $s_i = \alpha_i \sum_{j=1}^n a_{ij}(x_i - x_j)$. Let $v_k = [v_{k1}, \ldots, v_{kn}] \in \mathbb{R}^n$ be the stacked vector of the kth elements of all Ss_i, $i = 1, \ldots, n$. From Lemma 2.3, we have

$$s^T(\widehat{L} \otimes PBB^TP)s = \sum_{k=1}^p \lambda_k v_k^T \widehat{L} v_k. \tag{11}$$

Note that $Ss_i = \alpha_i \sum_{j=1}^n a_{ij}(Sx_i - Sx_j)$. we have

$$v_{ki} = \alpha_i \sum_{j=1}^n [(Sx_i)_k - (Sx_j)_k], \tag{12}$$

where $(Sx_i)_k$ denotes the kth element of the vector Sx_i, $k = 1, \ldots, p$. Define $\omega_k \triangleq [(Sx_1)_k, \ldots, (Sx_n)_k]^T \in \mathbb{R}^n$. From (12), we have $v_k = \Lambda \mathcal{L}_A \omega_k$. Therefore, $v_k^T \Lambda^{-1} \xi = \omega_k^T \mathcal{L}_A^T \xi = 0$. Since $\xi_i > 0$ and $\alpha_i > 0$, $\forall i = 1, \ldots, n$, the vector $\Lambda^{-1}\xi$ is positive. Under the condition that $\mathcal{G}$ is strongly connected, we can get from Lemma 2.2 that the n eigenvalues of $\widehat{L}$ can be arranged as $0 = \lambda_1(\widehat{L}) < \lambda_2(\widehat{L}) \leq \cdots \leq \lambda_n(\widehat{L})$ and thus

$$v_k^T \widehat{L} v_k \geq \frac{\lambda_2(\widehat{L})}{n} v_k^T v_k. \tag{13}$$

From Lemma 2.3 and (11), we can obtain

$$\begin{aligned} s^T(\widehat{L} \otimes PBB^TP)s &= \sum_{k=1}^p \lambda_k v_k^T \widehat{L} v_k \\ &\geq \frac{\lambda_2(\widehat{L})}{n} \sum_{k=1}^p \lambda_k v_k^T v_k \\ &= \frac{\lambda_2(\widehat{L})}{n} s^T(I_p \otimes PBB^TP)s, \end{aligned} \tag{14}$$

where we have used (13) to obtain the inequality. Substituting (14) into (10), we obtain

$$\dot{V} \leq -\sum_{i=1}^n \Big[\frac{\lambda_2(\widehat{L})}{n} - \frac{\xi_i}{\alpha_i}\Big] s_i^T PBB^TP s_i - \sum_{i=1}^n \frac{\xi_i}{\alpha_i} s_i^T s_i.$$

We then have the following result.

Theorem 3.2: Suppose that the directed graph $\mathcal{G}$ is strongly connected and Assumption 3.1 holds. Using (6) for (2) with $K = -B^TP$ where $P > 0$ is the unique solution of the ARE (4), if the heterogeneous control gains are chosen such that

$$\alpha_i > \frac{n \max_i \xi_i}{\lambda_2(\widehat{L})}, \tag{15}$$

the agents achieve consensus exponentially.

As shown in (15), α_i should be above a certain lower bound which is determined by some global information ($\widehat{L}$ and n). But the positive side is that we allow all agents to have heterogeneous control gains, which implies that the principle to increase the control gains also works as long as the gains are chosen large enough. This fact inspires us to introduce an adaptive strategy for the control gains which will be discussed in the following section.

C. Consensus algorithm with heterogeneous varying control gains

Here, we are ready to deal with the consensus problem for linear multi-agent systems with heterogeneous varying control gains. An intuitive algorithm is as follows

$$u_i = \alpha_i(t)\phi_i(v_i^TPv_i)K \sum_{j=1}^n a_{ij}(x_i - x_j), \tag{16}$$

$$\dot{\alpha}_i(t) = \gamma_i \varphi_i(\sum_{j=1}^n a_{ij}(x_i - x_j)), \tag{17}$$

where γ_i is a positive constant, $v_i \triangleq \sum_{j=1}^n a_{ij}(x_i - x_j)$, $\phi_i(w)$ is continuous and monotonically increasing with respect to w and satisfying $\phi_i(w) > 0$ when $w \geq 0$ to be determined later, and $\varphi_i(w)$ is continuous in w which satisfies $\varphi_i(w) \geq 0$ and $\varphi_i(w) = 0$ if and only if $w = 0$. Here we assume that $\alpha_i(0) > 0$.

Under the condition that $\mathcal{G}$ is strongly connected, since $\alpha_i(t)$ and $\phi_i(v_i^TPv_i)$ is varying, we consider the following integral-type Lyapunov function

$$V(t) = \sum_{i=1}^n \alpha_i(t) \int_0^{v_i^TPv_i} \xi_i \phi_i(\tau) \mathrm{d}\tau. \tag{18}$$

Note from (17) that $\alpha_i(t) \geq \alpha_i(0) > 0$, and P is positive definite. It implies that $V(t)$ will always be nonnegative and $V(t) = 0$ if and only if $\| \sum_{i=1}^n a_{ij}(x_i - x_j)\| = 0$, which also guarantees the consensus of all agents under a strongly connected directed graph. Therefore, $V(t)$ is a suitable Lyapunov function candidate. For conciseness, we denote by $\phi_i = \phi_i(v_i^TPv_i)$. Let $\Phi \triangleq \text{diag}\{\phi_1, \cdots, \phi_n\}$. The derivative of $V(t)$ is

$$\begin{aligned} \dot{V}(t) = &\sum_{i=1}^n \dot{\alpha}_i(t) \int_0^{v_i^TPv_i} \xi_i \phi_i(\tau) \mathrm{d}\tau \\ &+ 2[(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda\Phi \otimes P)(\mathcal{L}_A \otimes I_p)\dot{x}. \end{aligned} \tag{19}$$

Note that the time-varying vector $\Phi\Lambda\xi$ is always positive and $\phi_i(\omega)$ is continuous and monotonically increasing with respect to ω. Following the same steps in Section III-B, we have

$$\dot{V}(t) \leq \sum_{i=1}^n \xi_i \dot{\alpha}_i(t) \phi_i(v_i^TPv_i) v_i^TPv_i - \sum_{i=1}^n \xi_i \alpha_i(t) \phi_i v_i^T v_i$$

$$-\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2-\xi_i\alpha_i(t)\phi_i\Big]v_i^TPBB^TPv_i. \tag{20}$$

The first term of (20) inspires us to introduce the function $\varphi_i(t,\omega)$ as $\varphi_i(t,\omega) = \omega^TPBB^TP\omega$, and thus the time-varying control gains are updated by $\dot{\alpha}_i(t) = \gamma_i v_i^TPBB^TPv_i$. We have

$$\dot{V}(t) \leq -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2-\xi_i\alpha_i(t)\phi_i-\xi_i\gamma_i\phi_i v_i^TPv_i\Big] \cdot v_i^TPBB^TPv_i - \sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^Tv_i. \tag{21}$$

The challenge is how to derive an appropriate upper bound for the term $\xi_i\gamma_i\phi_i v_i^TPv_i$. Note that ϕ_i is assumed to be a function of $v_i^TPv_i$. Let $\phi_i(v_i^TPv_i) = (c_{1i}+c_{i2}v_i^TPv_i)^{r_i}$, where $c_{1i}>0$, $c_{i2}>0$, and $r_i\neq 1$ are positive constants[2]. Using Young's inequality, for positive real numbers q_{i1} and q_{i2} satisfying $\frac{1}{q_{i1}}+\frac{1}{q_{i2}}=1$, we have

$$\xi_i\gamma_i\phi_i v_i^TPv_i \leq \frac{(\xi_i\gamma_i)^{q_{i1}}}{q_{i1}(c_{i2}k_i)^{q_{i1}}}+\frac{k_i^{q_{i2}}\phi_i^{\frac{r_i+1}{r_i}q_{i2}}}{q_{i2}}, \tag{22}$$

where k_i is a positive constant satisfying $\frac{k_i^{q_{i2}}}{q_{i2}} = \frac{\lambda_2(\widehat{L})}{2n}\alpha_i^2(0)$. Let $\frac{r_i+1}{r_i}q_{i2}=2$. Since $r_i\neq 1$, we have $q_{i2}>1$ and thus the Young's inequality is valid for q_{i2} and q_{i2} satisfying $\frac{1}{q_{i1}}+\frac{1}{q_{i2}}=1$. In this case, the control algorithm (16) with (17) can be rewritten as

$$u_i = \alpha_i(t)(c_{1i}+c_{2i}v_i^TPv_i)^{r_i}K\sum_{j=1}^{n}a_{ij}(x_i-x_j), \tag{23}$$

$$\dot{\alpha}_i(t) = \gamma_i v_i^TPBB^TPv_i, \tag{24}$$

where $c_{1i}>0$, $c_{i2}>0$, $r_i\neq 1$, and γ_i are positive constants. For simplicity, choose $r_i=3$. We have

$$q_{i1}=3, \quad q_{i2}=\frac{3}{2}, \quad k_i=\Big[\frac{3\lambda_2(\widehat{L})\alpha_i^2(0)}{4n}\Big]^{\frac{2}{3}}. \tag{25}$$

We then have the following main result.

Theorem 3.3: Suppose that the directed graph $\mathcal{G}$ is strongly connected. Using (23) and (24) for (2) with $K=-B^TP$ where $P>0$ is the unique solution of the ARE (4), the following two statements hold.

(i) $\alpha_i(t)$ is monotonically increasing and will converge to a finite positive constant as $t\to\infty$, $\forall i=1,\ldots,n$.

(ii) The agents achieve consensus asymptotically with a common varying velocity.

Proof: Consider the following Lyapunov function candidate

$$V_0(t) = V(t)+\sum_{i=1}^{n}\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^{2r_i}}{8\gamma_i n}[\alpha_i-\bar{\alpha}]^2, \tag{26}$$

where $V(t)$ is defined as in (18), $r_i=3$, and $\bar{\alpha}$ is a large constant satisfying

$$\bar{\alpha} > \frac{4n^2}{\lambda_2(\widehat{L})\min_i\alpha_i(0)c_{i1}^6}+\frac{32n^2\max_i\gamma_i^3}{27\lambda_2^3(\widehat{L})\min_i\alpha_i^5(0)c_{1i}^6c_{2i}^3}. \tag{27}$$

From (21) and (22), the derivative of $V_0(t)$ is given as

$$\begin{aligned}\dot{V}_0(t) =& \dot{V}(t)+\sum_{i=1}^{n}\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}[\alpha_i(t)-\bar{\alpha}]v_i^TPBB^TPv_i\\ \leq& -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2-\xi_i\alpha_i(t)\phi_i-\frac{\lambda_2(\widehat{L})\alpha_i^2(0)}{2n}\phi_i^2\\ &-\frac{\xi_i^3\gamma_i^3}{3c_{i2}^3k_i^3}-\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}[\alpha_i(t)-\bar{\alpha}]\Big]v_i^TPBB^TPv_i\\ &-\sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^Tv_i.\end{aligned}$$

Note that $\alpha_i(t)\geq\alpha_i(0)>0$, $\phi_i\geq c_{1i}^3$, and

$$\xi_i\alpha_i(t)\phi_i \leq \frac{n\xi_i^2}{\lambda_2(\widehat{L})}+\frac{\lambda_2(\widehat{L})}{4n}\alpha_i^2(t)\phi_i^2.$$

We have

$$\begin{aligned}\dot{V}_0(t) \leq& -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}\bar{\alpha}-\frac{n\xi_i^2}{\lambda_2(\widehat{L})}-\frac{\xi_i^3\gamma_i^3}{3c_{2i}^3k_i^3}\Big]\\ &\cdot v_i^TPBB^TPv_i-\sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^Tv_i,\\ \leq& -\sum_{i=1}^{n}\xi_i\alpha_i(0)c_{1i}^3v_i^Tv_i,\end{aligned} \tag{28}$$

where we have used (27) and the fact that PBB^TP is positive semidefinite to obtain the last inequality. Therefore, we have $V_0(t)\leq V_0(0)$ and thus $\sum_{j=1}a_{ij}(x_i-x_j), \alpha_i-\bar{\alpha}\in\mathbb{L}_\infty$. Since $\bar{\alpha}$ is a constant, we have $\alpha_i\in\mathbb{L}_\infty$. Also note that α_i is monotonically increasing. Therefore, all α_i will converge to some finite constants and thus i) holds. Note that $\dot{V}_0=0$ implies that $\sum_{j=1}^{n}a_{ij}(x_i-x_j)=0$, $\forall i=1,\ldots,n$. Then we can conclude from LaSalle's invariance principle that $\lim_{t\to\infty}\|\sum_{j=1}^{n}a_{ij}[x_i(t)-x_j(t)]\|=0$, $\forall i=1,\ldots,n$. Since $\mathcal{G}$ is strongly connected, we can get that the agents will achieve consensus and thus ii) holds. ■

Here we use the example in Section III-A to show the effectiveness of the proposed algorithm. We choose $r_i=3$, $c_{1i}=1$, $c_{2i}=0.1$, and $\gamma_i=0.03$. The initial values are chosen as $\alpha_i(0)=0.01$. We can see from Fig. 2(a) that all agents achieve consensus. Fig. 2(b) shows the varying gains $\alpha_i(t)$, $i=1,\ldots,n$, which converge to different constants. An incredible observation is that some $\alpha_i(t)$ is even smaller than the lower bound by using a common control gain. This is because that the case with heterogeneous gains are more feasible than the case with a common gain. Actually, the latter can be seen as a special case of heterogeneous gains. An intuitive future work is to find the best control gains in the sense of, for example, minimum energy of the whole system, with respect to the traditional LQR problem.

[2]Here we give an example of ϕ_i to make the proof clear. There are other choices of ϕ_i, for example, $\phi_i(v_i^TPv_i)=c_{1i}+c_{i2}(v_i^TPv_i)^{r_i}$, with c_{1i}, c_{i2}, and $r_i\neq 1$ being positive constants. $r_i\neq 1$ is required for the use of Young's inequality.

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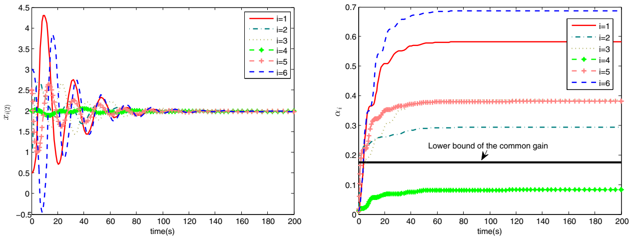

(a) Agents' velocities. (b) Agents' control gains.

Fig. 2. The agents' velocities and control gains using (23) and (24).

Another advantage of using varying heterogeneous control gains is that we allow uncertainties in the control input. For example, due to the disalignment of actuators attached to agent i, the control input might be $(1+\delta_i)u_i$ instead of u_i. Then from the control input (23) and the preceding analysis, we can see that the uncertainties δ_i on u_i will not effect the consensus convergence if $\|\delta_i\| < 1$. Moreover, we also allow δ_i to be time-varying as long as its derivative are relatively small. On the other hand, uncertainties in the dynamics as in [18] are also deserved special attention and will be conducted in the future by combining the idea in the present paper.

IV. Consensus Algorithm with Heterogeneous Varying Control Gains and Relative Output Measurements

In the section, we consider the consensus algorithm for agents with general linear dynamics using relative output measurements, where the dynamics of the agents are described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \qquad i = 1, \ldots, n, \end{aligned} \tag{29}$$

where $x_i \in \mathbb{R}^p$ is the state of agent i, $y_i \in \mathbb{R}^q$ is the output, and $u_i \in \mathbb{R}^m$ is the control input. $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$, and $C \in \mathbb{R}^{q\times p}$ are constant matrices.

The objective is to find an algorithm or a protocol that makes all agents consensus. The existing consensus algorithm for general linear multi-agent systems via output feedback in the literature is as follows [11], [14], [18]

$$\begin{aligned} \dot{\hat{x}}_i &= (A+BF)\hat{x}_i + \alpha G \sum_{j=1}^{n} a_{ij}[C(\hat{x}_i - \hat{x}_j) - (y_i - y_j)], \\ u_i &= F\hat{x}_i, \end{aligned} \tag{30}$$

where $\hat{x}_i$ is the observer state, and $F \in \mathbb{R}^{m\times p}$ and $G \in \mathbb{R}p \times q$ are the feedback gain matrices to be determined. A necessary and sufficient condition for the consensus problem using (30) has be proposed in [14].

Lemma 4.1: Suppose that the directed graph $\mathcal{G}$ contains a directed spanning tree. Using (30) for (29), the agents achieve consensus if and only if all matrices $A + BF$, $A + \alpha\lambda_i GC$, $i = 2, \ldots, n$, are Hurwitz, where λ_i are the nonzero eigenvalues of the Laplacian matrix $\mathcal{L}_A$.

The condition $A+BF$ being Hurwitz can be ensured if the pair (A, B) is stabilizable. For the condition $A + \alpha\lambda_i GC$, $i = 2, \ldots, n$, being Hurwitz, we can choose the feedback gain matrix $F = -Q^{-1}C^T$, where Q is a solution of the linear matrix inequality

$$A^T P + PA - 2C^T C < 0.$$

Then if the control gain α is chosen such that

$$\alpha > \frac{1}{\min_{\mu_i \neq 0} \mathscr{R}(\mu_i)}, \tag{31}$$

where μ_i is the eigenvalue of $\mathcal{L}_A$, all $A + \alpha\lambda_i GC$, $i = 2, \ldots, n$, are Hurwitz.

Therefore, throughout this section, we have the following assumption.

Assumption 4.1: The pair (A, B) is stabilizable, and the pair (C, A) is detectable.

Since (C, A) being detectable is equivalent to (A^T, C^T) being stabilizable [30], the following control algebraic Riccati equation

$$AP + PA^T - PC^T CP + I_p = 0.$$

has a unique solution $P > 0$. Then it is straightforward to get that $Q \triangleq P^{-1}$ is the unique solution of the following algebraic Riccati equation

$$QA + A^T Q - C^T C + I_p = 0. \tag{32}$$

A. Consensus algorithm with heterogeneous constant control gains and relative output measurements

Similar to Section III-B, we first investigate the consensus problem via output feedback with heterogeneous constant control gains. Here we still assume that the directed graph $\mathcal{G}$ is strongly connected. The consensus algorithm with heterogeneous constant control gains is proposed as

$$\begin{aligned} \dot{\hat{x}}_i &= (A+BF)\hat{x}_i + \alpha_i G \sum_{j=1}^{n} a_{ij}[C(\hat{x}_i - \hat{x}_j) - (y_i - y_j)], \\ u_i &= F\hat{x}_i. \end{aligned} \tag{33}$$

where α_i is a positive constant and $G = -Q^{-1}C^T \in \mathbb{R}^{p\times q}$ with $Q > 0$ being the unique solution of ARE (32). Define $\tilde{x}_i \triangleq x_i - \hat{x}_i$. Let x, $\hat{x}$, $\tilde{x}$, and y be, respectively, the stack vectors of x_i, $\hat{x}_i$, $\tilde{x}_i$, and y_i, $i = 1, \ldots, n$. We then have the following vector form

$$\dot{\hat{x}} = [I_n \otimes (A+BF)]\hat{x} - (\Lambda\mathcal{L}_A \otimes GC)\tilde{x}. \tag{34}$$

Clearly, if F is designed such that $A+BF$ is Hurwitz, then the system (34) is input-to-state stable with $(\Lambda\mathcal{L}_A \otimes GC)\tilde{x}$ as the input and $\hat{x}$ as the state. Next, we will check the dynamics of $\tilde{x}$, whose vector form can be written as

$$\dot{\tilde{x}} = (I_n \otimes A)\tilde{x} + (\Lambda\mathcal{L}_A \otimes GC)\tilde{x}. \tag{35}$$

The system (35) is very similar to the system (8). Therefore, we consider the following Lyapunov function candidate similar to (7)

$$V = [(\mathcal{L}_A \otimes I_p)\tilde{x}]^T (\Xi\Lambda \otimes Q)(\mathcal{L}_A \otimes I_p)\tilde{x}, \tag{36}$$

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where ξ_i is well defined as in Lemma 2.1 since $\mathcal{G}$ is strongly connected, and Q is defined as in (32). Using (32), and note that $G = -Q^{-1}C^T$, the derivative of $V(t)$ is given as

$$\begin{aligned}\dot{V}(t) =& \tilde{s}^T\Big[\Xi\Lambda^{-1}\otimes C^TC\Big]\tilde{s} - \tilde{s}^T(\widehat{L}\otimes C^TC)\tilde{s}\\ &- \tilde{s}^T(\Xi\Lambda^{-1}\otimes I_p)\tilde{s},\end{aligned} \tag{37}$$

where $\tilde{s} \triangleq (\Lambda\mathcal{L}_A\otimes I_p)\tilde{x}$ and $\widehat{L}$ is defined as in Lemma 2.2. Following the same steps in the analysis of Section III-B, we can get from Lemma 2.3 that

$$\tilde{s}^T(\widehat{L}\otimes C^TC)\tilde{s} \geq \frac{\lambda_2(\widehat{L})}{n}\tilde{s}^T(I_p\otimes C^TC)\tilde{s}. \tag{38}$$

Substituting (38) into (37), we obtain

$$\dot{V} \leq -\sum_{i=1}^n\Big[\frac{\lambda_2(\widehat{L})}{n} - \frac{\xi_i}{\alpha_i}\Big]\tilde{s}_i^TC^TC\tilde{s}_i - \sum_{i=1}^n\frac{\xi_i}{\alpha_i}\tilde{s}_i^T\tilde{s}_i. \tag{39}$$

We then have the following result.

Theorem 4.2: Suppose that the directed graph $\mathcal{G}$ is strongly connected. Using (33) for (29) with $G = -Q^{-1}C^T$ where $Q > 0$ is the unique solution of the ARE (32) and with F such that $A + BF$ is Hurwitz, if the heterogeneous control gains are chosen such that

$$\alpha_i > \frac{n\max_i\xi_i}{\lambda_2(\widehat{L})}, \tag{40}$$

the agents achieve consensus exponentially, *i.e.*, $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$.

Proof: From (39), it is easy to get that $\lim_{t\to\infty}\|\tilde{s}\| = 0$ and thus $\tilde{x}_i \to \tilde{x}_j$ as $t\to\infty$, $\forall i, j = 1,\ldots,n$. Since $A + BF$ is Hurwitz, the system (34) is input-to-state stable. We then can get $\lim_{t\to\infty}\|\hat{x}\| = 0$ from the fact that $(\Lambda\mathcal{L}_A\otimes GC)\tilde{x} = (I_n\otimes GC)\tilde{s}$ and $\lim_{t\to\infty}\|\tilde{s}\| = 0$. ■

We get a similar result to Theorem 3.2. As shown in (40), for the consensus with heterogeneous control gains via output feedback, we can still choose the control gains large enough to make all agents consensus.

B. Consensus algorithm with heterogeneous varying control gains and relative output measurements

Here, we are ready to deal with the consensus problem for linear multi-agent systems with heterogeneous varying control gains using relative output measurements. Motivated by the algorithm via state feedback with varying gains (23) and (24) and the algorithm via output feedback with constant gains (33), we propose the following consensus algorithm

$$\begin{aligned}\dot{\hat{x}}_i &= (A + BF)\hat{x}_i + \alpha_i(t)(c_{1i} + c_{2i}z_i^TQz_i)^{r_i}GCz_i,\\ u_i &= F\hat{x}_i, \qquad (41)\\ \dot{\alpha}_i &= \gamma_iz_i^TC^TCz_i, \qquad (42)\end{aligned}$$

where c_{1i}, c_{2i}, $r_i \neq 1$, and γ_i are positive constants, and $z_i \triangleq \sum_{j=1}^n a_{ij}[(\hat{x}_i - \hat{x}_j) - (x_i - x_j)]$. Here we assume that $\alpha_i(0) > 0$. Define $\phi(z_i^TQz_i) \triangleq (c_{1i} + c_{2i}z_i^TQz_i)^{r_i}$, $\Phi \triangleq \mathrm{diag}(\phi_1,\cdots,\phi_n)$, and $z \triangleq \mathrm{col}(z_1,\ldots,z_n)$. We have $z = (\mathcal{L}_A\otimes I_p)\tilde{x}$. The vector form of the dynamics for $\tilde{x}$ can be written as

$$\dot{\tilde{x}} = (I_n\otimes A)\tilde{x} + [\Lambda(t)\Phi\mathcal{L}_A\otimes GC]\tilde{x}. \tag{43}$$

The vector form of the dynamics for $\hat{x}$ can be written as

$$\dot{\hat{x}} = [I_n\otimes(A + BF)]\hat{x} - [\Lambda(t)\Phi\mathcal{L}_A\otimes GC]\tilde{x}. \tag{44}$$

We then have the following main result.

Theorem 4.3: Suppose that the directed graph $\mathcal{G}$ is strongly connected and Assumption 4.1 holds. Using (41) and (42) for (29) with $G = -Q^{-1}C^T$ where $Q > 0$ is the unique solution of the ARE (32) and with F such that $A + BF$ is Hurwitz, the following two statements hold.

(i) $\alpha_i(t)$ is monotonically increasing and will converge to a finite positive constant as $t\to\infty$, $\forall i = 1,\ldots,n$.
(ii) The agents achieve consensus asymptotically with a common varying velocity.

Proof: Similar to the analysis in Section III-C, we first consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^n\alpha_i(t)\int_0^{z_i^TQz_i}\xi_i\phi_i(\tau)\mathrm{d}\tau. \tag{45}$$

where z_i and $\phi_i(\cdot)$ are defined as in (41). The derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) =& \sum_{i=1}^n\dot{\alpha}_i(t)\int_0^{z_i^TQz_i}\xi_i\phi_i(\tau)\mathrm{d}\tau\\ &+ 2[(\mathcal{L}_A\otimes I_p)\tilde{x}]^T(\Xi\Lambda\Phi\otimes Q)(\mathcal{L}_A\otimes I_p)\dot{\tilde{x}}\\ \leq& -\sum_{i=1}^n\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2 - \xi_i\alpha_i(t)\phi_i - \xi_i\gamma_i\phi_iz_i^TQz_i\Big]\\ &\cdot z_i^TC^TCz_i - \sum_{i=1}^n\xi_i\alpha_i(t)\phi_iz_i^Tz_i.\end{aligned} \tag{46}$$

Similar to (22), we have

$$\xi_i\gamma_i\phi_iz_i^TQz_i \leq \frac{(\xi_i\gamma_i)^{q_{i1}}}{q_{i1}(c_{i2}k_i)^{q_{i1}}} + \frac{k_i^{q_{i2}}\phi_i^{\frac{r_i+1}{r_i}q_{i2}}}{q_{i2}}, \tag{47}$$

where k_i is a positive constant satisfying $\frac{k_i^{q_{i2}}}{q_{i2}} = \frac{\lambda_2(\widehat{L})}{2n}\alpha_i^2(0)$. Let $\frac{r_i+1}{r_i}q_{i2} = 2$. For simplicity, we choose the gains the same as (25). We then consider the following Lyapunov function candidate

$$V_0(t) = V(t) + \sum_{i=1}^n\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{8\gamma_in}[\alpha_i - \bar{\alpha}]^2, \tag{48}$$

where $V(t)$ is defined as in (45) and $\bar{\alpha}$ is a large constant satisfying

$$\bar{\alpha} > \frac{4n^2\xi_i^2}{\alpha_i(0)c_i^6\lambda_2(\widehat{L})} + \frac{32n^2\xi_i^3\gamma_i^3}{27\alpha_i^5(0)c_i^6\lambda_2^3(\widehat{L})}. \tag{49}$$

From (46), the derivative of $V_0(t)$ is given as

$$\dot{V}_0(t) = \dot{V}(t) + \sum_{i=1}^n\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}[\alpha_i(t) - \bar{\alpha}]z_i^TC^TCz_i$$

$$\leq -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}\bar{\alpha} - C\Big] z_i^T C^T C z_i - \sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i z_i^T z_i. \quad (50)$$

where we have used (49) and the fact that PBB^TP is nonnegative definite to obtain the inequality. From the same analysis in the proof of Theorem 3.3, we can conclude our result. ■

Remark 4.4: Here we highlight the difference between our results and [25]. The work in [25] focuses on the leader-following tracking problem. The authors utilize the property that the associated matrix (Laplacian matrix plus a diagonal matrix) for the coordinated tracking problem is positive stable. In contrast, the associated matrix (Laplacian matrix) for the consensus problem studied in the current paper is only semi-positive stable. Therefore, the results in [25] cannot be used for the leaderless consensus problem. In fact, the leaderless problem and the algorithm with relative output measurements are considered as future work in [25].

Remark 4.5: Note that all the results in this paper are obtained under a strongly connected directed graph. By using the Perron-Frobenius form, all the results can be extended to the case where the directed graph has a directed spanning tree following similar steps to those in [26]. And the result in [25] becomes a special case of our results when there exists one agent that has no neighbors and has directed paths to all other agents.

V. Conclusions

In this paper we have studied the distributed consensus problem for general linear multi-agent systems with heterogeneous control gains under a general directed graph. Fully distributed consensus algorithms have been proposed using, respectively, the relative and output measurements. A notable feature of the proposed consensus algorithms is that the control gains are heterogeneous for each agent and can be obtained with only local information from the neighbors. Novel integral-type Lyapunov functions have been proposed to solve the consensus problem.

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