MINIMAL WALKS AND COUNTRYMAN LINES

by

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ABSTRACT

I will give some basic definitions and facts surrounding Stevo Todorcevic's method of minimal walks, including a modification of the full lower trace, and show how these objects can be used to construct a Countryman line, an uncountable linear order whose square is decomposable into countably many chains. Although this construction is analogous to Todorcevic's construction in [12] it is (arguably) more accessible than that one or Shelah's construction in [9]. Finally, I will give further definitions, facts, and applications of minimal walks that are not needed for the Countryman line construction.

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PRELIMINARIES

1.1 **Introduction**

The method of minimal walks was developed by Stevo Todorcevic in 1984 and the work circulated as [10]. Soon after others started using the method in their own proofs and so Todorcevic published [11] to present the method in a more explicit form and to demonstrate the failure of the partition relation $\omega_1 \to [\omega_1]^2_{\omega_1}$. Since [11] the method has found numerous applications, including the construction of an L space in [7], answering a long time and important conjecture. The method itself involves an analysis of several two place functions on the set of countable ordinals, where said functions are recursively defined using C-sequences. These C-sequences, functions, and resulting facts will be the subject of Chapter 2.

The method of minimal walks was originally developed during Todorcevic's discovery of a new proof of the existence of a Countryman line, an uncountable linear order whose square is decomposable into countably many chains. The existence of a Countryman line was first questioned by Roger Countryman in [1], where he conjectured that such an order does not exist. Half a decade later in [9] Saharon Shelah was able to answer the conjecture in the negative using an Aronszajn-type construction. The purpose of Chapter 3 is to explain the significance of Countryman lines and then construct one with a modification of the full lower trace function from the method of minimal walks. It is hoped that this will make the construction easier to comprehend for those coming to the subject for the first time.

Not all of the two place functions associated with the method of minimal walks are needed for the construction of a Countryman line. The definitions of these additional functions are collected in Section 4.1. Section 4.2 gives certain facts associated with these new functions and facts that are associated to the functions in Section 2.2 but not referenced in Chapter 3. Section 4.3 is devoted to further applications of minimal walks such as proving a negative partition relation, the construction of a Hausdorff gap, and the construction of a topological L space. Finally, Appendix A gives a brief proof of the independence of the statement "There exists a C-sequence." from the Zermelo-Fraenkel axioms.

1.2 Background and Notation

The standard reference for the method of minimal walks is Todorcevic's [12] where the definitions are given recursively and many of the proofs use transfinite induction, especially when the subject matter involves cardinals larger than ω_1. The subject of this thesis takes place in the space of countable ordinals so we have been able to give closed form definitions for many of the functions associated with the method of minimal walks and have subsequently been able to reduce the instances of induction to standard induction over the natural numbers. Hence, this thesis can be read profitably by anyone possessing a strong familiarity with abstract arguments who has taken an upper division undergraduate course in set theory, say chapters 1-4, 6 and 9 of Hrbacek and Jech's [2].

We use standard notation for standard set theoretic objects and those terms that we leave undefined can be found in any standard reference for set theory, for example Jech [3] or Kunen [5]. If s and t are sequences of natural numbers we write $s \sqsubseteq t$ and say s is an *initial segment* of t if $s(i) = t(i)$ for all i in the domain of s. We use $<_{lex}$ for the *right lexicographic ordering* for finite sequences of natural numbers, that is $s <_{lex} t$ if $t \sqsubseteq s$ or $s(j) < t(j)$ where j is the least i such that $s(i) \neq t(i)$. In addition, if A and B are sets of ordinals we write $A < B$ if every element of A is less than every element of B and write $A \sqsubseteq B$ and say A is an *initial segment* of B if $A \subseteq B$ and $A < B \setminus A$ (the two notions of initial segment agree if we identify sets of ordinals with strictly increasing sequences of ordinals). Finally, we use $\hat{}$ for the concatentation operator.

THE SPACE OF COUNTABLE ORDINALS

2.1 Fundamental Sequences

The underlying set of the space of countable ordinals is ω_1, the least uncountable ordinal. The structure imposed on ω_1 is the usual ordering relation along with a system of sets indexed by the countable ordinals $\langle C_\alpha : \alpha < \omega_1 \rangle$, which has the following two properties:

(a) $C_{\alpha+1} = \{\alpha\}$.

(b) If α is a nonzero limit ordinal then C_α is a cofinal subset of α containing 0 such that $|\{\xi \in C_\alpha : \xi < \gamma\}|$ is finite for all $\gamma < \alpha$.

Each C_α in $\langle C_\alpha : \alpha < \omega_1 \rangle$ is a *fundamental sequence* and a system of fundamental sequences will be called a *C-sequence*, at times denoted by $\vec{C}$. We note that a C-sequence is an example of a *ladder system* and while ladder systems are usually used for climbing up in recursive constructions here they will be used as a tool for walking down from one ordinal to a smaller one. We also note that as a structure $\left(\omega_1, \leq, \vec{C}\right)$ is (imperfectly) analogous to the more familiar structure $(\omega, \leq, P(n))$, the space of finite ordinals with the usual ordering and predecessor function.

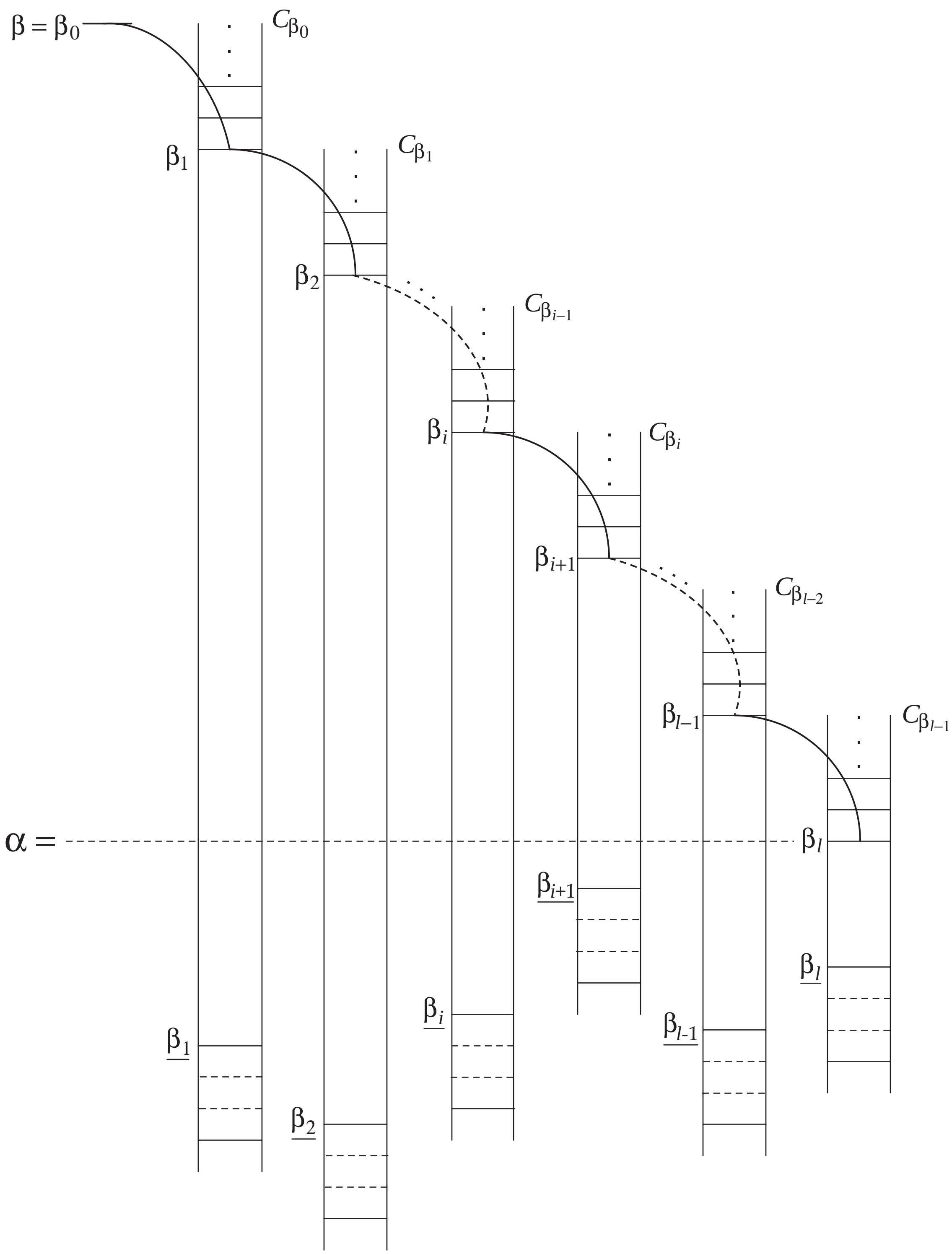

Figure 1. The Walk from β to α

2.2 The Walk from β to α

Referring to Figure 1 on page 5 will facilitate much of the discussion and many of the definitions in this section. For the remainder of the section we fix a C-sequence and nonzero countable ordinals $\alpha < \beta$.

2.2.1 The Steps and Stops of the Walk from β to α

The basic concept when walking from a countable ordinal β to a smaller ordinal α is to find the minimal element of C_β that is greater than or equal to α. This element is the first step from β towards α and is denoted β_1. The greatest element of C_β that is less than α is defined to be the first stop in the walk from β to α and is denoted $\underline{\beta_1}$. The next step in the walk is then defined to be the least element of C_{β_1} that is greater than or equal to α and the next stop is the greatest element of C_{β_1} that is less than α. One continues this process until the next step is α itself. This intuitive description is captured formally by the following definitions and facts.

Definition 1 *The* $\boldsymbol{i^{th}}$ ***step*** *from β to α is given by*

$$\beta_0 = 0$$

$$\beta_i = \min\left\{\xi \in C_{\beta_{i-1}} : \xi \geq \alpha\right\} \qquad (i > 0).$$

Fact 2 *Since $\beta_i > \beta_{i+1}$ it must be that there is an $l < \omega$ such that $\beta_l = \alpha$, otherwise $(\beta_i)_{i<\omega}$ would be an infinite strictly decreasing set of ordinals.*

Definition 3 *The* $\boldsymbol{i^{th}}$ ***stop*** *from* β *to* α *is given by*

$$\underline{\beta_0} = \emptyset$$

$$\underline{\beta_i} = \max\left\{\xi \in C_{\beta_{i-1}} : \xi < \alpha\right\} \qquad (i > 0).$$

Fact 4 $\underline{\beta_i}$ *and* β_i *are the unique ordinals of* $C_{\beta_{i-1}}$ *such that*

$$\underline{\beta_i} < \alpha \leq \beta_i \qquad \textit{and} \qquad C_{\beta_{i-1}} \cap \left(\underline{\beta_i}, \beta_i\right) = \emptyset.$$

2.2.2 Upper Trace of the Walk from β to α

Definition 5 *The* ***upper trace*** *of the walk from* α *to* β *is the functions*

$$\operatorname{Tr} : [\omega_1]^2 \to \mathcal{P}(\omega_1)$$

given by

$$\operatorname{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_l\},$$

where β_i *is as in Definition 1 and* l *is as in Fact 2.*

In [12] Todorcevic defines the *minimal walk from* β *to* α to be the increasing enumeration of $\operatorname{Tr}(\alpha, \beta)$. Unless otherwise noted we will identify the walk from β to α with the upper trace of the walk. $\operatorname{Tr}(\alpha, \beta)$ can be defined recursively (see Definition 30 in section 4.1) but we will use Definition 5 throughout this thesis.

2.2.3 Full Code of the Walk from β to α

The *full code* of the walk from β to α is a finite sequence of natural numbers where the i^{th} entry of the sequence is the number of rungs on the ladder C_{β_i} that are below α (see Figure 1). Formally the definition is as follows.

Definition 6 *The **full code** of the walk is the function*

$$\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$$

such that for $0 \leq i < l$

$$\rho_0 (\alpha, \beta) (i) = \left| \left\{ \xi \in C_{\beta_i} : \xi < \alpha \right\} \right| ,$$

where β_i *is as in Definition 1 and* l *is as in Fact 2.*

We will use the full code of the walk as one of the main building blocks in the construction of a Countryman line in Chapter 3. $\rho_0 (\alpha, \beta)$ can also be defined recursively (see Definition 30 in Section 4.1) but we will not need that definition for our present purposes. The following fact about ρ_0 will be useful in the sequel.

Fact 7 *Given a system of fundamental sequences and a countable ordinal* β *the upper trace of the walk from* β *to* α *can be recovered from* $\rho_0 (\alpha, \beta)$ *. To see this let* $\langle C_\alpha (i) : i < \omega \rangle$ *be the natural enumeration of* C_α *for* $\alpha \in \omega_1$*, then*

$$\beta_i = C_{\beta_{i-1}} (\rho_0 (\alpha, \beta) (i - 1) + 1) .$$

2.2.4 Maximum Stop of the Walk from β to α

The next definition will be used in the proof of several facts that will facilitate the construction of a Countryman line in Chapter 3. Referring to Figure 1, page 5, the maximum stop of the walk from β to α, denoted $\lambda (\alpha, \beta)$, is the highest rung of all the rungs that are below α. The formal definition is as follows.

Definition 8 *The **maximum stop** of the walk is the function*

$$\lambda : [\omega_1]^2 \to \omega$$

given by

$$\lambda(\alpha, \beta) = \max_{i \le l} \underline{\beta_i},$$

where $\underline{\beta_i}$ is as in Definition 3 and l is as in Fact 2.

We note that $\lambda(\alpha, \beta)$ is the maximum element of the *lower trace* of the walk from β to α (see Definition 31, section 4.1) but the lower trace will not be needed for our present purposes.

2.2.5 The Support of the Walk from β to α

The *support* of the walk from β to α, denoted $S(\alpha, \beta)$, is the lynch pin of our construction of a Countryman line. It doesn't appear explicitly in [12] but it is a proper subset of thes *full lower trace* (see Definition 34, Section 4.1 below). In order to picture $S(\alpha, \beta)$ intuitively it is illustrative to think of walking up from an ordinal ζ to both α and β (see Figure 2, page 10). As one walks up there will be an ordinal ξ from which the two traces $\mathrm{Tr}(\zeta, \alpha)$ and $\mathrm{Tr}(\zeta, \beta)$ will split. That is, while walking up from ζ to ξ the walks from ζ to α and from ζ to β step on exactly the same ordinals and after ξ they never step on the same ordinal again. $S(\alpha, \beta)$ is the set of all such ξ and is defined formally as follows.

Definition 9 *The **support** of the walk from β to α is the set*

$$S(\alpha, \beta) = \{\xi \le \alpha : \mathrm{Tr}(\xi, \alpha) \cap \mathrm{Tr}(\xi, \beta) = \{\xi\}\}.$$

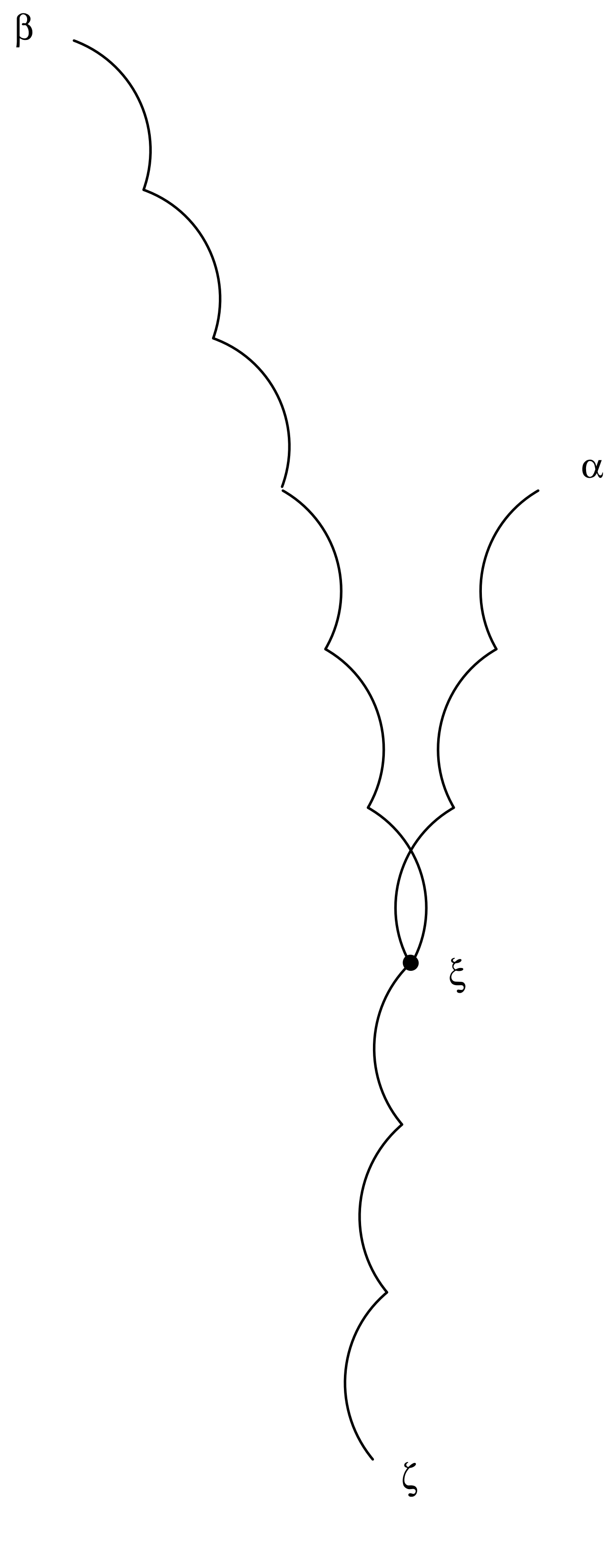

Figure 2. $\xi \in S(\alpha,\beta)$

2.3 Basic Facts

Most of the facts in this section are intuitively clear after a moments reflection on Figure 1, page 5, although in this section we are often dealing with $\alpha < \beta < \gamma$ rather than just α and β. The proofs of the facts are included both for completeness and to aid in the development of the afore mentioned intuition. They are relatively simple using induction on the natural numbers and/or naive set theory. The following remark sets notation for the remainder of this section.

Remark 10 *Unless otherwise noted, for each of the following facts we fix nonzero countable ordinals $\alpha < \beta < \gamma$ and set*

$$\begin{aligned} \mathrm{Tr}\,(\alpha, \gamma) &= \{\gamma = \gamma_0 > \gamma_1 > \cdots > \gamma_l = \alpha\}, \\ \mathrm{Tr}\,(\beta, \gamma) &= \{\gamma = \gamma_0' > \gamma_1' > \cdots > \gamma_m' = \beta\}, \\ \mathrm{Tr}\,(\alpha, \beta) &= \{\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha\}, \end{aligned}$$

and define the stops for each walk accordingly, that is $\underline{\gamma_i}$ is the i^{th} stop in the walk from γ to α, $\underline{\gamma_i'}$ is the i^{th} stop in the walk from γ to β, and $\underline{\beta_i}$ is the i^{th} stop in the walk from β to α.

Fact 11 below is to the method of minimal walks as $\sin^2\theta + \cos^2\theta = 1$ is to the method of trigonometry. Indeed, just as one uses facts and identities following from $\sin^2\theta + \cos^2\theta = 1$ to traverse the unit circle, Fact 11 is used to prove many of the basic facts that allow one to traverse ω_1. Although it requires two instances of induction to prove, Fact 11 is a really a consequence of Definition 1 and Fact 4 and can readily be seen from Figure 1.

Fact 11 $\beta \in \mathrm{Tr}(\alpha, \gamma) \Leftrightarrow \mathrm{Tr}(\alpha, \gamma) = \mathrm{Tr}(\alpha, \beta) \cup \mathrm{Tr}(\beta, \gamma)$.

Proof. The 'if' part is immediate since $\beta \in \mathrm{Tr}(\alpha, \beta)$. For the 'only if' part assume $\beta \in \mathrm{Tr}(\alpha, \gamma)$. Then there is $k < l$ such that $\gamma_k = \beta$. We will show by induction on l that

$$i \leq k \Rightarrow \gamma_i = \gamma_i' \qquad \textit{and} \qquad k \leq i \leq l \Rightarrow \gamma_i = \beta_{i-k}.$$

The base case for $i \leq k$ is $\gamma_0 = \gamma = \gamma_0'$. Assume $\gamma_j = \gamma_j'$, with $j < k$. Then $C_{\gamma_j} = C_{\gamma_j'}$. Since $\underline{\gamma_{j+1}} < \alpha < \beta \leq \gamma_{j+1}$, we have by Fact 4 that $\gamma_{j+1} = \gamma_{j+1}'$.

The base case for $k \leq i \leq l$ is $\gamma_k = \beta = \beta_{k-k}$. Assume $\gamma_j = \beta_{j-k}$, with $k \leq j < l$. Then

$$\gamma_{j+1} = \min\left\{\xi \in C_{\gamma_i} : \xi \geq \alpha\right\} = \min\left\{\xi \in C_{\beta_{j-k}} : \xi \geq \alpha\right\} = \beta_{j-k+1}.$$

The desired conclusion follows from the Principal of Mathematical Induction. ∎

The following corollary is used solely to facilitate subsequent proofs within this section and is dependent on the set-up in Remark 10.

Corollary 12 *If* $\beta \in \mathrm{Tr}(\alpha, \gamma)$, *then for* $i \leq m$,

$$\gamma_i = \gamma_i' \qquad \textit{and} \qquad \underline{\gamma_i} = \underline{\gamma_i'},$$

and for $m < i \leq l$,

$$\gamma_i = \beta_{i-m} \qquad \textit{and} \qquad \underline{\gamma_i} = \underline{\beta_{i-m}}.$$

The next fact is telling us that an ordinal β is in the walk from γ to α if and only if each stop of the walk from γ to β lies below α. Again, this is intuitively clear from Figure 1 and follows from Definition 8 and Facts 4 and 11.

Fact 13 $\beta \in \mathrm{Tr}(\alpha,\gamma) \Leftrightarrow \lambda(\beta,\gamma) < \alpha.$

Proof. For the 'only if' direction assume $\beta \in \mathrm{Tr}(\alpha,\gamma)$. By Corollary 12 we have that for $i \leq m$, $\underline{\gamma_i'} = \underline{\gamma_i} < \alpha$ and therefore

$$\lambda(\beta,\gamma) = \max_{i \leq m} \underline{\gamma_i'} < \alpha.$$

For the 'if' direction assume $\lambda(\beta,\gamma) < \alpha$. We will show by induction on m that for $i \leq m$, $\gamma_i = \gamma_i'$. In particular, $\gamma_m = \beta$ so that $\beta \in \mathrm{Tr}(\alpha,\gamma)$. For the base case we have $\gamma_0 = \gamma = \gamma_0'$. Assume $\gamma_j = \gamma_j'$, with $j < m$. Then $C_{\gamma_j} = C_{\gamma_j'}$. Since

$$\underline{\gamma_{j+1}'} \leq \lambda(\beta,\gamma) < \alpha < \beta \leq \gamma_{j+1}',$$

we have by Fact 4 that $\gamma_{j+1} = \gamma_{j+1}'$. The desired conclusion follows from the Principal of Mathematical Induction. ■

The following fact will be needed in Appendix A and does not use the set-up in Remark 10. It follows immediately from Fact 7.

Fact 14 $\rho_0(\cdot,\beta) : \beta \to \omega_1$ *is a one-to-one function.*

Proof. Suppose $\alpha, \alpha' < \beta$ and $\rho_0(\alpha,\beta) = \rho_0(\alpha',\beta)$. Let $\mathrm{Tr}(\alpha,\beta) = \{\beta_0,\ldots,\beta_l\}$ and $\mathrm{Tr}(\alpha',\beta) = \{\beta_0',\ldots,\beta_{l'}'\}$. We will prove by induction on $l = |\rho_0(\alpha,\beta)| = |\rho_0(\alpha',\beta)| = l'$ that $\beta_i = \beta_i'$. In particular, $\alpha = \beta_l = \beta_{l'}' = \alpha'$. For the base case we have $\beta_0 = \beta = \beta_0'$. Assume $\beta_j = \beta_j'$ for $j < l$. Then $C_{\beta_j} = C_{\beta_j'}$ so from Fact 7

$$\beta_{j+1} = C_{\beta_j}(\rho_0(\alpha,\beta)(j)+1) = C_{\beta_j'}(\rho_0(\alpha',\beta)(j)+1) = \beta_{j+1}'.$$

The desired conclusion follows from the Principal of Mathematical Induction. ■

The following two facts, which are consequences of Definition 6 and Facts 4, 7, 11 and 13, are needed to prove Facts 22 and 44 below.

Fact 15 $\beta \in \mathrm{Tr}(\alpha, \gamma) \Leftrightarrow \rho_0(\beta, \gamma) \sqsubseteq \rho_0(\alpha, \gamma)$.

Proof. For the 'only if' direction assume $\beta \in \mathrm{Tr}(\alpha, \gamma)$. Since $\alpha < \beta$ we have $|\rho_0(\beta, \gamma)| < |\rho_0(\alpha, \gamma)|$. Further, from Fact 4 and Corollary 12 we have for $i \leq m$,

$$\rho_0(\beta, \gamma)(i) = \left|\{\xi \in C_{\gamma_i'} : \xi < \beta\}\right| = \left|\left\{\xi \in C_{\gamma_i'} : \xi \leq \underline{\gamma_{i+1}'}\right\}\right|$$
$$= \left|\left\{\xi \in C_{\gamma_i} : \xi \leq \underline{\gamma_{i+1}}\right\}\right| = \left|\{\xi \in C_{\gamma_i} : \xi < \alpha\}\right| = \rho_0(\alpha, \gamma)(i),$$

so $\rho_0(\beta, \gamma) \sqsubseteq \rho_0(\alpha, \gamma)$.

For the 'if' direction assume $\rho_0(\beta, \gamma) \sqsubseteq \rho_0(\alpha, \gamma)$. We will show by induction on m that for $i \leq m$, $\gamma_i = \gamma_i'$. In particular, $\gamma_m = \beta$ so that $\beta \in \mathrm{Tr}(\alpha, \gamma)$. For the base case we have $\gamma_0 = \gamma = \gamma_0'$. Assume $\gamma_j = \gamma_j'$, with $j < m$ and let $\langle C_\alpha(i) : i < \omega\rangle$ be the natural enumeration of C_α for $\alpha \in \omega_1$. Then by Fact 7

$$\gamma_{j+1}' = C_{\gamma_j'}(\rho_0(\beta, \gamma)(j) + 1) = C_{\gamma_j}(\rho_0(\alpha, \gamma)(j) + 1) = \gamma_{j+1}.$$

The desired conclusion follows from the Principal of Mathematical Induction. ■

Fact 16 $\lambda(\beta, \gamma) < \alpha \Rightarrow \rho_0(\alpha, \gamma) = \rho_0(\beta, \gamma)\,\hat{}\,\rho_0(\alpha, \beta)$.

Proof. Assume $\lambda(\beta, \gamma) < \alpha$. By Facts 13 and 15 $\rho_0(\beta, \gamma) \sqsubseteq \rho_0(\alpha, \gamma)$, so that $\rho_0(\alpha, \gamma)(i) = \rho_0(\beta, \gamma)(i)$ for $i < m$. On the other hand, by Corollary 12

$$\rho_0(\alpha, \gamma)(i) = \left|\{\xi \in C_{\gamma_i} : \xi < \alpha\}\right| = \left|\{\xi \in C_{\beta_{i-m}} : \xi < \alpha\}\right| = \rho_0(\alpha, \beta)(i - m).$$

for $m \leq i < l$. The desired conclusion follows. ■

This next fact will be in defining a linear order in Chapter 3. Note that if j is the least i such that $\rho_0(\beta,\gamma)(i) \neq \rho_0(\alpha,\gamma)(i)$ then the $j+1^{st}$ step from γ to α must be between α and β, else the steps would agree (since $C_{\gamma_j} = C_{\gamma'_j}$).

Fact 17 $\rho_0(\alpha,\gamma) <_{lex} \rho_0(\beta,\gamma)$

Proof. If $\rho_0(\beta,\gamma) \sqsubseteq \rho_0(\alpha,\gamma)$ then we are done. Otherwise, let j be the least i such that $\rho_0(\beta,\gamma)(i) \neq \rho_0(\alpha,\gamma)(i)$. Then

$$\alpha \leq \gamma_{j+1} < \beta \leq \gamma'_{j+1},$$

so

$$\begin{aligned}\rho_0(\beta,\gamma)(j) = \left|\left\{\xi \in C_{\beta_j} : \xi < \beta\right\}\right| \geq \left|\left\{\xi \in C_{\beta_j} : \xi < \alpha\right\} \cup \{\gamma_{j+1}\}\right| \\ > \left|\left\{\xi \in C_{\beta_j} : \xi < \alpha\right\}\right| = \rho_0(\alpha,\gamma)(j).\end{aligned}$$

Hence, $\rho_0(\alpha,\gamma) <_{lex} \rho_0(\beta,\gamma)$. ■

Fact 18 is just a rewording of what was said in the preamble to Definition 9 on page 9 and is immediately clear from Figure 2 on, page 10.

Fact 18 *If* $\xi = \max(\mathrm{Tr}(\alpha,\gamma) \cap \mathrm{Tr}(\alpha,\beta))$, *then* $\mathrm{Tr}(\alpha,\gamma) \cap \mathrm{Tr}(\alpha,\beta) = \mathrm{Tr}(\alpha,\xi)$.

Proof. We have by Fact 11 and the maximality of ξ

$$\begin{aligned}\mathrm{Tr}(\alpha,\gamma) \cap \mathrm{Tr}(\alpha,\beta) = (\mathrm{Tr}(\alpha,\xi) \cup \mathrm{Tr}(\xi,\gamma)) \cap (\mathrm{Tr}(\alpha,\xi) \cup \mathrm{Tr}(\xi,\beta)) \\ = \mathrm{Tr}(\alpha,\xi) \cup (\mathrm{Tr}(\xi,\gamma) \cap \mathrm{Tr}(\xi,\beta)) = \mathrm{Tr}(\alpha,\xi) \cup \{\xi\} = \mathrm{Tr}(\alpha,\xi).\end{aligned}$$

■

Fact 19 will be used to show the "subadditivity" of the support of α and β in Section 4.2 (see Fact 42), but it is related to Fact 18 above so we include it here. The proof is simply a matter of applying the definition of initial segment for sets of ordinals given on page 3.

Fact 19 *If* $\xi = \max(\mathrm{Tr}(\alpha,\gamma) \cap \mathrm{Tr}(\alpha,\beta))$, *then*

$$\mathrm{Tr}(\alpha,\xi) \sqsubseteq \mathrm{Tr}(\alpha,\gamma) \qquad \text{and} \qquad \mathrm{Tr}(\alpha,\xi) \sqsubseteq \mathrm{Tr}(\alpha,\beta).$$

Proof. It follows from Fact 18 that

$$\mathrm{Tr}(\alpha,\gamma) = \mathrm{Tr}(\alpha,\xi) \cup \mathrm{Tr}(\xi,\gamma) \qquad \text{and} \qquad \mathrm{Tr}(\alpha,\beta) = \mathrm{Tr}(\alpha,\xi) \cup \mathrm{Tr}(\xi,\beta).$$

The desired conclusion now follows since $\delta \leq \xi$ for all $\delta \in \mathrm{Tr}(\alpha,\xi)$ and $\xi < \zeta$ for all $\zeta \in \mathrm{Tr}(\alpha,\gamma) \setminus \mathrm{Tr}(\alpha,\xi) = \mathrm{Tr}(\xi,\gamma)$ and $\zeta \in \mathrm{Tr}(\alpha,\beta) \setminus \mathrm{Tr}(\alpha,\xi) = \mathrm{Tr}(\xi,\beta)$. ■

The next fact is the first we have seen that is not immediately clear from either definition or picture. Note that for each ζ below α there is exactly one $\xi \in S(\alpha,\beta)$ such that $\xi \in \mathrm{Tr}(\zeta,\alpha) \cap \mathrm{Tr}(\zeta,\beta)$. Now, if $\omega \leq \alpha$ there are infinitely many ζ for which we can form the set $\mathrm{Tr}(\zeta,\alpha) \cap \mathrm{Tr}(\zeta,\beta)$. Hence, one might be lead to believe that $S(\alpha,\beta)$ is an infinite set. The proof of Fact 20 showing otherwise exhibits a general pattern that is encountered often in the method of minimal walks.

Fact 20 $S(\alpha,\beta)$ *is finite.*

Proof. Suppose $S(\alpha,\beta)$ is infinite and let δ be a limit point of $S(\alpha,\beta)$. Set $\delta' = \max(\lambda(\delta,\alpha), \lambda(\delta,\beta))$ and suppose $\xi \in (\delta',\delta)$. Then by Fact 13, $\delta \in \mathrm{Tr}(\xi,\alpha)$ and $\delta \in \mathrm{Tr}(\xi,\beta)$, so

$$\{\xi,\delta\} \subseteq \mathrm{Tr}(\xi,\alpha) \cap \mathrm{Tr}(\xi,\beta).$$

Hence, $\xi \notin S(\alpha, \beta)$ for any $\xi \in (\delta', \delta)$, contradicting the choice of δ. ■

Fact 22, which follows immediately from Fact 21, will be the main tool in proving that the linear order defined in Chapter 3 is a Countryman line. The proof of Fact 21 is the main use of Fact 13 above.

Fact 21 *If* $\xi = \min S(\beta, \gamma) \setminus \alpha$, *then*

$$\mathrm{Tr}(\alpha, \beta) = \mathrm{Tr}(\alpha, \xi) \cup \mathrm{Tr}(\xi, \beta) \qquad \textit{and} \qquad \mathrm{Tr}(\alpha, \gamma) = \mathrm{Tr}(\alpha, \xi) \cup \mathrm{Tr}(\xi, \gamma).$$

Proof. Set $\delta = \max(\mathrm{Tr}(\alpha, \beta) \cap \mathrm{Tr}(\alpha, \gamma))$. By the maximality of δ, $\mathrm{Tr}(\delta, \beta) \cap \mathrm{Tr}(\delta, \gamma) = \{\delta\}$. Further, $\delta \geq \alpha$ so $\delta \in S(\beta, \gamma) \setminus \alpha$. Now suppose there is $\delta' \in S(\beta, \gamma) \setminus \alpha$ such that $\delta' < \delta$. Applying Fact 13 we have $\lambda(\delta, \beta) < \alpha \leq \delta'$ and $\lambda(\delta, \gamma) < \alpha \leq \delta'$. We can then apply Fact 13 again to obtain $\delta \in \mathrm{Tr}(\delta', \beta) \cap \mathrm{Tr}(\delta', \gamma)$ so $\{\delta, \delta'\} \subseteq \mathrm{Tr}(\delta', \beta) \cap \mathrm{Tr}(\delta', \gamma)$, contrary to our choice of δ'. Thus, there is no $\delta' \in S(\beta, \gamma) \setminus \alpha$ less than δ so $\delta = \xi$. ■

Fact 22 *If* $\alpha \leq \beta < \gamma$ *and* $\xi = \min(S(\beta, \gamma) \setminus \alpha)$, *then*

$$\rho_0(\alpha, \beta) = \rho_0(\xi, \beta)\,\hat{}\,\rho_0(\alpha, \xi) \qquad \textit{and} \qquad \rho_0(\alpha, \gamma) = \rho_0(\xi, \gamma)\,\hat{}\,\rho_0(\alpha, \xi).$$

Proof. This follows from facts, 21, 13, and 16. ■

REFERENCES

[1] Roger S. Countryman, Jr. Spaces having a σ-monotone base. Preprint, 1970.

[2] Hrbacek, Karel; Jech, Thomas. Introduction to set theory. Third edition. Monographs and Textbooks in Pure and Applied Mathematics, 220. Marcel Dekker, Inc., New York, 1999. xii+291 pp. ISBN: 0-8247-7915-0 MR1697766 (2000c:03001)

[3] Jech, Thomas. Set theory. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xiv+769 pp. ISBN: 3-540-44085-2 MR1940513 (2004g:03071)

[4] Komjáth, Péter; Totik, Vilmos. Problems and theorems in classical set theory. Problem Books in Mathematics. Springer, New York, 2006. xii+514 pp. ISBN: 978-0387-30293-5; 0-387-30293-X MR2220838

[5] Kunen, Kenneth. Set theory. An introduction to independence proofs. Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980. xvi+313 pp. ISBN: 0-444-85401-0 MR0597342 (82f:03001)

[6] Moore, Justin Tatch. A five element basis for the uncountable linear orders. Ann. of Math. (2) 163 (2006), no. 2, 669–688. MR2199228 (Review)

[7] Moore, Justin Tatch. A solution to the L space problem. J. Amer. Math. Soc. 19 (2006), no. 3, 717–736 (electronic). MR2220104

[8] Justin Tatch Moore. The method of minimal walks. Unpublished note, 2006.

[9] Shelah, Saharon. Decomposing uncountable squares to countably many chains. J. Combinatorial Theory Ser. A 21 (1976), no. 1, 110–114. MR0409196 (53 #12958)

[10] Stevo Todorcevic. Colouring pairs of countable ordinals. Berkeley Seminar Notes, 1985.

[11] Todorvcevic, Stevo. Partitioning pairs of countable ordinals. Acta Math. 159 (1987), no. 3-4, 261–294. MR0908147 (88i:04002)

[12] Stevo Todorcevic. Coherent sequences, in: M. Foreman, M. Magidor, A.Kanamori (eds.), Handbook of Set Theory, Springer-Verlag, forthcoming.

Independence of “There Exists a C-Sequence”

It is easily checked that in ZF one can construct a fundamental sequence as follows. If α is a successor ordinal set C_α to be the set containing the predecessor of α, and if α is a limit ordinal then let $(\alpha_i)_{i=0}^{\infty}$ be an enumeration of α and set

$$\beta_n = \max_{i<n} \alpha_i + n \qquad \textit{and} \qquad C_\alpha = \{\beta_n : n < \infty\}.$$

In ZFC one can then use the axiom of choice to *choose a system* of fundamental sequences, i.e. AC implies there exists a C-sequence. Thus,

$$CON\,(\mathrm{ZF} + \mathrm{AC}) \Rightarrow CON\,(\mathrm{ZF} + \text{“}\textit{There exists a } C\textit{-sequence}\text{”}).$$

On the other hand, given a C-sequence one can define a system of one-to-one functions $\langle e_\beta : \beta \to \omega : \beta < \omega_1 \rangle$ using Fact 14 and the Fundamental Theorem of Arithmetic as follows. Let $(p_i)_{i<\infty}$ be the increasing enumeration of the primes and set

$$e_\beta(\alpha) = p_0^{\rho_0(\alpha,\beta)(0)} \cdot p_1^{\rho_0(\alpha,\beta)(1)} \cdot \ldots \cdot p_{l-1}^{\rho_0(\alpha,\beta)(l-1)},$$

where l is the length of the walk from β to α. By the work of Ulam, such a system of one-to-one functions implies that ω_1 is not measurable. Since under the axiom of determinacy ω_1 *is* measurable we have that AD implies that a C-sequence can not exist. Thus,

$$CON\,(\mathrm{ZF} + \mathrm{AD}) \Rightarrow CON\,(\mathrm{ZF} + \text{“}\textit{There does not exist a } C\textit{-sequence}\text{”}).$$

so “*There exists a C-sequence*” is independent of the Zermelo-Fraenkel axioms.