ULAM STABILITY OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

The basic statements of data dependence in the theory of ordinary differential equations are the following (see for example [2], [5], [6], [8], [17], [20], [23], [24]): monotony w.r.t. data, continuity w.r.t. data, differentiability w.r.t. parameters, Lyapunov stability, asymptotic behavior, structural stability, analiticity of solutions, regularity of solutions, G-convergences. On the other hand, in the theory of functional equations, there are some special kind of data dependence (see [9], [10], [4], [7], [3], [18], [19]). There are some results of this type for some differential equations ([8], [11], [12], [14]-[16]) and some integral equations ([13], [21] and [22]).

With these results in mind we shall present, in this paper, four types of Ulam stability for ordinary differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some examples and some counterexamples are given.
2. General definitions and remarks

Let \((\mathbb{B}, |·|)\) be a (real or complex) Banach space, \(a \in \mathbb{R}, b \in \mathbb{R}, a < b \leq +\infty, \varepsilon\) a positive real number, \(f : [a, b) \times \mathbb{B} \to \mathbb{B}\) a continuous operator and \(\varphi : [a, b) \to \mathbb{R}_+\) be a continuous function. We consider the following differential equation

\[
x'(t) = f(t, x(t)), \quad \forall \ t \in [a, b)
\]  

(2.1)

and the following differential inequations

\[
|y'(t) - f(t, y(t))| \leq \varepsilon, \quad \forall \ t \in [a, b)
\]  

(2.2)

and

\[
|y'(t) - f(t, y(t))| \leq \varphi(t), \quad \forall \ t \in [a, b)
\]  

(2.3)

and

\[
|y'(t) - f(t, y(t))| \leq \varepsilon \varphi(t), \quad t \in [a, b).
\]  

(2.4)

**Definition 2.1.** The equation (2.1) is Ulam-Hyers stable if there exists a real number \(c_f > 0\) such that for each \(\varepsilon > 0\) and for each solution \(y \in C^1([a, b), \mathbb{B})\) of (2.2) there exists a solution \(x \in C^1([a, b), \mathbb{B})\) of (2.1) with

\[
|y(t) - x(t)| \leq c_f \varepsilon, \quad \forall \ t \in [a, b).
\]

**Definition 2.2.** The equation (2.1) is generalized Ulam-Hyers stable if there exists \(\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+), \theta_f(0) = 0\), such that for each solution \(y \in C^1([a, b), \mathbb{B})\) of the inequation (2.2) there exists a solution \(x \in C^1([a, b), \mathbb{B})\) of the equation (2.1) with

\[
|y(t) - x(t)| \leq \theta_f(\varepsilon), \quad \forall \ t \in [a, b).
\]

**Definition 2.3.** The equation (2.1) is Ulam-Hyers-Rassias stable with respect to \(\varphi\) if there exists \(c_{f,\varphi} > 0\) such that for each \(\varepsilon > 0\) and for each solution \(y \in C^1([a, b), \mathbb{B})\) of (2.4) there exists a solution \(x \in C^1([a, b), \mathbb{B})\) of (2.1) with

\[
|y(t) - x(t)| \leq c_{f,\varphi} \varepsilon \varphi(t), \quad \forall \ t \in [a, b).
\]

**Definition 2.4.** The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to \(\varphi\) if there exists \(c_{f,\varphi} > 0\) such that for each solution \(y \in C^1([a, b), \mathbb{B})\) of
(2.3) there exists a solution $x \in C^1([a, b], \mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_{f, \varphi}(t), \, \forall \, t \in [a, b).$$

**Remark 2.1.** A function $y \in C^1([a, b], \mathbb{B})$ is a solution of (2.2) if and only if there exists a function $g \in C([a, b], \mathbb{B})$ (which depend on $y$) such that

(i) $|g(t)| \leq \varepsilon, \, \forall \, t \in [a, b)$

(ii) $y'(t) = f(t, y(t)) + g(t), \, \forall \, t \in [a, b)$.

We have similar remarks for the inequations (2.3) and (2.4).

So, the Ulam stabilities of the differential equations are some special types of data dependence of the solutions of differential equations.

**Remark 2.2.** If $y \in C^1([a, b], \mathbb{B})$ is a solution of the inequation (2.2), then $y$ is a solution of the following integral inequation

$$\left| y(t) - y(a) - \int_a^t f(s, y(s))ds \right| \leq (t - a)\varepsilon, \, \forall \, t \in [a, b).$$

Indeed, by Remark 2.1 we have that

$$y'(t) = f(t, y(t)) + g(t), \, t \in [a, b).$$

This implies that

$$y(t) = y(a) + \int_a^t f(s, y(s))ds + \int_a^t g(s)ds, \, t \in [a, b).$$

From this it follows that

$$\left| y(t) - y(a) - \int_a^t f(s, y(s))ds \right| \leq \left| \int_a^t g(s)ds \right|$$

$$\leq \int_a^t |g(s)|ds \leq \varepsilon(t - a).$$

We have similar remarks for the solutions of the inequations (2.3) and (2.4).

**Remark 2.3.** A solution of the inequation (2.2) is called an $\varepsilon$-solution of the equation (2.1) (see for example [2], p. 94-95; [8], p. 14-18; [24], p. 233).

**Remark 2.4.** The case $b < +\infty$ and the case $b = +\infty$ are two distinct cases as the following example shows.
Example 2.1. We consider in the case $B := \mathbb{R}$ the equation

$$x'(t) = 0, \quad t \in [a, b)$$

and the inequation

$$|y'(t)| \leq \varepsilon, \quad t \in [a, b).$$

Let $y \in C^1[a, b)$ be a solution of (2.6). Then there exists $g \in C[a, b]$ such that:

(i) $|g(t)| \leq \varepsilon, \quad t \in [a, b)$

(ii) $y'(t) = g(t), \quad t \in [a, b)$.

We have, for all $c \in \mathbb{R}$,

$$|y(t) - c| \leq |y(0) - c| + \int_a^t |g(s)| \, ds$$

$$\leq |y(0) - c| + \varepsilon(t-a), \quad t \in [a, b).$$

If we take $c := y(0)$, then

$$|y(t) - y(0)| \leq \varepsilon(t-a), \quad t \in [a, b).$$

If $b < +\infty$, then

$$|y(t) - y(0)| \leq (b-a)\varepsilon.$$

So, the equation (2.5) is Ulam-Hyers stable.

Let $b = +\infty$. The function $y(t) = \varepsilon t$ is a solution of the inequation (2.6) and

$$|y(t) - c| = |\varepsilon t - c| \to +\infty \text{ as } t \to +\infty.$$

So, the equation (2.5) is not Ulam-Hyers stable on the interval $[a, +\infty)$.

Let us consider the inequation

$$|y'(t)| \leq \varphi(t), \quad t \in [a, +\infty).$$

(2.7)

Let $y$ be a solution of (2.7) and $x(t) = y(0), \quad t \in [a, +\infty)$ a solution of (2.5).

We have that

$$|y(t) - x(t)| = |y(t) - y(0)| \leq \int_a^t \varphi(s) \, ds, \quad t \in [a, +\infty).$$
If there exists $c_\varphi \in \mathbb{R}_+$ such that
\[ \int_a^t \varphi(s)ds \leq c_\varphi \varphi(t), \quad t \in [a, +\infty) \]
then the equation (2.5) is generalized Ulam-Hyers-Rassias stable on $[a, +\infty)$ with respect to $\varphi$.

**Remark 2.5.** For the Ulam-Hyers-Rassias stability of the differential equation
\[ y' - \lambda y = 0 \]
in a Banach space see [16]. For other results see [1], [11], [12], [14] and [15].

3. **Generalized Ulam-Hyers-Rassias stability**

Let us consider the equation (2.1) and the inequation (2.3) in the case $b = \infty$.

We suppose that:

(i) $f \in C([a, +\infty) \times \mathbb{B}, \mathbb{B})$ and $\varphi \in C([a, +\infty), \mathbb{R}_+)$ be an increasing function;

(ii) there exists $l_f \in L^1[a, +\infty)$ such that
\[ |f(t, u) - f(t, v)| \leq l_f(t)|u - v|, \quad \forall \, u, v \in \mathbb{B},\, \forall \, t \in [a, +\infty); \]

(iii) there exists $\lambda_\varphi > 0$ such that
\[ \int_a^t \varphi(s)ds \leq \lambda_\varphi \varphi(t), \quad \forall \, t \in [0, a + \infty). \]

We have

**Theorem 3.1.** In the conditions (i), (ii), (iii) the equation (2.1) $(b = +\infty)$ is generalized Ulam-Hyers-Rassias stable.

**Proof.** Let $y \in C^1([a, +\infty), \mathbb{B})$ be a solution of the inequation (2.3) $(b = +\infty)$.

Denote by $x$ the unique solution of the Cauchy problem
\[ x'(t) = f(t, x(t)), \quad t \in [a, +\infty) \]
\[ x(a) = y(a). \]

We have that
\[ x(t) = y(a) + \int_a^t f(s, x(s))ds, \quad t \in [a, +\infty) \]
and
\[
|y(t) - y(a) - \int_a^t f(s, y(s))ds| \leq \int_a^t \varphi(s)ds \leq \lambda \varphi(t), \quad t \in [a, +\infty).
\]

From these relation it follows
\[
|y(t) - x(t)| \leq |y(t) - y(a) - \int_a^t f(s, y(s))ds|
+ \int_a^t |f(s, y(s)) - f(s, x(s))|ds
\leq \lambda \varphi(t) + \int_a^t l_f(s)|y(s) - x(s)|ds.
\]

By a Gronwall lemma (see [22], [23], [5]) we have that
\[
|y(t) - x(t)| \leq \lambda \varphi(t)e^{\int_a^t l_f(s)ds}
\leq \left[\lambda e^{\int_a^{+\infty} l_f(s)ds}\right] \varphi(t) = c_{f, \varphi}(t), \quad t \in [a, +\infty),
\]
i.e. the equation (2.1) \((b = +\infty)\) is generalized Ulam-Hyers-Rassias stable.

**Remark 3.1.** For the case \(B := \mathbb{C}\) see [13], [15].

**Remark 3.2.** If we take \(B\) a Banach space of sequences in \(K = \mathbb{R} \lor \mathbb{C}\) \((C(K), C_0(K), L^p(K), \ldots)\) then we have some results for an infinite system of differential equations.

**Remark 3.3.** For the Ulam stability of some integral equations see [13] and [21].

**Remark 3.4.** If we have a differential equation of \(n\)-order in a Banach space \(B\) then we reduce it to a differential equation of first order in the Banach space \(B^n\). If the order \(n\) is even we can use the Green function technique as the following example shows.

For simplicity we shall consider the following second order differential equation
\[
-x''(t) = f(t, x(t)), \quad t \in [a, b]
\]
where \(a < b < +\infty\) and \(f \in C([a, b] \times \mathbb{R})\).
Let us denote by $G$ the Green function of the following boundary value problem (see [6], [17], [20], [23])

$$-y'' = h(t)$$
$$y(a) = 0, \; y(b) = 0$$

The function $G : [a, b] \times [a, b] \to \mathbb{R}$ is defined by

$$G(t, s) := \begin{cases} 
\frac{(s - a)(b - t)}{b - a} & \text{if } s \leq t, \\
\frac{(t - a)(b - s)}{b - a} & \text{if } s \geq t.
\end{cases}$$

We have

**Theorem 3.2.** We suppose that:

(i) $f \in C([a, b] \times \mathbb{R})$;

(ii) there exists $L_f > 0$ such that

$$|f(t, u) - f(t, v)| \leq L_f |u - v|, \; \forall \; t \in [a, b], \; \forall \; u, v \in \mathbb{R};$$

(iii) $L_f \frac{(b - a)^2}{4} < 1$.

Then the equation (3.1) is Ulam-Hyers stable.

**Proof.** Let $y \in C^2[a, b]$ be a solution of the inequation

$$| - y'' - f(t, y(t))| \leq \varepsilon, \; \forall \; t \in [a, b].$$

First of all we remark that $y$ is a solution of the following inequation

$$\left| y(t) - \frac{t - a}{b - a} y(b) - \frac{b - t}{b - a} y(a) - \int_a^b G(t, s) f(s, y(s))\,ds \right|$$

$$\leq \varepsilon \left[ \frac{t^2}{2} - \frac{a + b}{2} t + \frac{ab}{2} \right], \; \forall \; t \in [a, b].$$

Now we take $x$ the solution of the following boundary value problem ([8], p. 186; [20], p. 99)

$$-x''(t) = f(t, x(t)), \; t \in [a, b];$$

$$x(a) = y(a), \; x(b) = y(b).$$
It is clear that
\[
x(t) = \frac{t - a}{b - a} y(b) + \frac{b - t}{b - a} y(a) + \int_a^b G(t, s)f(s, x(s))ds, \quad t \in [a, b]
\]
and we estimate \(|y(t) - x(t)|\) in a similar way as in the proof of Theorem 3.1.

References


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