Dynamic Analysis of a Reward Process
Defined on a Cyclic Renewal Process
with Application to Maintenance Problems

Kazuki Takahashi
200520846
(Master’s Program in Social Systems and Engineering)
Advised by Professor Ushio Sumita

Submitted to the Graduate School of
Systems and Information Engineering
in Fulfillment of the Requirements
for the Degree of Master of Engineering
at the
University of Tsukuba

March 2007
Abstract

The purpose of this thesis is to introduce a cyclic renewal process as an extension of an alternating renewal process, where each of underlying i.i.d. nonnegative random increments is composed of multiple stages, i.e. \( Y_n = \sum_{j=1}^{J} X_j, \ n \geq 1 \), where \( Y_n \) denotes the lifetime of the \( n \)-th cycle and \( Y_n \)'s are i.i.d. with respect to \( n \). Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, we also introduce a reward process with jumps defined on the cyclic renewal process. When the system is running in stage \( j \), the profit grows linearly at the rate of \( \rho(j) \). Upon a minor failure, the subsequent minimal repair in stage \( (j + 1) \) incurs the linear cost at the rate of \( \rho(j + 1) \). In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval \((0, T]\). A multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management.
List of Figures

2.1 Typical Sample Path of \([N(t), J(t), X(t)]\) ........................................ 5
2.2 Typical Sample Path of \([X(t), Z(t), J(t)]\) with Jumps .................. 5

5.1 Typical Sample Path of \([N(t), J(t), X(t)]\) for Preventive Maintenance Model . . . . . . 21
5.2 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (2, 10)\) .................. 24
5.3 \(E[Z(T)|J(0) = 1]/T(T= 10, 20, 30, 40, 50)\) for \((\mu, \rho_{\text{DOWN}}) = (2, 10)\) .............. 24
5.4 \(E[Z(T)|J(0) = 1]/T\) \((K = 5, 10, 15, 20)\) for \((\mu, \rho_{\text{DOWN}}) = (2, 10)\) .......... 24
5.5 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (1, 5)\) .................. 26
5.6 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (5, 5)\) .................. 26
5.7 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (10, 5)\) .................. 26
5.8 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (20, 5)\) .................. 26
5.9 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (1, 10)\) .................. 27
5.10 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (5, 10)\) .................. 27
5.11 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (10, 10)\) .................. 27
5.12 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (20, 10)\) .................. 27
5.13 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (1, 15)\) .................. 28
5.14 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (5, 15)\) .................. 28
5.15 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (10, 15)\) .................. 28
5.16 \(E[Z(T)|J(0) = 1]/T\) for \((\mu, \rho_{\text{DOWN}}) = (20, 15)\) .................. 28
5.17 \(E[Z(T)|J(0) = 1]/T\) \((T = 10, 20, 30, 40, 50)\) for \((\mu, \rho_{\text{DOWN}}) = (1, 5)\) .............. 29
5.18 \(E[Z(T)|J(0) = 1]/T\) \((T = 10, 20, 30, 40, 50)\) for \((\mu, \rho_{\text{DOWN}}) = (5, 5)\) .............. 29
5.19 \(E[Z(T)|J(0) = 1]/T\) \((T = 10, 20, 30, 40, 50)\) for \((\mu, \rho_{\text{DOWN}}) = (10, 5)\) .............. 29
5.20 \(E[Z(T)|J(0) = 1]/T\) \((T = 10, 20, 30, 40, 50)\) for \((\mu, \rho_{\text{DOWN}}) = (20, 5)\) .............. 29
5.21 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (1,10)$ . . . . . . . . . . 30

5.22 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (5,10)$ . . . . . . . . . . 30

5.23 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (10,10)$ . . . . . . . . . . 30

5.24 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (20,10)$ . . . . . . . . . . 30

5.25 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (1,15)$ . . . . . . . . . . 31

5.26 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (5,15)$ . . . . . . . . . . 31

5.27 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (10,15)$ . . . . . . . . . . 31

5.28 $E[Z(T)|J(0) = 1]/T \ (T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{\text{DOWN}}) = (20,15)$ . . . . . . . . . . 31

5.29 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (1,5)$ . . . . . . . . . . . 32

5.30 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (5,5)$ . . . . . . . . . . . 32

5.31 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (10,5)$ . . . . . . . . . . . 32

5.32 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (20,5)$ . . . . . . . . . . . 32

5.33 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (1,10)$ . . . . . . . . . . . 33

5.34 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (5,10)$ . . . . . . . . . . . 33

5.35 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (10,10)$ . . . . . . . . . . . 33

5.36 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (20,10)$ . . . . . . . . . . . 33

5.37 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (1,15)$ . . . . . . . . . . . 34

5.38 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (5,15)$ . . . . . . . . . . . 34

5.39 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (10,15)$ . . . . . . . . . . . 34

5.40 $E[Z(T)|J(0) = 1]/T \ (K = 5, 10, 15, 20)$ for $(\mu, \rho_{\text{DOWN}}) = (20,15)$ . . . . . . . . . . . 34
## List of Tables

5.1 Parameter Values for $\lambda$, $\rho_{\text{UP}}$, $D$, $i$, $K(0)$, $K(1)$ and $c$ ........................................ 23

5.2 $K^*_T$ for $(\mu, \rho_{\text{DOWN}})=(2,10)$ .................................................. 25

5.3 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(2,10)$ ........................................ 25

5.4 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(1,5)$ ......................................... 35

5.5 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(5,5)$ ......................................... 35

5.6 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(10,5)$ ........................................ 35

5.7 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(20,5)$ ......................................... 35

5.8 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(1,10)$ ......................................... 36

5.9 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(5,10)$ ......................................... 36

5.10 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(10,10)$ ....................................... 36

5.11 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(20,10)$ ....................................... 36

5.12 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(1,15)$ ......................................... 37

5.13 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(5,15)$ ......................................... 37

5.14 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(10,15)$ ....................................... 37

5.15 $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(20,15)$ ....................................... 37
Chapter 1

Introduction

Renewal theory is the branch of probability theory concerning a variety of problems related to the partial sums of a sequence of independent and identically distributed (i.i.d.) nonnegative random variables. More specifically, let \((Y_n)_{n=1}^{\infty}\) be a sequence of i.i.d. nonnegative random variables and define \(S_n = \sum_{j=1}^{n} Y_j\). The renewal process \(\{N(t) : t \geq 0\}\) associated with \((Y_n)_{n=1}^{\infty}\) is a counting process defined by \(N(t) = n\) if and only if \(S_n \leq t < S_{n+1}\). Of interest are the renewal function \(H(t) = E[N(t)]\), the renewal density \(h(t) = \frac{d}{dt}H(t)\) if it exists, and other related probabilistic entities. As the name “renewal theory” indicates, the study stemmed from a class of applications involving successive replacements of items subject to failure. Here, \(Y_n\) denotes the lifetime of the \(n\)-th item and \(N(t)\) is the number of replacements that took place by time \(t\).

The renewal theory has been extended in many ways. A delayed renewal process, for example, has the distribution of \(Y_1\) different from that of \(Y_n (n > 1)\), and an alternating renewal process deals with a situation where \(Y_n\) consists of two stages: the system uptime and the system repair time, see e.g. Cox [4]. A Markov renewal process considers a case where distributions of interfailure times are governed by a Markov chain \(\{J(n) : n = 0, 1, 2, \cdots\}\) in discrete time, i.e. if \(J(n-1) = i\) and \(J(n) = j\), then the distribution of \(Y_n\) is given by \(A_{ij}(x)\). The reader is referred to Keilson [12], Keilson and Rao [13, 14], and an excellent survey paper by Çinlar [2] for the study of Markov renewal processes. Keener [11] develops a general renewal theory where i.i.d. increments have support on full continuum. In Kijima and Sumita [15], the renewal theory is extended in that the distribution of the \(X_{n+1}\) depends on the partial sum \(S_n\) up to the \(n\)-th increment.

The purpose of this thesis is to introduce a cyclic renewal process as an extension of an alternating renewal process, where each of underlying i.i.d. nonnegative random increments is composed
of multiple stages, i.e. \( Y_n = \sum_{j=1}^{J} X_j, \ n \geq 1 \), where \( Y_n \) denotes the lifetime of the \( n \)-th cycle and \( Y_n \)'s are i.i.d. with respect to \( n \). Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, we also introduce a reward process with jumps defined on the cyclic renewal process. When the system is running in stage \( j \), the profit grows linearly at the rate of \( \rho(j) \). Upon a minor failure, the subsequent minimal repair in stage \((j + 1)\) incurs the linear cost at the rate of \( \rho(j + 1) \). In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval \((0, T]\). A multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management.

When the renewal aspect is suppressed, the above model is reduced to a semi-Markov process. The study of semi-Markov processes can be dated back to the middle of 1950s, originated by works of Lévy [16], Smith [24] and Takács [28]. Subsequently the scope of the study has been expanded through a series of papers by Pyke [21, 22], Pyke and Schaufele [23], and Moore and Pyke [20]. Since the early 1960s, the field attracted many researchers resulting in a collection of quite extensive results. The reader is referred to two excellent survey papers by Çinlar [1, 2] and references therein for extensive analysis of semi-Markov and related processes. Reward processes defined on semi-Markov processes also have been studied extensively, including the original works by Jewell [8, 9, 10] followed by Howard [5], Mclean and Neuts [19], Çinlar [3], Hunter [6], Sumita and Masuda [25], Masuda and Sumita [18] and Igaki, Sumita and Kowada [7] to name a few. However, the joint distribution of the cyclic renewal process, the underlying semi-Markov process and the reward process has never been studied in the literature to the best knowledge of the author.

The structure of this thesis is as follows. A cyclic renewal process \( \{N(t) : t \geq 0\} \) is formally introduced in Chapter 2 based on a cyclic semi-Markov process \( \{J(t) : t \geq 0\} \) describing multiple
stages to constitute system lifetimes. The associated age process \( \{X(t) : t \geq 0\} \) and the reward process \( \{Z(t) : t \geq 0\} \) are also introduced so that the multivariate process \([N(t), J(t), X(t), Z(t)]\) becomes Markov. Chapter 3 is devoted to analysis of the multivariate process \([N(t), J(t), X(t), Z(t)]\) by examining its probabilistic flow in its state space, yielding various transform results. In Chapter 4, the asymptotic expansion of \( E[Z(t)] \) as \( t \to \infty \) is derived, which is used to numerically explore optimal preventive maintenance policies for production management in Chapter 5. Finally, concluding remarks are given in Chapter 6, addressing the agenda for future research. Some mathematical details are deferred to Appendix for enhancing the readability of the thesis.
Chapter 2

Model Description

We consider a cyclic renewal process \( \{N(t); t \geq 0\} \) defined on \( \mathcal{N} = \{0, 1, 2, \ldots\} \) where the underlying lifetime consists of \( J \) stages and \( N(t) \) denotes the number of failures by time \( t \). More specifically, let \( \mathcal{J} = \{1, 2, \ldots, J\} \) be the set of the stages and let the dwell time in stage \( j \in \mathcal{J} \) be a nonnegative random variable denoted by \( X_j \). Throughout the paper, we assume that \( X_j \ (j \in \mathcal{J}) \) are independent of the failure count and also mutually independent. For each \( j \in \mathcal{J} \), it is assumed that \( X_j \) is absolutely continuous characterized by

\[
\begin{align*}
\bar{A}_j(x) &= P[X_j > x]; \\
 a_j(x) &= -\frac{d}{dx} \bar{A}_j(x); \\
 \eta_j(x) &= \frac{a_j(x)}{A_j(x)}; \\
 \alpha_j(v) &= \int_0^\infty e^{-vx} a_j(x) dx
\end{align*}
\]

(2.1)

where \( \bar{A}_j(x) \), \( a_j(x) \), \( \eta_j(x) \) and \( \alpha_j(v) \) are the survival function, the probability density function, the hazard function and the Laplace transform of \( a_j(x) \) respectively. Here \( v \) takes values from the complex plane satisfying \( \text{Re}(v) > 0 \) so that \( \alpha_j(v) \) is well defined. A lifetime associated with the cyclic renewal process is given by

\[
Y = \sum_{j=1}^{J} X_j .
\]

(2.2)

Let \( Y_k \) be the lifetime of the \( k \)-th renewal cycle where \( Y_k \)'s are i.i.d. with common structure of (2.2). For \( k = 0 \), one then sees that

\[
P[N(t) = 0] = P[0 \leq t < Y_1]
\]

(2.3)

and for \( k \geq 1 \),

\[
P[N(t) = k] = P[\sum_{m=1}^{k} Y_m \leq t < \sum_{m=1}^{k+1} Y_m] .
\]

(2.4)
Let \( \{J(t) ; t \geq 0\} \) be a stochastic process describing the stage at time \( t \). Since the bivariate process \([N(t), J(t)]\) is not Markov, we introduce an additional process \( \{X(t) ; t \geq 0\} \) on \( \mathbb{R}^+ \) denoting the elapsed time since the last entry into the current stage at time \( t \), where \( \mathbb{R}^+ \) is the set of nonnegative real numbers. This process is called the age process. The trivariate process \([N(t), J(t), X(t)]\) then becomes Markov. A typical sample path of \([N(t), J(t), X(t)]\) is depicted in Figure 2.1 where \( N(0) = 0, J(0) = i \) and \( X(0) = 0 \).

From an application point of view, of particular interest is a reward process \( \{Z(t) ; t \geq 0\} \) with jumps defined on \([N(t), J(t), X(t)]\). We assume that the reward increases or decreases linearly at the rate of \( \rho(j) \) when \( J(t) \) is in state \( j \in \mathcal{J} \). Furthermore, the reward process jumps in the random amount of \( D_j \) when \( J(t) \) moves from \( j \) to \( j + 1 \) for \( j \in \mathcal{J} \setminus \{J\} \), and \( D_J \) for a transition from \( J \) to 1. Accordingly, \( Z(t) \) takes a value from \( \mathcal{R} \) where \( \mathcal{R} \) is the set of real numbers. As for \( X_j (j \in \mathcal{J}) \), it is assumed that \( D_j (j \in \mathcal{J}) \) are independent of the failure count, mutually independent, and absolutely continuous having

\[
\tilde{B}_j(z) = P[D_j > z] ; \quad b_j(z) = -\frac{d}{dz}\tilde{B}_j(z) ; \quad \beta_j(w) = \int_{-\infty}^{\infty} e^{-wz}b_j(z)dz ,
\]

where \( w \) takes values on the unit circle on the complex plane so that \( \beta_j(w) \) is well defined.

In order to describe the reward process \( \{Z(t) ; t \geq 0\} \) more formally, let \( \{M_j(t) ; t \geq 0\} \) be the stochastic process counting the number of transitions of \( J(t) \) from \( j \) to \( j + 1 \) by time \( t \) for \( j \in \mathcal{J} \setminus \{J\} \). The stochastic process \( \{M_J(t) ; t \geq 0\} \) is defined similarly for transitions of \( J(t) \) from \( J \) to 1. One then has

\[
Z(t) = \int_0^t \rho(J(\tau))d\tau + \sum_{j=1}^J \sum_{m=1}^{M_j(t)} D_{j,m} ,
\]
where $D_{j:m}$ denotes the jump amount associated with the $m$-th transition from $j$ to $j + 1$ for $j \in \mathcal{J} \setminus \{J\}$, and from $J$ to 1 for $j = J$. Following the mathematical convention, we define $\sum_{m=a}^{b} c_m = 0$ whenever $a > b$. It should be noted that, by the assumptions discussed above, $D_{j:m}(m = 1, \cdots, M_j(t))$ are i.i.d. with respect to $m$. When $J(t)$ is a general semi-Markov process, the expectation of the semi-Markov reward process with jumps is given in Howard [5]. The transform results of $[J(t), Z(t)]$ are derived in McLean and Neuts [19]. The trivariate Markov process $[J(t), X(t), Z(t)]$ is also studied in detail in Sumita and Masuda [25, 26] and Masuda [17]. The thrust of this paper is to analyze the multivariate process $[N(t), J(t), X(t), Z(t)]$, which is new. The results are then used to numerically explore optimal preventive maintenance policies for production management.
Chapter 3

Dynamic Analysis of Multivariate Process \([N(t), J(t), X(t), Z(t)]\)

In this section, we analyze the multivariate process \([N(t), J(t), X(t), Z(t)]\) by describing its probabilistic flow in the state space \(\mathcal{N} \times J \times \mathbb{R}^+ \times \mathbb{R}\). For this purpose, let \(F_{k;ij}(x, z, t)\) be the joint distribution function of \([N(t), J(t), X(t), Z(t)]\) given \(J(0) = i, X(0) = Z(0) = 0\). More formally, we define

\[
F_{k;ij}(x, z, t) = P[N(t) = k, J(t) = j, X(t) \leq x, Z(t) \leq z| J(0) = i, X(0) = Z(0) = 0].
\]

The corresponding joint probability density function is given by

\[
f_{k;ij}(x, z, t) = \frac{\partial^2}{\partial x \partial z} F_{k;ij}(x, z, t).
\]

For the process \([N(t), J(t), X(t), Z(t)]\) to be at \((0, j, x, z)\) at time \(t > 0\) given \(J(0) = i\), either no transition of \(J(t)\) has occurred in the time interval \([0, t]\) with \(j = i\), or at least one transition of \(J(t)\) from \(J(0) = i\) occurred in \([0, t]\), the process entered the state \((0, j, 0^+, z - \rho(j)x)\) at time \(t - x\), and no transition of \(J(t)\) has occurred since then. Accordingly, one has

\[
f_{0;ij}(x, z, t) = \delta_{\{j=i\}} \delta(z - \rho(j)t)\delta(t - x)\bar{A}_j(x)
+ \delta_{\{j>i\}} f_{0;ij}(0^+, z - \rho(j)x, t - x)\bar{A}_j(x), \quad x > 0, \ j = i, \cdots, \ J.
\]

Here, \(\delta_{\{P\}} = 1\) if the statement \(P\) holds true, \(\delta_{\{P\}} = 0\) otherwise, and \(\delta(t)\) is the delta function defined as the unit function associated with the convolution operation, i.e., \(f(x) = \int f(y)\delta(x - y)dy\) for any integrable function \(f\). Similarly, for \(k > 0\), to be at \((k, j, x, z)\) at time \(t > 0\), the process
should have entered the state \((k, j, 0+, z - \rho(j)x)\) at time \(t-x\) and no transition of \(J(t)\) has occurred since then. This then yields

\[
(3.4) \quad f_{k:ij}(x, z, t) = f_{k:ij}(0+, z - \rho(j)x, t-x)A_j(x), \quad x > 0, \quad k \geq 1.
\]

In order to determine the boundary conditions \(f_{k:ij}(0+, z, t)\) associated with the age process \(X(t)\), we first consider the case that \(k = 0, z = 0+\) and \(t = 0+\). One then sees that \(f_{0:ij}(0+, 0+, 0+) = \delta_{[j=i]} \delta(z) \delta(t)\). For \(t > 0\) and \(j \geq i\), the process \([N(t), J(t), X(t), Z(t)]\) just enters the state \((0, j, 0+, z)\) at time \(t\) only if the dwell time of \(J(t)\) in state \(j-1\) expires at time \(t\) with the reward at \(z - D_{j-1}\) followed by the instantaneous jump of size \(D_{j-1}\) so that \(Z(t) = z\). Combining the two cases, one observes that

\[
(3.5) \quad f_{0:ij}(0+, z, t) = \delta_{[j=i]} \delta(z) \delta(t) + \sum_{j > i} \int_0^\infty f_{0:i-1,j}(x, z - z', t) \eta_{j-1}(x)b_{j-1}(z') dz' dx, \quad j = i, \ldots, J.
\]

For \(k \geq 1\), similar arguments lead to

\[
(3.6) \quad f_{k:ij}(0+, z, t) = \begin{cases} \int_0^\infty \int_{-\infty}^\infty f_{k-1:i-1}(x, z - z', t) \eta_{j-1}(x)b_{j-1}(z') dz' dx, & j = 1 \\ \int_0^\infty \int_{-\infty}^\infty f_{k:i-1}(x, z - z', t) \eta_{j-1}(x)b_{j-1}(z') dz' dx, & 2 \leq j \leq J \end{cases}.
\]

We are now in a position to prove the key theorems of this thesis. For notational convenience, the following matrix Laplace transforms are introduced.

\[
(3.7) \quad \hat{\phi}_k(x, z, s) \overset{\text{def}}{=} [\hat{\phi}_{k:ij}(x, z, s)]; \quad \hat{\phi}_{k:ij}(x, z, s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f_{k:ij}(x, z, t) dt,
\]

\[
(3.8) \quad \hat{\phi}_k(x, w, s) \overset{\text{def}}{=} [\hat{\phi}_{k:ij}(x, w, s)]; \quad \hat{\phi}_{k:ij}(x, w, s) \overset{\text{def}}{=} \int_{-\infty}^\infty e^{-wz} \hat{\phi}_{k:ij}(x, z, s) dz,
\]

\[
(3.9) \quad \hat{\phi}_k(v, w, s) \overset{\text{def}}{=} [\hat{\phi}_{k:ij}(v, w, s)]; \quad \hat{\phi}_{k:ij}(v, w, s) \overset{\text{def}}{=} \int_0^\infty e^{-vx} \hat{\phi}_{k:ij}(x, w, s) dx,
\]

\[
(3.10) \quad \hat{\xi}_k(0+, z, s) \overset{\text{def}}{=} [\hat{\xi}_{k:ij}(0+, z, s)]; \quad \hat{\xi}_{k:ij}(0+, z, s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f_{k:ij}(0+, z, t) dt,
\]

\[
(3.11) \quad \hat{\xi}_k(0+, w, s) \overset{\text{def}}{=} [\hat{\xi}_{k:ij}(0+, w, s)]; \quad \hat{\xi}_{k:ij}(0+, w, s) \overset{\text{def}}{=} \int_0^\infty e^{-wz} \hat{\xi}_{k:ij}(0+, z, s) dz,
\]

\[
(3.12) \quad \beta_D(w,s) \overset{\text{def}}{=} \left[ \delta_{[i=j]} \frac{1 - \alpha_j(s + \rho(j)w)}{s + \rho(j)w} \right],
\]

\[
(3.13) \quad \zeta_{ij}(w, s) \overset{\text{def}}{=} \prod_{n=i}^j \alpha_n(s + \rho(n)w) \beta_n(w) ; \quad \zeta_{ij}(w, s) = 1 \quad \text{for} \quad i > j,
\]
Similarly, substitution of (3.4) into (3.6) yields

\[ \hat{\Phi}(v, w, s) = \begin{bmatrix} \zeta_{11}(w, s) & 0 & \cdots & 0 \\ 0 & \zeta_{22}(w, s) & \cdots & 0 \\ \vdots & \cdots & \cdots & \zeta_{J-1,J-1}(w, s) \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \]

(3.14)

\[ \hat{\Phi}_D(v, w, s) \overset{\text{def}}{=} \begin{bmatrix} \zeta_{1J}(w, s) \\ \zeta_{2J}(w, s) \\ \vdots \\ \zeta_{J,J}(w, s) \end{bmatrix}, \]

(3.15)

Proof

Theorem 3.1

For the multivariate process \([N(t), J(t), X(t), Z(t)]\) with \(N(0) = X(0) = Z(0) = 0\) and \(J(0) = i\), let \(\hat{\Phi}(v, w, s)\) be defined as in (3.9). Then

\[ \hat{\Phi}(v, w, s) = \begin{cases} \left[ I - \hat{\phi}^*(w, s)\right]^{-1} \hat{\phi}_D(w, v + s), & k = 0 \\ \{\zeta_{1J}(w, s)\} \hat{\phi}^*(w, s) \hat{\phi}_D^{-1}(w, v + s), & k \geq 1 \end{cases} \]

(3.18)

Substituting (3.3) into (3.5), it can be seen that

\[ f_{0:i,j}(0+, z, t) \]

(3.19)

\[ = \delta_{(j=i)} \delta(z) \delta(t) + \delta_{(j>i)} \int_0^\infty \int_0^\infty \{\delta_{(j-1=i)} \delta(z - z' - \rho(j-1)t) \delta(t - x) \hat{A}_{j-1}(x) \\
+ \delta_{(j>j-1)} f_{0:i,j-1}(0+, z - z' - \rho(j-1)x, t - x) \hat{A}_{j-1}(x)\} \eta_{j-1}(x) b_{j-1}(z') dz' dx. \]

Similarly, substitution of (3.4) into (3.6) yields

\[ f_{k:i,j}(0+, z, t) \]

(3.20)

\[ = \begin{cases} \int_0^\infty \int_{-\infty}^\infty f_{k-1:i,j}(0+, z - z' - \rho(J)x, t - x) a_j(x) b_j(z') dz' dx, & j = 1 \\
\int_0^\infty \int_{-\infty}^\infty f_{k:i,j+1}(0+, z - z' - \rho(j-1)x, t - x) a_{j-1}(x) b_{j-1}(z') dz' dx, & 2 \leq j \leq J. \end{cases} \]

By taking Laplace transforms with respect to \(t\) in (3.19) and (3.20), it then follows that

\[ \hat{\xi}_{0:i,j}(0+, z, s) \]

(3.21)

\[ = \delta_{(j=i)} \delta(z) + e^{-sx} \delta_{(j>i)} \int_0^\infty \int_{-\infty}^\infty \{\delta_{(j-1=i)} \delta(z - z' - \rho(j-1)x) \hat{A}_{j-1}(x) \\
+ \delta_{(j>1)} e^{-sx} \hat{\xi}_{0:i,j-1}(0+, z - z' - \rho(j-1)x, s) \hat{A}_{j-1}(x)\} \eta_{j-1}(x) b_{j-1}(z') dz' dx, \]
and

\[ (3.22) \quad \tilde{\xi}_{k:ij}(0+, z, s) = \begin{cases} \int_0^{\infty} \int_0^{\infty} e^{-sx} \hat{\xi}_{k-1:i, j}(0+, z - z', -\rho(J)x, s) a_J(x) b_J(z') dz' dx, & j = 1 \\ \int_0^{\infty} \int_0^{\infty} e^{-sx} \hat{\xi}_{k,i, j-1}(0+, z - z', -\rho(j-1)x, s) a_{j-1}(x) b_{j-1}(z') dz' dx, & 2 \leq j \leq J \end{cases} \]

If we again take Laplace transforms with respect to \( z \) in (3.21) and (3.22), one has

\[ (3.23) \quad \hat{\xi}_{0:ij}(0+, w, s) = \delta_{(j=1)} + \delta_{(j=1)} \alpha_{j-1}(s + \rho(j-1)w) \beta_{j-1}(w) + \delta_{(j=1)} \hat{\xi}_{0:1, j-1}(0+, w, s) \alpha(s + \rho(j-1)w) \beta_{j-1}(w), \]

and

\[ (3.24) \quad \hat{\xi}_{k:ij}(0+, w, s) = \begin{cases} \hat{\xi}_{k-1:i, j}(0+, w, s) \alpha_{j}(s + \rho(J)w) \beta_{j}(w), & j = 1 \\ \hat{\xi}_{k,i, j-1}(0+, w, s) \alpha_{j-1}(s + \rho(j-1)w) \beta_{j-1}(w), & 2 \leq j \leq J \end{cases} \]

Equations in (3.23) and (3.24) can be rewritten in matrix form using \( \hat{\xi}_{\underline{k}}(0+, w, s) \) defined in (3.11) in the following manner. From (3.23), we first note that

\[
\hat{\xi}_{\underline{k}}(0+, w, s) = \begin{bmatrix} 1 & \hat{\xi}_{0,1:2}(0+, w, s) & \cdots & \hat{\xi}_{0,1:j}(0+, w, s) \\
 & 1 & \cdots & \hat{\xi}_{0,2:j}(0+, w, s) \\
 & & \ddots & \vdots \\
 & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & \hat{\xi}_{0,1:2}(0+, w, s) & \cdots & \hat{\xi}_{0,1:j}(0+, w, s) \\
 & 1 & \cdots & \hat{\xi}_{0,2:j}(0+, w, s) \\
 & & \ddots & \vdots \\
 & & & 1 \end{bmatrix}
\]

The last matrix in the above expression can be written from (3.14) as \( I + \alpha^*(w, s) \alpha^*(w, s) \), so that

\[ (3.25) \quad \hat{\xi}_{\underline{k}}(0+, w, s) = [I - \alpha^*(w, s)]^{-1} \]

For \( k = 1 \), one sees that

\[
\hat{\xi}_{\underline{1}}(0+, w, s) = \begin{bmatrix} \hat{\xi}_{1,1:1}(0+, w, s) & \cdots & \hat{\xi}_{1,1:1}(0+, w, s) \\
& \ddots & \vdots \\
& \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \hat{\xi}_{0,1:2}(0+, w, s) \alpha_{j}(s + \rho(J)w) \beta_{j}(w) & \cdots & \hat{\xi}_{0,1:j}(0+, w, s) \alpha_{j-1}(s + \rho(j-1)w) \beta_{j-1}(w) \\
& \ddots & \vdots \\
& \ddots & \ddots \end{bmatrix}
\]

10
By employing (3.23) repeatedly in the above expression, it follows that

\[
\hat{\xi}_{11}(0+, w, s) = \begin{bmatrix}
\zeta_1,1(w, s) & \zeta_1,2(w, s) & \cdots & \zeta_1,1(w, s) \\
\zeta_2,1(w, s) & \zeta_2,2(w, s) & \cdots & \zeta_2,1(w, s) \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_{J,1}(w, s) & \zeta_{J,2}(w, s) & \cdots & \zeta_{J,1}(w, s)
\end{bmatrix}.
\]

From (3.15) and (3.16), this then leads to

\[
\hat{\xi}_{11}(0+, w, s) = \frac{\alpha_{D}^*}{\alpha_{D}}(w, s)\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\zeta_{11}(w, s) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta_{1,1}(w, s)
\end{bmatrix}.
\]

It should be noted that

\[
\alpha_{D}^*(s, w)\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\zeta_{11}(w, s) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta_{1,1}(w, s)
\end{bmatrix} = \zeta_{1,1}(w, s)I,
\]

so that one has

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\zeta_{11}(w, s) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta_{1,1}(w, s)
\end{bmatrix} = \zeta_{1,1}(w, s)\alpha_{D}^{*-1}(w, s).
\]

Substituting (3.27) into (3.26), one concludes that

\[
\hat{\xi}_{11}(0+, w, s) = \zeta_{1,1}(w, s)\alpha_{D}^*(w, s)\frac{\alpha_{D}}{\alpha_{D}}\alpha_{D}^{*-1}(w, s).
\]

For \( k \geq 2 \), it can be seen from (3.24) that

\[
\hat{\xi}_{11}(0+, w, s) = \begin{bmatrix}
\hat{\xi}_{k,11}(0+, w, s) & \cdots & \hat{\xi}_{k,1J}(0+, w, s) \\
\hat{\xi}_{k,21}(0+, w, s) & \cdots & \hat{\xi}_{k,2J}(0+, w, s) \\
\vdots & \vdots & \ddots \\
\hat{\xi}_{k,J1}(0+, w, s) & \cdots & \hat{\xi}_{k,JJ}(0+, w, s)
\end{bmatrix} = \begin{bmatrix}
\hat{\xi}_{k-1,11}(0+, w, s)\alpha_{J}(s + \rho(J)w)\beta_{J}(w) & \cdots & \hat{\xi}_{k-1,1J}(0+, w, s)\alpha_{J-1}(s + \rho(J-1)w)\beta_{J-1}(w) \\
\hat{\xi}_{k-1,21}(0+, w, s)\alpha_{J}(s + \rho(J)w)\beta_{J}(w) & \cdots & \hat{\xi}_{k-1,2J}(0+, w, s)\alpha_{J-1}(s + \rho(J-1)w)\beta_{J-1}(w) \\
\vdots & \vdots & \ddots \\
\hat{\xi}_{k-1,J1}(0+, w, s)\alpha_{J}(s + \rho(J)w)\beta_{J}(w) & \cdots & \hat{\xi}_{k-1,JJ}(0+, w, s)\alpha_{J-1}(s + \rho(J-1)w)\beta_{J-1}(w)
\end{bmatrix}.
\]
The last matrix in the above expression can be written in matrix product form as

\[
\hat{\xi}_k(0+, w, s) = \begin{bmatrix}
\hat{\xi}_{k-1; 1}(0+, w, s) & \hat{\xi}_{k; 11}(0+, w, s) & \cdots & \hat{\xi}_{k; 1,k-1}(0+, w, s) \\
\hat{\xi}_{k-1; 2}(0+, w, s) & \hat{\xi}_{k; 21}(0+, w, s) & \cdots & \hat{\xi}_{k; 2,k-1}(0+, w, s) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\xi}_{k-1; k}(0+, w, s) & \hat{\xi}_{k; k1}(0+, w, s) & \cdots & \hat{\xi}_{k; k,k-1}(0+, w, s) \\
\hat{\zeta}_{k; 1}(w, s) & \cdots & \hat{\zeta}_{k; k,k-1}(w, s) & \zeta_{k-1; k}(w, s)
\end{bmatrix}.
\]

By applying (3.24) to the above expression repeatedly, it follows that

\[
\hat{\xi}_k(0+, w, s) = \zeta_{k; 1}(w, s) \hat{\xi}_{k-1}(0+, w, s).
\]

By induction, Equation (3.29) leads to

\[
\hat{\xi}_k(0+, w, s) = \left\{ \zeta_{k; 1}(w, s) \right\}^{k-1} \hat{\xi}_1(0+, w, s), \quad k \geq 1.
\]

Substitution of (3.28) into (3.30) then yields

\[
\hat{\xi}_k(0+, w, s) = \left\{ \zeta_{k; 1}(w, s) \right\}^k \hat{\alpha}_D^{-1}(w, s), \quad k \geq 1.
\]

Putting (3.25) and (3.31) together, one finally concludes that

\[
\hat{\xi}_k(0+, w, s) = \left\{ \zeta_{k; 1}(w, s) \right\}^k \hat{\alpha}_D^{-1}(w, s), \quad k \geq 1.
\]

We now turn our attention to analysis of \(\hat{\varphi}_{k; ij}(v, w, s)\). By taking Laplace transforms of (3.3) and (3.4) with respect to \(t\), one sees that

\[
\hat{\varphi}_{k; ij}(x, z, s) = \begin{cases}
\delta_{(j=i)} (z - \rho(j)x)e^{-sx} \tilde{A}_j(x), & k = 0 \\
\delta_{(j>i)} e^{-sx} \xi_{0; ij}(0+, z - \rho(j)x, s) \tilde{A}_j(x), & k \geq 1,
\end{cases}
\]

If Laplace transforms are taken again with respect to \(z\) in (3.33), one has

\[
\hat{\varphi}_{k; ij}(x, w, s) = \begin{cases}
\delta_{(j=i)} + \delta_{(j>i)} \hat{\xi}_{0; ij}(0+, w, s) e^{-(s+\rho(j)w)x} \tilde{A}_j(x), & k = 0 \\
\hat{\xi}_{k; ij}(0+, w, s) e^{-(s+\rho(j)w)x} \tilde{A}_j(x), & k \geq 1.
\end{cases}
\]

By taking Laplace transforms one more time with respect to \(x\) in (3.34), it follows that

\[
\hat{\varphi}_{k; ij}(v, w, s) = \begin{cases}
\delta_{(j=i)} + \delta_{(j>i)} \hat{\xi}_{0; ij}(0+, w, s) \frac{1-\alpha_j(v+\rho(j)w+s)}{v+\rho(j)w+s}, & k = 0 \\
\hat{\xi}_{k; ij}(0+, w, s) \cdot \frac{1-\alpha_j(v+\rho(j)w+s)}{v+\rho(j)w+s}, & k \geq 1.
\end{cases}
\]
which can be rewritten in matrix form as

\[ (3.36) \quad \hat{\varphi}_k(v, w, s) = \hat{\gamma}_k(0+, w, s) \beta_{D}(w, v + s), \quad k \geq 0. \]

Substituting (3.32) into (3.36), it can be seen that

\[ \hat{\varphi}_k(v, w, s) = \begin{cases} \left[ I - \alpha^*(s, w) \right]^{-1} \beta_{D}(w, v + s), & k = 0 \\ \left\{ \xi_{1J}(w, s) \right\} \alpha^*_D(w, s) \alpha^{*-1}(w, s) \beta_{D}(w, v + s), & k \geq 1 \end{cases}, \]

completing the proof. \( \square \)

By taking the generating function of \( \hat{\varphi}_k(v, w, s) \) in (3.18) with respect to \( k \) \((k = 0, 1, 2 \cdots)\), the joint transform of \([N(t), J(t), X(t), Z(t)]\) can be obtained as we show next.

**Theorem 3.2** Let \( \hat{\varphi}(v, w, s, u) \) be the matrix generating function of \( \hat{\varphi}_k(v, w, s) \) in (3.18) defined by

\[ (3.37) \quad \hat{\varphi}(v, w, s, u) \stackrel{\text{def}}{=} [\hat{\varphi}_{ij}(v, w, s, u)]; \quad \hat{\varphi}_{ij}(v, w, s, u) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \hat{\varphi}_{k;ij}(v, w, s) u^k. \]

Then one has

\[ (3.38) \quad \hat{\varphi}(v, w, s, u) = \chi(w, s, u) \beta_{D}(w, v + s), \]

where

\[ (3.39) \quad \chi(w, s, u) \stackrel{\text{def}}{=} \left[ I - \alpha^*(w, s) \right]^{-1} + \frac{u \xi_{1J}(w, s)}{1 - u \xi_{1J}(w, s)} \alpha^*_D(w, s) \alpha^{*-1}(w, s). \]

**Proof** Multiplying \( u^k \) to both sides of (3.37) and then summing from \( k = 0 \) to \( \infty \), one has

\[
\hat{\varphi}(v, w, s, u) = \left[ I - \alpha^*(w, s) \right]^{-1} \beta_{D}(w, v + s) + u \xi_{1J}(w, s) \sum_{k=1}^{\infty} \left\{ u \xi_{1J}(w, s) \right\}^{k-1} \alpha^*_D(w, s) \alpha^{*-1}(w, s) \beta_{D}(w, v + s) \\
= \left[ I - \alpha^*(w, s) \right]^{-1} + \frac{u \xi_{1J}(w, s)}{1 - u \xi_{1J}(w, s)} \alpha^*_D(w, s) \alpha^{*-1}(w, s) \beta_{D}(w, v + s) \\
= \chi(w, s, u) \beta_{D}(w, v + s),
\]

proving the theorem. \( \square \)
Chapter 4

Asymptotic Expansion of $E[Z(t)|J(0) = i]$

The purpose of this chapter is to establish an asymptotic expansion of $E[Z(t)|J(0) = i]$ as $t \to \infty$.

We first note from (3.37) that

\[ E[Z(t)|J(0) = i] = \sum_{j=1}^{L-1} \left\{ \frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \right\}_{w=0}, \]

(4.1)

where $\mathcal{L}^{-1}$ means the inversion of the Laplace transform, i.e., $\mathcal{L}^{-1}\{\alpha(s)\} = a(t)$ with $\alpha(s) = \int_0^\infty e^{-st} a(t) dt$. Analytical derivation of the asymptotic expansion of $E[Z(t)|J(0) = i]$ as $t \to \infty$ based on (4.1) is rather lengthy. In what follows, this derivation is shown through multiple steps.

Some theorems are deferred to Appendix for enhancing the readability.

**Step 1. Key Notation**

\[ \mu_{j,k} \overset{\text{def}}{=} E[X^k_j] = (-1)^k \frac{\partial}{\partial s} \alpha_j(s) \bigg|_{s=0} \]

(4.2)

\[ S_{ij:1} \overset{\text{def}}{=} \sum_{n=i}^{j} \mu_{n:1} \]

(4.3)

\[ \mathcal{J}_i^j \overset{\text{def}}{=} \{i, i+1, i+2, \ldots, j-1, j\} \]

(4.4)

\[ S_{ij} \overset{\text{def}}{=} \sum_{n=i}^{j} \mu_{n:2} + \sum_{n=i}^{j} \sum_{m \in \mathcal{J}_i^j \setminus \{n\}} \mu_{m:1} \mu_{n:1} \]

(4.5)

\[ C_{0:ij} \overset{\text{def}}{=} -\sum_{n=i}^{j} \{\rho(n)\mu_{n:1} + E[D_n]\} \]

(4.6)

\[ C_{1:ij} \overset{\text{def}}{=} \sum_{n=i}^{j} \left[ E[D_n]S_{ij:1} + \rho(n)\left\{\mu_{n:1}(S_{i,n-1:1} + S_{n+1,j:1}) + \mu_{n:2}\right\}\right] \]

(4.7)
From (3.12) and (3.39), Equation (4.11) can be rewritten as

\[
\hat{C}_{0:ij} \overset{\text{def}}{=} \frac{1}{S_{1:11}^2} \left\{ \mu_{j:1} (C_{0:i,j-1} S_{1:11} + C_{1:11} S_{i,j-1}) - \frac{1}{2} C_{0:11} \mu_{j:2} \right\} - \frac{\rho(j) \mu_{j:2}}{2S_{1:11}}
\]

(4.9) \hspace{1cm} \hat{C}_{1:ij} \overset{\text{def}}{=} \frac{\mu_{j:1} C_{0:11}}{S_{1:11}^2}

(4.10) \hspace{1cm} \hat{C}_{0:ij}'' \overset{\text{def}}{=} \frac{1}{S_{1:11}^2} \left[ \mu_{j:1} \left( C_{1:11} + C_{0:11} \right) + C_{1:11} - C_{0:11} \left( S_{1:11} + S_{1:11-1:1} \right) \right]

\hspace{1cm} - \frac{1}{2} \mu_{j:2} C_{0:11} - \frac{\rho(j) \mu_{j:2}}{2S_{1:11}}

**Step 2.** Derivation of \( \hat{\nu}(0, w, s, 1) \)

From (3.38) with \( v = 0 \) and \( u = 1 \), one sees that

\[
\hat{\nu}(0, w, s, 1) = \chi(w, s, 1) \beta_j(w, s).
\]

From (3.12) and (3.39), Equation (4.11) can be rewritten as

\[
\hat{\nu}(0, w, s, 1) = \begin{pmatrix}
1 & \hat{\zeta}_{11}(w, s) & \hat{\zeta}_{12}(w, s) & \cdots & \hat{\zeta}_{1:1}(w, s) \\
1 & \hat{\zeta}_{22}(w, s) & \cdots & \hat{\zeta}_{2:1}(w, s) \\
0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
+ \frac{\hat{\zeta}_{1J}(w, s)}{1 - \hat{\zeta}_{1J}(w, s)} \begin{pmatrix}
\hat{\zeta}_{1J}(w, s) \\
\hat{\zeta}_{2J}(w, s) \\
\hat{\zeta}_{J:1}(w, s)
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hat{\zeta}_{1J}(w, s) \\
\hat{\zeta}_{2J}(w, s) \\
\hat{\zeta}_{J:1}(w, s)
\end{pmatrix}
\begin{pmatrix}
\frac{1 - \alpha_1(s + \rho(1)w)}{s + \rho(1)w} \\
\frac{1 - \alpha_1(s + \rho(J)w)}{s + \rho(J)w}
\end{pmatrix}
\]
Substitution of (3.13) into the above expression then yields

\[
\begin{align*}
\hat{\varphi}(0, w, s, 1) &= \\
&= \left[
\begin{array}{cccc}
\frac{1-\alpha_1(s+\rho(1)w)}{s+\rho(1)w} & \zeta_{11}(w, s) & \frac{1-\alpha_2(s+\rho(2)w)}{s+\rho(2)w} & \cdots & \frac{1-\alpha_J(s+\rho(J)w)}{s+\rho(J)w} \\
\zeta_{1J}(w, s) & \frac{1-\alpha_1(s+\rho(1)w)}{s+\rho(1)w} & \zeta_{11}(w, s) & \frac{1-\alpha_2(s+\rho(2)w)}{s+\rho(2)w} & \cdots & \frac{1-\alpha_J(s+\rho(J)w)}{s+\rho(J)w} \\
\zeta_{12}(w, s) & \frac{1-\alpha_1(s+\rho(1)w)}{s+\rho(1)w} & \zeta_{12}(w, s) & \frac{1-\alpha_2(s+\rho(2)w)}{s+\rho(2)w} & \cdots & \frac{1-\alpha_J(s+\rho(J)w)}{s+\rho(J)w} \\
\zeta_{1J-1}(w, s) & \frac{1-\alpha_1(s+\rho(1)w)}{s+\rho(1)w} & \zeta_{1J-1}(w, s) & \frac{1-\alpha_2(s+\rho(2)w)}{s+\rho(2)w} & \cdots & \frac{1-\alpha_J(s+\rho(J)w)}{s+\rho(J)w}
\end{array}
\right] + \frac{1}{1-\zeta_{1J}(w, s)}
\end{align*}
\]

It may be worth noting that Equation (4.12) coincides with Theorem 2.1 of Sumita and Masuda [25] with \( v = 0 \).

If we rewrite Equation (4.12) componentwise, one finds that

\[
\hat{\varphi}_{ij}(0, w, s, 1) = \begin{cases} 
\zeta_{i,j-1}(w, s) \frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w} & \text{if } i \leq j \\
\frac{1-\zeta_{1J}(w, s)}{\zeta_{1J-1}(w, s)} \frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w} & \text{if } i > j
\end{cases}
\]

**Step 3.** Taylor Expansion of \( \frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \) at \( s = 0 \) for \( i \leq j \)

By differentiating (4.13) for \( i \leq j \) with respect to \( w \) and setting \( w = 0 \), one finds that

\[
\begin{align*}
\frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \bigg|_{w=0} &= \frac{\partial}{\partial w} \left\{ \begin{array}{c}
\zeta_{i,j-1}(w, s) \\
1 - \zeta_{1J}(w, s)
\end{array} \right\} \bigg|_{w=0} \cdot \frac{1-\alpha_j(s)}{s} \\
&\quad + \frac{\zeta_{i,j-1}(0, s)}{1 - \zeta_{1J}(0, s)} \cdot \frac{\partial}{\partial w} \left\{ \begin{array}{c}
\frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w}
\end{array} \right\} \bigg|_{w=0},
\end{align*}
\]

where by using Theorems A.3, A.4 and A.5, it can be seen that

\[
\begin{align*}
\frac{\partial}{\partial w} \left\{ \begin{array}{c}
\zeta_{i,j-1}(w, s) \\
1 - \zeta_{1J}(w, s)
\end{array} \right\} \bigg|_{w=0} &\cdot \frac{1-\alpha_j(s)}{s} \\
&= \mu_{j;1}C_{0;1J} + s\left\{ \mu_{j;1}(C_{0;i,j-1}S_{1J,1} + C_{1;1J} - C_{0;1J}S_{i,j-1}) - \frac{1}{2}C_{0;1J} \mu_{j;2} \right\} + o(s) \\
&= \frac{\mu_{j;1}C_{0;1J} + s\left\{ \mu_{j;1}(C_{0;i,j-1}S_{1J,1} + C_{1;1J} - C_{0;1J}S_{i,j-1}) - \frac{1}{2}C_{0;1J} \mu_{j;2} \right\} + o(s)}{s^2S_{1J,1}^2 + o(s^2)}
\end{align*}
\]

and

\[
\zeta_{i,j-1}(0, s) \cdot \frac{\partial}{\partial w} \left\{ \begin{array}{c}
1-\alpha_j(s+\rho(j)w)
\end{array} \right\} \bigg|_{w=0} = -\frac{1}{2} \rho(j) \mu_{j;2} + o(1) - \frac{1}{2} \frac{\rho(j) \mu_{j;2} + o(1)}{sS_{1J,1} - \frac{1}{2}s^2S_{1J} + o(s^2)}.
\]
Substituting (4.15) and (4.16) into (4.14) yields that

\[
\frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \bigg|_{w=0} = \mu_{j;1} C_{0:1,J} + \frac{1}{2} \mu_{j;2} \left\{ C_{0:1,J} - C_{0:1,J} S_{1:J,1} + C_{1:1,J} - C_{0:1,J} S_{1,j-1} \right\} - \frac{1}{2} \rho(j) C_{0:1,J} + o(1) \]

\[
= \frac{1}{s^2} \frac{\partial}{\partial w} \left\{ \mu_{j;1} C_{0:1,J} + \frac{1}{2} \mu_{j;2} \left\{ C_{0:1,J} - C_{0:1,J} S_{1:J,1} + C_{1:1,J} - C_{0:1,J} S_{1,j-1} \right\} \right\} + o(1) .
\]

By conducting polynomial division and simplifying in the above expression, one finally concludes that

\[
\frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \bigg|_{w=0} = \frac{1}{s} C_{1;j} + \frac{1}{s} C_{0:ij} + o(1) .
\]

**Step 4.** Taylor Expansion of \( \frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \bigg|_{w=0} \) at \( s = 0 \) for \( i > j \)

By differentiating (4.13) for \( i > j \) with respect to \( w \) and setting \( w = 0 \), one sees that

\[
\frac{\partial}{\partial w} \hat{\varphi}_{ij}(0, w, s, 1) \bigg|_{w=0} = \frac{\partial}{\partial w} \left\{ \frac{\zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s)}{1 - \zeta_{1,j}(w, s)} \right\} \bigg|_{w=0} \cdot \frac{1 - \alpha_j(s)}{s} \]

\[
+ \frac{\zeta_{i,j}(0, s) \zeta_{1,j-1}(0, s)}{1 - \zeta_{1,j}(0, s)} \cdot \frac{\partial}{\partial w} \left\{ \frac{1 - \alpha_j(s + \rho(j) w)}{s + \rho(j) w} \right\} \bigg|_{w=0} ,
\]

where

\[
\frac{\partial}{\partial w} \left\{ \frac{\zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s)}{1 - \zeta_{1,j}(w, s)} \right\} \bigg|_{w=0} = \frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \bigg|_{w=0} \cdot \left\{ 1 - \zeta_{1,j}(0, s) \right\}^2 .
\]

As to \( \frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \bigg|_{w=0} \) in (4.20), we note that

\[
\frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \bigg|_{w=0} = \frac{\partial}{\partial w} \zeta_{i,j}(w, s) \bigg|_{w=0} \cdot \zeta_{1,j-1}(0, s) + \zeta_{i,j}(0, s) \cdot \frac{\partial}{\partial w} \zeta_{1,j-1}(w, s) \bigg|_{w=0} .
\]

Using Theorems A.3 and A.4, this in turn leads to

\[
\frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \bigg|_{w=0} = \{ C_{0:ij} + s C_{1:ij} + o(s) \} \left\{ 1 - s S_{1,j-1:1} + \frac{1}{2} s^2 S_{1,j-1} + o(s^2) \right\} + \{ 1 - s S_{i,J,1} + \frac{1}{2} s^2 S_{i,j} + o(s^2) \} \{ C_{0:1,j-1} + s C_{1:1,j-1} + o(s) \} .
\]
By simplifying the right hand side of the above expression, it can be seen that
\[
\left. \frac{\partial}{\partial w} \{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \} \right|_{w=0} = (C_{0:1,j} + C_{0:1,j-1}) + s \left\{ C_{1:1,j} + C_{1:1,j-1} - (S_{i,j:1} C_{0:1,j-1} + S_{1,j-1:1} C_{0:1,j}) \right\} + o(s).
\]
Accordingly, one has
\[
\left. \frac{\partial}{\partial w} \{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \} \right|_{w=0} \cdot \left\{ 1 - \zeta_{1,j}(0, s) \right\} = \left[ (C_{0:1,j} + C_{0:1,j-1}) + s \left\{ C_{1:1,j} + C_{1:1,j-1} - (S_{i,j:1} C_{0:1,j-1} + S_{1,j-1:1} C_{0:1,j}) \right\} + o(s) \right]
\cdot \left\{ sS_{1,j:1} - \frac{1}{2} s^2 S_{1,j} + o(s^2) \right\} = sS_{1,j:1} (C_{0:1,j} + C_{0:1,j-1}) + o(s).
\]
From Theorems A.3 and A.4, we can rewrite \( \zeta_{i,j}(0, s) \zeta_{1,j-1}(0, s) \frac{\partial}{\partial w} \zeta_{i,j}(w, s) \bigg|_{w=0} \) in (4.20) as follows.
\[
\left. \frac{\partial}{\partial w} \{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \} \right|_{w=0} = \{ 1 - sS_{1,j:1} + \frac{1}{2} s^2 S_{1,j} + o(s^2) \} \left\{ 1 - sS_{1,j-1:1} + \frac{1}{2} s^2 S_{1,j-1} + o(s^2) \right\} (C_{0:1,j} + sC_{1:1,j} + o(s)) = C_{0:1,j} \left\{ S_{1,j:1} (C_{0:1,j} + C_{0:1,j-1}) + C_{1:1,j} - C_{0:1,j} (S_{i,j:1} + S_{1,j-1:1}) \right\} + o(s)
\]
By substituting (4.21) and (4.22) into (4.20), one has
\[
\left. \frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \right|_{w=0} = \frac{C_{0:1,j} + s \left\{ S_{1,j:1} (C_{0:1,j} + C_{0:1,j-1}) + C_{1:1,j} - C_{0:1,j} (S_{i,j:1} + S_{1,j-1:1}) \right\} + o(s)}{s^2 S_{1,j:1}^2 + o(s^2)}.
\]
By conducting polynomial division in the right hand side of (4.23), one yields that
\[
\left. \frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \right|_{w=0} = \frac{1}{s^2} \cdot \frac{C_{0:1,j}}{S_{1,j:1}^2} + \frac{1}{s} \cdot \frac{1}{S_{1,j:1}^2} \left\{ S_{1,j:1} (C_{0:1,j} + C_{0:1,j-1}) + C_{1:1,j} - C_{0:1,j} (S_{i,j:1} + S_{1,j-1:1}) \right\} + o\left( \frac{1}{s} \right).
\]
Hence the first term of the right hand side in (4.19) can be rewritten as
\[
\left. \frac{\partial}{\partial w} \left\{ \zeta_{i,j}(w, s) \zeta_{1,j-1}(w, s) \right\} \right|_{w=0} \cdot \frac{1 - \alpha_j(s)}{s} = \frac{1}{s^2} \cdot \frac{C'_{1,j}}{S_{1,j:1}^2} + \frac{1}{s} \cdot \frac{1}{S_{1,j:1}^2} \left\{ \mu_{j:1} \left\{ S_{1,j:1} (C_{0:1,j} + C_{0:1,j-1}) + C_{1:1,j} \right\} - C_{0:1,j} (S_{i,j:1} + S_{1,j-1:1}) \right\} - \frac{1}{s} \cdot \frac{1}{S_{1,j:1}^2} C_{0:1,j} + o\left( \frac{1}{s} \right).
\]
18
For the second term of the right hand side in (4.19), one finds from Theorem A.5 that

\[
(4.26) \quad \frac{\zeta_{i,j}(0,s)\zeta_{1,j-1}(0,s)}{1 - \zeta_{1,J}(0,s)} . \quad \frac{\partial}{\partial w} \left\{ \frac{1 - \alpha_j(s + \rho(j)w)}{s + \rho(j)w} \right\} \bigg|_{w=0} = \frac{-\frac{1}{2} \rho(j)\mu_{j:2} + o(1)}{sS_{1,J:1} - \frac{1}{2}s^2 S_{1,J} + o(s^2)} .
\]

If we employ polynomial division in (4.26) in a manner similar to (4.24), it can be seen that

\[
(4.27) \quad \frac{\zeta_{i,J}(0,s)\zeta_{1,j-1}(0,s)}{1 - \zeta_{1,J}(0,s)} . \quad \frac{\partial}{\partial w} \left\{ \frac{1 - \alpha_j(s + \rho(j)w)}{s + \rho(j)w} \right\} \bigg|_{w=0} = -\frac{1}{s} \cdot \frac{\rho(j)\mu_{j:2}}{2S_{1,J:1}} + o\left(\frac{1}{s}\right) .
\]

By substituting (4.25) and (4.27) into (4.19) and simplifying, one concludes that

\[
(4.28) \quad \frac{\partial}{\partial w} \hat{\varphi}_{ij}(0,w,s,1) \bigg|_{w=0} = \frac{1}{s^2} C_{1:ij} + \frac{1}{s} C_{0:ij} + o\left(\frac{1}{s}\right) .
\]

**Step 5. Statement of the Theorem**

Combining (4.18) with (4.28) yields that

\[
(4.29) \quad \frac{\partial}{\partial w} \hat{\varphi}_{ij}(0,w,s,1) \bigg|_{w=0} = \frac{1}{s^2} C_{1:ij} + \frac{1}{s} \left\{ \frac{1}{s} \left[ \delta_{\{i\leq j\}} C_{0:ij} + \delta_{\{i>j\}} C_{0:ij} \right] + o\left(\frac{1}{s}\right) \right\} .
\]

By taking the inverse Laplace transform with respect to \(s\) and summing from \(j = 1\) to \(J\), the following theorem holds true.

**Theorem 4.1**

\[
(4.30) \quad E[Z(t)|J(0) = i] = \sum_{j=1}^{J} \left[ \delta_{\{i\leq j\}} C_{0:ij} + \delta_{\{i>j\}} C_{0:ij} + tC_{1:ij} \right] + o(1) \quad \text{as} \quad t \to \infty .
\]

19
Chapter 5

Numerical Exploration of Optimal Preventive Maintenance Policies for Production Management

We consider a production system where the system down cost is huge. A typical example may be the production of semi-conductor chips because the production machines are extremely expensive and the repair takes a long time since vendor engineers often have to be called in once the system fails. In such a situation, preventive maintenance is widely practiced where minimal repairs take place as minor problems occur, which can be addressed by on-site engineers. A complete overhaul demanding the presence of vendor engineers is conducted only after minimal repairs are repeated certain many times. The question then is to determine when to conduct a complete overhaul. The reward process defined on the cyclic renewal process proposed in this thesis provides a useful computational vehicle for numerically exploring optimal preventive maintenance policies of this sort in a dynamic environment. In this chapter, we demonstrate this claim using Theorem 4.1.

The idea behind minimal repairs is to prolong the availability of the system in the time interval $(0, T]$ by accommodating a partial system adjustment from time to time which can be done at much lower cost and in much shorter time in comparison with a complete overhaul. Starting with a fresh system lifetime, it is natural to assume that the time until the next minimal repair becomes shorter while the subsequent minimal repair time becomes longer as this cycle is repeated. When it is decided to conduct a complete overhaul, the system is brought back to its original fresh state upon completion of the overhaul.

In order to formulate this problem, let $\{\hat{X}_i\}_{i=1}^{\infty}$ and $\{\tilde{X}_i\}_{i=1}^{\infty}$ be sequences of i.i.d. exponential
random variables with parameters $\lambda$ and $\mu$ respectively. We assume that the system lifetime when it is in the fresh state, denoted by $X_1$ following (2.2) and Figure 5.1, is the Gamma variate of integral order $K(0)$ with scaling parameter $\lambda$, i.e.,

$$X_1 = \sum_{i=1}^{K(0)} \tilde{X}_i .$$  \hspace{1cm} (5.1)

We also assume that the time required for conducting a complete overhaul is the Gamma variate of integral order $K(1)$ with scaling parameter $\mu$. Assuming that $K$ minimal repairs would take place, one has

$$X_{2(K+1)} = \sum_{i=1}^{K(1)} \tilde{X}_i .$$  \hspace{1cm} (5.2)

So as to reflect the fact that the time until the next minimal repair becomes shorter while the subsequent minimal repair time becomes longer as this cycle is repeated, we define

$$X_j = \begin{cases} \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{K+2-(j+1)/2} & \text{if } j = 3, 5, \cdots, 2K+1 \\ \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{j/2} & \text{if } j = 2, 4, \cdots, 2K \end{cases} .$$  \hspace{1cm} (5.3)

Here, for $j$ odd, $X_j$ is the time until the next minimal repair which decreases stochastically. For $j$ even, $X_j$ is the subsequent minor repair time which increases stochastically. The structure of $X_j$’s is depicted in Figure 5.1 below.

![Figure 5.1: Typical Sample Path of $[N(t), J(t), X(t)]$ for Preventive Maintenance Model](image)

Let $\alpha_j(s)$ be the Laplace transform of the p.d.f of $X_j$. From (5.1), (5.2) and (5.3), it can be seen that

$$\alpha_j(s) = \begin{cases} \left( \frac{\lambda}{s+\lambda} \right)^{K(0)} & \text{if } j = 1 \\ \left( \frac{\lambda}{s+\lambda} \right)^{K(1)} & \text{if } j = 2(K+1) \\ \left( \frac{\lambda}{s+\lambda} \right)^{K+1+\frac{K+1}{2}} & \text{if } j = 3, 5, \cdots, 2K+1 \\ \left( \frac{\mu}{s+\mu} \right)^{\frac{j}{2}} & \text{if } j = 2, 4, \cdots, 2K \end{cases} .$$  \hspace{1cm} (5.4)
By differentiating (5.4) with respect to \( s \) once or twice and setting \( s = 0 \), one finds that

\[
E[X_j] = \begin{cases} 
\frac{K(0)}{\lambda} & \text{if } j = 1 \\
\frac{K(1)}{\mu} & \text{if } j = 2(K + 1) \\
\frac{K + 2 - \frac{j+1}{2}}{\lambda} & \text{if } j = 3, 5, \cdots, 2K + 1 \\
\frac{j}{2\mu} & \text{if } j = 2, 4, \cdots, 2K 
\end{cases}
\]

(5.5)

and

\[
E[X_j^2] = \begin{cases} 
\frac{1}{\lambda}K(0)(K(0) + 1) & \text{if } j = 1 \\
\frac{1}{\mu}K(1)(K(1) + 1) & \text{if } j = 2(K + 1) \\
\frac{1}{\lambda}\left(K + 2 - \frac{j+1}{2}\right)\left(K + 3 - \frac{j+1}{2}\right) & \text{if } j = 3, 5, \cdots, 2K + 1 \\
\frac{j}{2\mu}\left(\frac{j}{2} + 1\right) & \text{if } j = 2, 4, \cdots, 2K 
\end{cases}
\]

(5.6)

We next turn our attention to the reward structure. In production systems involving chemical processes, which include production systems for semi-conductor chips, it is often observed that the longer the time since a complete overhaul passes, the higher the defective rate is. Accordingly, the reward per unit time decreases as a function of the number of minimal repairs that took place. Conversely, the cost per unit time increases as a function of the number of minimal repairs that took place. Then the reward rate function \( \rho(j) \) may be defined as

\[
\rho(j) = \begin{cases} 
\rho_{\text{UP}}\left(1 - \frac{j-1}{2c}\right) & \text{if } j = 1, 3, \cdots, 2K + 1 \\
-\rho_{\text{DOWN}}\left(1 + \frac{j}{2c}\right) & \text{if } j = 2, 4, \cdots, 2K + 2 
\end{cases}
\]

(5.7)

where \( c \) is a parameter satisfying \( c > K \) since \( \rho_{\text{UP}} > 0 \). The fixed cost for calling in on-site engineers for a minimal repair and that for calling in vendor engineers for a complete overhaul can be expressed in terms of random reward jumps \( D_j \). The associated means are defined as

\[
E[D_j] = \begin{cases} 
-D & \text{if } j = 1, 3, \cdots, 2K + 1 \\
0 & \text{if } j = 2, 4, \cdots, 2K \\
-10D & \text{if } j = 2K + 2 
\end{cases}
\]

(5.8)

In what follows, a set of parameter values for \( \lambda, \rho_{\text{UP}}, D, i, K(0), K(1) \) and \( c \) would be fixed as specified in Table 5.1 below. Numerical experiments are then conducted to explore the optimal
value of $K$, which maximizes the expected reward per unit time in the time interval $(0,T]$, as a function of $K$ and $T$ for given values of $\mu$ and $\rho_{\text{DOWN}}$.

Table 5.1: Parameter Values for $\lambda$, $\rho_{\text{UP}}$, $D$, $i$, $K(0)$, $K(1)$ and $c$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho_{\text{UP}}$</th>
<th>$D$</th>
<th>$i$</th>
<th>$K(0)$</th>
<th>$K(1)$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>100</td>
<td>1</td>
<td>20</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

More specifically, let $C_0(K)$ and $C_1(K)$ be defined as

$$C_0(K) = \sum_{j=1}^{2K+2} \left\{ \delta_{\{i\leq j\}} C'_{0;ij} + \delta_{\{i>j\}} C''_{0;ij} \right\}$$  (5.9)

and

$$C_1(K) = \sum_{j=1}^{2K+2} C'_{1;j},$$  (5.10)

so that one has from (4.30)

$$\frac{E[Z(T)|J(0) = 1]}{T} = C_1(K) + \frac{1}{T} C_0(K) + \frac{1}{T} o(1).$$  (5.11)

The optimal number of minimal repairs, denoted by $K^*_T$, is now given as

$$K^*_T \overset{\text{def}}{=} \arg \max_K \left\{ C_1(K) + \frac{1}{T} C_0(K) + \frac{1}{T} o(1) \right\}. $$  (5.12)

Of interest is to understand the behavior of $K^*_T$ as $K$ and $T$ are varied for given values of $\mu$ and $\rho_{\text{DOWN}}$.

As a typical example, $E[Z(T)|J(0) = 1]/T$ is plotted in Figure 5.2 as a function of $K$ and $T$ for $(\mu, \rho_{\text{DOWN}}) = (2,10)$. In order to facilitate the understanding of the functional behavior, the marginal functions for $K$ and $T$ are exhibited in Figures 5.3 and 5.4 respectively. The values of the optimal number of minimal repairs $K^*_T$ are given in Table 5.2 for $T = 10, 20, 30, 40, 50$. One observes that $K^*_T$ decreases as $T$ increases. Since $K^*_T$ is determined based on the asymptotic expansion given in (4.30), one has to be careful in understanding $K^*_T$ especially when $T$ is small. In order to clarify this concern, $C_0(K)$ and $C_1(K)$ are computed in Table 5.3. It should be noted that if no minimal repair takes place with $K = 0$, the cyclic renewal model is reduced to an alternating renewal process with the long run average reward per unit time of 3.2000. Since $C_1(1) = 4.6818$ is the largest for
all $1 \leq K \leq 20$, it is safe to say that the optimal preventive maintenance policy is to set $K^*_T = 1$ when $T$ is sufficiently large.

Figures 5.5 through 5.16 exhibit $E[Z(T)|J(0) = 1]/T$ as a function of $K$ and $T$ for each pair of $\mu = 1, 5, 10, 20$ and $\rho_{\text{DOWN}} = 5, 10, 15$ arranged in lexicographic order. Figures 5.17 through 5.40 illustrate $E[Z(T)|J(0) = 1]/T$ as a marginal function of $K$ or $T$ in the same order. The counterparts for Table 5.3 are given in Tables 5.4 through 5.15 respectively. The following observations can be made.

- From Tables 5.4 through 5.15, one finds that $C_1(1)$ is the largest for all $1 \leq K \leq 20$ if $\mu = 1$. 

24
Table 5.2: $K_T^*$ for $(\mu, \rho_{DOWN})=(2,10)$

<table>
<thead>
<tr>
<th>$T$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_T^*$</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 5.3: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{DOWN})=(2,10)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13.8875</td>
<td>3.2000</td>
</tr>
<tr>
<td>1</td>
<td>21.0630</td>
<td>4.6818</td>
</tr>
<tr>
<td>2</td>
<td>38.2932</td>
<td>4.4000</td>
</tr>
<tr>
<td>3</td>
<td>52.3524</td>
<td>4.1615</td>
</tr>
<tr>
<td>4</td>
<td>63.7674</td>
<td>3.9571</td>
</tr>
<tr>
<td>5</td>
<td>72.9615</td>
<td>3.7800</td>
</tr>
<tr>
<td>6</td>
<td>80.2777</td>
<td>3.6250</td>
</tr>
<tr>
<td>7</td>
<td>85.9964</td>
<td>3.4882</td>
</tr>
<tr>
<td>8</td>
<td>90.3484</td>
<td>3.3667</td>
</tr>
<tr>
<td>9</td>
<td>93.5250</td>
<td>3.2579</td>
</tr>
<tr>
<td>10</td>
<td>95.6856</td>
<td>3.1600</td>
</tr>
<tr>
<td>11</td>
<td>96.9643</td>
<td>3.0714</td>
</tr>
<tr>
<td>12</td>
<td>97.4739</td>
<td>2.9909</td>
</tr>
<tr>
<td>13</td>
<td>97.3104</td>
<td>2.9174</td>
</tr>
<tr>
<td>14</td>
<td>96.5554</td>
<td>2.8500</td>
</tr>
<tr>
<td>15</td>
<td>95.2788</td>
<td>2.7880</td>
</tr>
<tr>
<td>16</td>
<td>93.5408</td>
<td>2.7308</td>
</tr>
<tr>
<td>17</td>
<td>91.3935</td>
<td>2.6778</td>
</tr>
<tr>
<td>18</td>
<td>88.8816</td>
<td>2.6286</td>
</tr>
<tr>
<td>19</td>
<td>86.0446</td>
<td>2.5828</td>
</tr>
<tr>
<td>20</td>
<td>82.9165</td>
<td>2.5400</td>
</tr>
</tbody>
</table>

For $\mu \neq 1$, $C_1(0)$ is the largest for all $1 \leq K \leq 20$. Consequently, if the average system performance is of concern for sufficiently large $T$, it is optimal to set $K_T^* = 1$ when $\mu = 1$ and $K_T^* = 0$ for $\mu \neq 1$.

- In Figures 5.17 through 5.20 corresponding to $\mu = 1, 5, 10, 20$ and $\rho_{DOWN} = 5$, it can been seen that $E[Z(T)|J(0) = 1]/T$ is monotonically increasing as a function of $K$ for all values of $T = 10, 20, 30, 40, 50$. This may suggest that a complete overhaul may not be necessary when the downtime cost of $\rho_{DOWN}$ is relatively small. When $\rho_{DOWN}$ is increased to 10 or 15, $E[Z(T)|J(0) = 1]/T$ appears to be a concave function of $K$ as can be seen in Figures 5.21 through 5.28.

- The behavior of $E[Z(T)|J(0) = 1]/T$ as a function of $T$ can be observed in Figures 5.29
through 5.40 following the lexicographic order of pairs $(\mu, \rho_{\text{DOWN}})$ for $\mu = 1, 5, 10, 20$ and $\rho_{\text{DOWN}} = 5, 10, 15$. One observes that $\mathbb{E}[Z(T)|J(0) = 1]/T$ is basically monotonically decreasing in $T$ except that it is monotonically increasing in $T$ for $K = 20$ and $\rho_{\text{DOWN}} = 15$.

Figure 5.5: $\mathbb{E}[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\text{DOWN}}) = (1, 5)$

Figure 5.6: $\mathbb{E}[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\text{DOWN}}) = (5, 5)$

Figure 5.7: $\mathbb{E}[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\text{DOWN}}) = (10, 5)$

Figure 5.8: $\mathbb{E}[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\text{DOWN}}) = (20, 5)$
Figure 5.9: $E[Z(T) | J(0) = 1] / T$ for $(\mu, \rho_{DOWN}) = (1, 10)$

Figure 5.10: $E[Z(T) | J(0) = 1] / T$ for $(\mu, \rho_{DOWN}) = (5, 10)$

Figure 5.11: $E[Z(T) | J(0) = 1] / T$ for $(\mu, \rho_{DOWN}) = (10, 10)$

Figure 5.12: $E[Z(T) | J(0) = 1] / T$ for $(\mu, \rho_{DOWN}) = (20, 10)$
Figure 5.13: $\mathbb{E}[Z(T)|J(0) = 1] / T$ for $(\mu, \rho_{\text{DOWN}}) = (1, 15)$

Figure 5.14: $\mathbb{E}[Z(T)|J(0) = 1] / T$ for $(\mu, \rho_{\text{DOWN}}) = (5, 15)$

Figure 5.15: $\mathbb{E}[Z(T)|J(0) = 1] / T$ for $(\mu, \rho_{\text{DOWN}}) = (10, 15)$

Figure 5.16: $\mathbb{E}[Z(T)|J(0) = 1] / T$ for $(\mu, \rho_{\text{DOWN}}) = (20, 15)$
Figure 5.17: $E[Z(T)|J(0) = 1]/T$ $(T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{DOWN}) = (1, 5)$

Figure 5.18: $E[Z(T)|J(0) = 1]/T$ $(T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{DOWN}) = (5, 5)$

Figure 5.19: $E[Z(T)|J(0) = 1]/T$ $(T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{DOWN}) = (10, 5)$

Figure 5.20: $E[Z(T)|J(0) = 1]/T$ $(T = 10, 20, 30, 40, 50)$ for $(\mu, \rho_{DOWN}) = (20, 5)$
Figure 5.21: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (1,10)$

Figure 5.22: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (5,10)$

Figure 5.23: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (10,10)$

Figure 5.24: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (20,10)$
Figure 5.25: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (1,15)$

Figure 5.26: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (5,15)$

Figure 5.27: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (10,15)$

Figure 5.28: $E[Z(T)|J(0) = 1]/T$ ($T = 10, 20, 30, 40, 50$) for $(\mu, \rho_{\text{DOWN}}) = (20,15)$
Figure 5.29: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{\text{DOWN}}) = (1, 5)$

Figure 5.30: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{\text{DOWN}}) = (5, 5)$

Figure 5.31: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{\text{DOWN}}) = (10, 5)$

Figure 5.32: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{\text{DOWN}}) = (20, 5)$
Figure 5.33: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (1, 10)$

Figure 5.34: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (5, 10)$

Figure 5.35: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (10, 10)$

Figure 5.36: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (20, 10)$
Figure 5.37: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (1, 15)$

Figure 5.38: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (5, 15)$

Figure 5.39: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (10, 15)$

Figure 5.40: $E[Z(T)|J(0) = 1]/T$ ($K = 5, 10, 15, 20$) for $(\mu, \rho_{DOWN}) = (20, 15)$
Table 5.4: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(1,5)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.2879</td>
<td>2.4000</td>
</tr>
<tr>
<td>1</td>
<td>20.4267</td>
<td>4.7022</td>
</tr>
<tr>
<td>2</td>
<td>37.7803</td>
<td>4.6531</td>
</tr>
<tr>
<td>3</td>
<td>52.7073</td>
<td>4.6113</td>
</tr>
<tr>
<td>4</td>
<td>65.71</td>
<td>4.5754</td>
</tr>
<tr>
<td>5</td>
<td>77.1827</td>
<td>4.5443</td>
</tr>
<tr>
<td>6</td>
<td>87.4363</td>
<td>4.5169</td>
</tr>
<tr>
<td>7</td>
<td>96.7182</td>
<td>4.4928</td>
</tr>
<tr>
<td>8</td>
<td>105.2258</td>
<td>4.4712</td>
</tr>
<tr>
<td>9</td>
<td>113.1179</td>
<td>4.4519</td>
</tr>
<tr>
<td>10</td>
<td>120.5227</td>
<td>4.4346</td>
</tr>
<tr>
<td>11</td>
<td>127.5439</td>
<td>4.4188</td>
</tr>
<tr>
<td>12</td>
<td>134.2659</td>
<td>4.4045</td>
</tr>
<tr>
<td>13</td>
<td>140.7575</td>
<td>4.3914</td>
</tr>
<tr>
<td>14</td>
<td>147.075</td>
<td>4.3794</td>
</tr>
<tr>
<td>15</td>
<td>153.2643</td>
<td>4.3683</td>
</tr>
<tr>
<td>16</td>
<td>159.3632</td>
<td>4.3581</td>
</tr>
<tr>
<td>17</td>
<td>165.4027</td>
<td>4.3486</td>
</tr>
<tr>
<td>18</td>
<td>171.4052</td>
<td>4.3398</td>
</tr>
<tr>
<td>19</td>
<td>177.4005</td>
<td>4.3316</td>
</tr>
<tr>
<td>20</td>
<td>183.3968</td>
<td>4.324</td>
</tr>
</tbody>
</table>

Table 5.5: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(5,5)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21.6579</td>
<td>7.4667</td>
</tr>
<tr>
<td>1</td>
<td>21.9014</td>
<td>5.0673</td>
</tr>
<tr>
<td>2</td>
<td>20.4267</td>
<td>4.7022</td>
</tr>
<tr>
<td>3</td>
<td>37.7803</td>
<td>4.6531</td>
</tr>
<tr>
<td>4</td>
<td>52.7073</td>
<td>4.6113</td>
</tr>
<tr>
<td>5</td>
<td>65.71</td>
<td>4.5754</td>
</tr>
<tr>
<td>6</td>
<td>77.1827</td>
<td>4.5443</td>
</tr>
<tr>
<td>7</td>
<td>87.4363</td>
<td>4.5169</td>
</tr>
<tr>
<td>8</td>
<td>96.7182</td>
<td>4.4928</td>
</tr>
<tr>
<td>9</td>
<td>105.2258</td>
<td>4.4712</td>
</tr>
<tr>
<td>10</td>
<td>113.1179</td>
<td>4.4519</td>
</tr>
<tr>
<td>11</td>
<td>120.5227</td>
<td>4.4346</td>
</tr>
<tr>
<td>12</td>
<td>127.5439</td>
<td>4.4188</td>
</tr>
<tr>
<td>13</td>
<td>134.2659</td>
<td>4.4045</td>
</tr>
<tr>
<td>14</td>
<td>140.7575</td>
<td>4.3914</td>
</tr>
<tr>
<td>15</td>
<td>147.075</td>
<td>4.3794</td>
</tr>
<tr>
<td>16</td>
<td>153.2643</td>
<td>4.3683</td>
</tr>
<tr>
<td>17</td>
<td>159.3632</td>
<td>4.3581</td>
</tr>
<tr>
<td>18</td>
<td>165.4027</td>
<td>4.3486</td>
</tr>
<tr>
<td>19</td>
<td>171.4052</td>
<td>4.3398</td>
</tr>
<tr>
<td>20</td>
<td>177.4005</td>
<td>4.3316</td>
</tr>
</tbody>
</table>

Table 5.6: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(10,5)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22.0877</td>
<td>8.6182</td>
</tr>
<tr>
<td>1</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>2</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>3</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>4</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>5</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>6</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>7</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>8</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>9</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>10</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>11</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>12</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>13</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>14</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>15</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>16</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>17</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>18</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>19</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
<tr>
<td>20</td>
<td>22.0943</td>
<td>5.1148</td>
</tr>
</tbody>
</table>

Table 5.7: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}})=(20,5)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22.1236</td>
<td>9.2762</td>
</tr>
<tr>
<td>1</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>2</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>3</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>4</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>5</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>6</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>7</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>8</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>9</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>10</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>11</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>12</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>13</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>14</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>15</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>16</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>17</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>18</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>19</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
<tr>
<td>20</td>
<td>22.1914</td>
<td>5.1387</td>
</tr>
</tbody>
</table>
Table 5.8: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{DOWN})=(1,10)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3.7634</td>
<td>-0.2000</td>
</tr>
<tr>
<td>1</td>
<td>20.0248</td>
<td>4.3467</td>
</tr>
<tr>
<td>2</td>
<td>36.3952</td>
<td>4.0980</td>
</tr>
<tr>
<td>3</td>
<td>49.7333</td>
<td>3.8868</td>
</tr>
<tr>
<td>4</td>
<td>60.5348</td>
<td>3.7053</td>
</tr>
<tr>
<td>5</td>
<td>69.2016</td>
<td>3.5475</td>
</tr>
<tr>
<td>6</td>
<td>76.0593</td>
<td>3.4092</td>
</tr>
<tr>
<td>7</td>
<td>81.3751</td>
<td>3.2871</td>
</tr>
<tr>
<td>8</td>
<td>85.3694</td>
<td>3.1781</td>
</tr>
<tr>
<td>9</td>
<td>88.2257</td>
<td>3.0805</td>
</tr>
<tr>
<td>10</td>
<td>90.0971</td>
<td>2.9926</td>
</tr>
<tr>
<td>11</td>
<td>91.1124</td>
<td>2.9129</td>
</tr>
<tr>
<td>12</td>
<td>91.3808</td>
<td>2.8404</td>
</tr>
<tr>
<td>13</td>
<td>90.9947</td>
<td>2.7742</td>
</tr>
<tr>
<td>14</td>
<td>90.0331</td>
<td>2.7134</td>
</tr>
</tbody>
</table>

Table 5.9: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{DOWN})=(5,10)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.1334</td>
<td>6.6000</td>
</tr>
<tr>
<td>1</td>
<td>21.7064</td>
<td>4.8903</td>
</tr>
<tr>
<td>2</td>
<td>39.4665</td>
<td>4.5873</td>
</tr>
<tr>
<td>3</td>
<td>53.9693</td>
<td>4.3315</td>
</tr>
<tr>
<td>4</td>
<td>65.7600</td>
<td>4.1126</td>
</tr>
<tr>
<td>5</td>
<td>75.2764</td>
<td>3.9232</td>
</tr>
<tr>
<td>6</td>
<td>82.8722</td>
<td>3.7577</td>
</tr>
<tr>
<td>7</td>
<td>88.836</td>
<td>3.6119</td>
</tr>
<tr>
<td>8</td>
<td>93.4052</td>
<td>3.4824</td>
</tr>
<tr>
<td>9</td>
<td>96.7758</td>
<td>3.3666</td>
</tr>
</tbody>
</table>

Table 5.10: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{DOWN})=(10,10)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21.4265</td>
<td>8.1455</td>
</tr>
<tr>
<td>1</td>
<td>21.9237</td>
<td>4.9611</td>
</tr>
<tr>
<td>2</td>
<td>39.8836</td>
<td>4.6508</td>
</tr>
<tr>
<td>3</td>
<td>54.5161</td>
<td>4.3891</td>
</tr>
<tr>
<td>4</td>
<td>66.6334</td>
<td>4.1652</td>
</tr>
<tr>
<td>5</td>
<td>76.0581</td>
<td>3.9716</td>
</tr>
<tr>
<td>6</td>
<td>83.7478</td>
<td>3.8025</td>
</tr>
<tr>
<td>7</td>
<td>89.7940</td>
<td>3.6536</td>
</tr>
<tr>
<td>8</td>
<td>94.4359</td>
<td>3.5213</td>
</tr>
<tr>
<td>9</td>
<td>97.8716</td>
<td>3.4032</td>
</tr>
<tr>
<td>10</td>
<td>100.2659</td>
<td>3.2972</td>
</tr>
<tr>
<td>11</td>
<td>101.7568</td>
<td>3.2012</td>
</tr>
<tr>
<td>12</td>
<td>102.4608</td>
<td>3.1138</td>
</tr>
<tr>
<td>13</td>
<td>102.4762</td>
<td>3.0342</td>
</tr>
<tr>
<td>14</td>
<td>101.8871</td>
<td>2.9613</td>
</tr>
<tr>
<td>15</td>
<td>100.7651</td>
<td>2.8944</td>
</tr>
<tr>
<td>16</td>
<td>98.1718</td>
<td>2.8326</td>
</tr>
<tr>
<td>17</td>
<td>97.1605</td>
<td>2.7754</td>
</tr>
<tr>
<td>18</td>
<td>94.7772</td>
<td>2.7223</td>
</tr>
<tr>
<td>19</td>
<td>92.0619</td>
<td>2.6729</td>
</tr>
<tr>
<td>20</td>
<td>89.0495</td>
<td>2.6268</td>
</tr>
</tbody>
</table>

Table 5.11: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{DOWN})=(20,10)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21.9186</td>
<td>9.0286</td>
</tr>
<tr>
<td>1</td>
<td>22.0333</td>
<td>4.9968</td>
</tr>
<tr>
<td>2</td>
<td>40.0632</td>
<td>4.6828</td>
</tr>
<tr>
<td>3</td>
<td>54.7908</td>
<td>4.4180</td>
</tr>
<tr>
<td>4</td>
<td>66.7717</td>
<td>4.1917</td>
</tr>
<tr>
<td>5</td>
<td>76.4509</td>
<td>3.9959</td>
</tr>
<tr>
<td>6</td>
<td>84.1877</td>
<td>3.8250</td>
</tr>
<tr>
<td>7</td>
<td>90.2751</td>
<td>3.6745</td>
</tr>
<tr>
<td>8</td>
<td>94.9535</td>
<td>3.5409</td>
</tr>
<tr>
<td>9</td>
<td>98.4218</td>
<td>3.4216</td>
</tr>
<tr>
<td>10</td>
<td>100.8454</td>
<td>3.3143</td>
</tr>
<tr>
<td>11</td>
<td>102.363</td>
<td>3.2173</td>
</tr>
<tr>
<td>12</td>
<td>103.0913</td>
<td>3.1293</td>
</tr>
<tr>
<td>13</td>
<td>103.1292</td>
<td>3.0490</td>
</tr>
<tr>
<td>14</td>
<td>102.5608</td>
<td>2.9754</td>
</tr>
<tr>
<td>15</td>
<td>101.4582</td>
<td>2.9078</td>
</tr>
<tr>
<td>16</td>
<td>99.883</td>
<td>2.8454</td>
</tr>
<tr>
<td>17</td>
<td>97.8888</td>
<td>2.7877</td>
</tr>
<tr>
<td>18</td>
<td>95.5215</td>
<td>2.7341</td>
</tr>
<tr>
<td>19</td>
<td>92.8214</td>
<td>2.6843</td>
</tr>
<tr>
<td>20</td>
<td>89.8234</td>
<td>2.6378</td>
</tr>
</tbody>
</table>
Table 5.12: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}}) = (1, 15)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-18.8146</td>
<td>-2.8000</td>
</tr>
<tr>
<td>1</td>
<td>19.6229</td>
<td>3.9911</td>
</tr>
<tr>
<td>2</td>
<td>35.0101</td>
<td>3.5429</td>
</tr>
<tr>
<td>3</td>
<td>46.7586</td>
<td>3.1623</td>
</tr>
<tr>
<td>4</td>
<td>55.3596</td>
<td>2.8351</td>
</tr>
<tr>
<td>5</td>
<td>61.2206</td>
<td>2.5508</td>
</tr>
<tr>
<td>6</td>
<td>64.6823</td>
<td>2.3015</td>
</tr>
<tr>
<td>7</td>
<td>66.032</td>
<td>2.0812</td>
</tr>
<tr>
<td>8</td>
<td>65.5131</td>
<td>1.8849</td>
</tr>
<tr>
<td>9</td>
<td>63.3334</td>
<td>1.7091</td>
</tr>
<tr>
<td>10</td>
<td>59.6714</td>
<td>1.5506</td>
</tr>
<tr>
<td>11</td>
<td>54.681</td>
<td>1.4071</td>
</tr>
<tr>
<td>12</td>
<td>48.4956</td>
<td>1.2764</td>
</tr>
<tr>
<td>13</td>
<td>41.2318</td>
<td>1.157</td>
</tr>
<tr>
<td>14</td>
<td>32.9913</td>
<td>1.0474</td>
</tr>
<tr>
<td>15</td>
<td>23.8637</td>
<td>0.9465</td>
</tr>
<tr>
<td>16</td>
<td>13.9278</td>
<td>0.8533</td>
</tr>
<tr>
<td>17</td>
<td>3.2538</td>
<td>0.767</td>
</tr>
<tr>
<td>18</td>
<td>-8.0963</td>
<td>0.6867</td>
</tr>
<tr>
<td>19</td>
<td>-20.0671</td>
<td>0.612</td>
</tr>
<tr>
<td>20</td>
<td>-32.6089</td>
<td>0.5421</td>
</tr>
</tbody>
</table>

Table 5.13: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}}) = (5, 15)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18.6089</td>
<td>5.7333</td>
</tr>
<tr>
<td>1</td>
<td>21.5105</td>
<td>4.7134</td>
</tr>
<tr>
<td>2</td>
<td>38.473</td>
<td>4.189</td>
</tr>
<tr>
<td>3</td>
<td>51.5466</td>
<td>3.7463</td>
</tr>
<tr>
<td>4</td>
<td>61.2704</td>
<td>3.3675</td>
</tr>
<tr>
<td>5</td>
<td>68.0889</td>
<td>3.0397</td>
</tr>
<tr>
<td>6</td>
<td>72.3723</td>
<td>2.7533</td>
</tr>
<tr>
<td>7</td>
<td>74.431</td>
<td>2.5009</td>
</tr>
<tr>
<td>8</td>
<td>74.527</td>
<td>2.2768</td>
</tr>
<tr>
<td>9</td>
<td>72.8832</td>
<td>2.0764</td>
</tr>
<tr>
<td>10</td>
<td>69.6903</td>
<td>1.8962</td>
</tr>
<tr>
<td>11</td>
<td>65.1122</td>
<td>1.7333</td>
</tr>
<tr>
<td>12</td>
<td>59.2907</td>
<td>1.5854</td>
</tr>
<tr>
<td>13</td>
<td>52.3491</td>
<td>1.4503</td>
</tr>
<tr>
<td>14</td>
<td>44.3948</td>
<td>1.3266</td>
</tr>
<tr>
<td>15</td>
<td>35.5223</td>
<td>1.2129</td>
</tr>
<tr>
<td>16</td>
<td>25.8145</td>
<td>1.1079</td>
</tr>
<tr>
<td>17</td>
<td>15.345</td>
<td>1.0108</td>
</tr>
<tr>
<td>18</td>
<td>4.1786</td>
<td>0.9206</td>
</tr>
<tr>
<td>19</td>
<td>-7.6266</td>
<td>0.8367</td>
</tr>
<tr>
<td>20</td>
<td>-20.0189</td>
<td>0.7585</td>
</tr>
</tbody>
</table>

Table 5.14: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}}) = (10, 15)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.7653</td>
<td>7.6727</td>
</tr>
<tr>
<td>1</td>
<td>21.7532</td>
<td>4.8074</td>
</tr>
<tr>
<td>2</td>
<td>38.9175</td>
<td>4.2729</td>
</tr>
<tr>
<td>3</td>
<td>52.1603</td>
<td>3.8219</td>
</tr>
<tr>
<td>4</td>
<td>62.0269</td>
<td>3.4362</td>
</tr>
<tr>
<td>5</td>
<td>68.967</td>
<td>3.1027</td>
</tr>
<tr>
<td>6</td>
<td>73.3543</td>
<td>2.8114</td>
</tr>
<tr>
<td>7</td>
<td>75.5025</td>
<td>2.5548</td>
</tr>
<tr>
<td>8</td>
<td>76.6759</td>
<td>2.327</td>
</tr>
<tr>
<td>9</td>
<td>74.0994</td>
<td>2.1234</td>
</tr>
<tr>
<td>10</td>
<td>70.9653</td>
<td>1.9404</td>
</tr>
<tr>
<td>11</td>
<td>66.4387</td>
<td>1.775</td>
</tr>
<tr>
<td>12</td>
<td>60.6626</td>
<td>1.6248</td>
</tr>
<tr>
<td>13</td>
<td>53.761</td>
<td>1.4877</td>
</tr>
<tr>
<td>14</td>
<td>45.8423</td>
<td>1.3622</td>
</tr>
<tr>
<td>15</td>
<td>37.0014</td>
<td>1.2468</td>
</tr>
<tr>
<td>16</td>
<td>27.3219</td>
<td>1.1403</td>
</tr>
<tr>
<td>17</td>
<td>16.8776</td>
<td>1.0418</td>
</tr>
<tr>
<td>18</td>
<td>5.7338</td>
<td>0.9504</td>
</tr>
<tr>
<td>19</td>
<td>-6.051</td>
<td>0.8653</td>
</tr>
<tr>
<td>20</td>
<td>-18.4249</td>
<td>0.7859</td>
</tr>
</tbody>
</table>

Table 5.15: $C_0(K)$ and $C_1(K)$ for $(\mu, \rho_{\text{DOWN}}) = (20, 15)$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_0(K)$</th>
<th>$C_1(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21.6137</td>
<td>8.7810</td>
</tr>
<tr>
<td>1</td>
<td>21.8751</td>
<td>4.8548</td>
</tr>
<tr>
<td>2</td>
<td>39.1407</td>
<td>4.3151</td>
</tr>
<tr>
<td>3</td>
<td>52.4684</td>
<td>3.8599</td>
</tr>
<tr>
<td>4</td>
<td>62.4067</td>
<td>3.4708</td>
</tr>
<tr>
<td>5</td>
<td>69.4077</td>
<td>3.1343</td>
</tr>
<tr>
<td>6</td>
<td>73.8471</td>
<td>2.8406</td>
</tr>
<tr>
<td>7</td>
<td>76.0401</td>
<td>2.5818</td>
</tr>
<tr>
<td>8</td>
<td>76.2522</td>
<td>2.3522</td>
</tr>
<tr>
<td>9</td>
<td>74.7094</td>
<td>2.147</td>
</tr>
<tr>
<td>10</td>
<td>71.6047</td>
<td>1.9626</td>
</tr>
<tr>
<td>11</td>
<td>67.1039</td>
<td>1.7959</td>
</tr>
<tr>
<td>12</td>
<td>61.3504</td>
<td>1.6445</td>
</tr>
<tr>
<td>13</td>
<td>54.4689</td>
<td>1.5065</td>
</tr>
<tr>
<td>14</td>
<td>46.568</td>
<td>1.38</td>
</tr>
<tr>
<td>15</td>
<td>37.7428</td>
<td>1.2638</td>
</tr>
<tr>
<td>16</td>
<td>28.0774</td>
<td>1.1565</td>
</tr>
<tr>
<td>17</td>
<td>17.6456</td>
<td>1.0573</td>
</tr>
<tr>
<td>18</td>
<td>6.5132</td>
<td>0.9653</td>
</tr>
<tr>
<td>19</td>
<td>-5.2615</td>
<td>0.8796</td>
</tr>
<tr>
<td>20</td>
<td>-17.6263</td>
<td>0.7997</td>
</tr>
</tbody>
</table>
Chapter 6

Concluding Remarks

In this thesis, a cyclic renewal process is considered as an extension of an alternating renewal process where each of underlying i.i.d. nonnegative random increments is composed of multiple stages. Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, also introduced is a reward process with jumps defined on the cyclic renewal process. When the system is running in stage $j$, the profit grows linearly at the rate of $\rho(j)$. Upon a minor failure, the subsequent minimal repair in stage $(j + 1)$ incurs the linear cost at the rate of $\rho(j + 1)$. In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval $(0, T]$. A multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management, demonstrating the usefulness of the cyclic renewal model.

While a variety of transform results could be obtained, they have not been utilized fully. For the multivariate process $[N(t), J(t), X(t), Z(t)]$, for example, it is possible to capture the time-dependent correlation structure numerically. Such studies would be continued and reported elsewhere in due course.
Bibliography

   

   


   

   

   

   
   ORC 64-16, Univ. California, Berkeley, 1964

   

   


**Acknowledgements**

I wish to thank Professor Ushio Sumita of Graduate School of Systems and Information Engineering, University of Tsukuba for supporting not only my research but also my life. I am also grateful to the members of my laboratory and Gotoh laboratory for their help. Finally, I owe my deepest gratitude to my parents.
Appendix A

Proofs of expressions used in this thesis

We provide the proofs of expressions used in this thesis.

**Theorem A.1** For $\zeta_{ij}(w,s)$ and $\alpha^*(w,s)$ defined in (3.13) and (3.14), one has

(A.1) \[
[I - \alpha^*(w,s)]^{-1} = \begin{bmatrix}
1 & \zeta_{11}(w,s) & \cdots & \zeta_{1,J-1}(w,s) \\
1 & \zeta_{21}(w,s) & \cdots & \zeta_{2,J-1}(w,s) \\
& \ddots & \ddots & \ddots \\
0 & & \cdots & \zeta_{J-1,J-1}(w,s) \\
\end{bmatrix}.
\]

**Proof** We show $[I - \alpha^*(w,s)]^{-1} = I$. By definition of $\zeta_{ij}(w,s)$, it can been seen that

\[
[I - \alpha^*(w,s)]^{-1} = \begin{bmatrix}
1 & -\zeta_{11}(w,s) & 0 & \cdots & 0 \\
1 & -\zeta_{21}(w,s) & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & & \cdots & -\zeta_{J-1,J-1}(w,s) & 0 \\
\end{bmatrix} = I,
\]

completing the proof.

**Theorem A.2** We denote the $k$-th moment of $X_j$ ($j \in J$) by $\mu_{j,k}$, i.e.,

(A.2) \[
\mu_{j,k} \overset{\text{def}}{=} \int_0^\infty x^k a_j(x) dx; \quad \mu_{j,0} = 1.
\]
Then
\begin{equation}
\alpha_j(s) = 1 - s\mu_{j:1} + \frac{1}{2}s^2\mu_{j:2} + o(s^2) .
\end{equation}

\textbf{Proof} By applying the second order asymptotic expansion of $e^x$, one observes that
\[
\alpha_j(s) = - \int_0^\infty \left(1 - sx + \frac{1}{2}s^2x^2 + o(s^2)\right)a_j(x)dx = 1 - s\mu_{j:1} + \frac{1}{2}s^2\mu_{j:2} + o(s^2) .
\]
\hfill \Box

\textbf{Theorem A.3} Let $\mathcal{J}_i^j \equiv \{i, i + 1, i + 2, \cdots, j\}$ for $i \leq j$ and
\[
S_{i:j:1} \equiv \sum_{n=i}^{j} \mu_{n:1},
\]
\[
S_{i:j} \equiv \sum_{n=i}^{j} \mu_{n:2} + \sum_{n=i}^{j} \sum_{m \in \mathcal{J}_i^j \setminus \{n\}} \mu_{n:1}\mu_{m:1} .
\]

Then one has
\begin{equation}
\zeta_{ij}(0, s) = - sS_{i:j:1} + \frac{1}{2}s^2S_{i:j} + o(s^2) .
\end{equation}

\textbf{Proof} we prove (A.4) by induction. For $j = i$, one sees easily that
\[
\sum_{n=i}^{i} \mu_{n:1} = \mu_{i:1} , \quad \sum_{n=i}^{i} \mu_{n:2} = \mu_{i:2} , \quad \sum_{n=i}^{i} \sum_{m \in \mathcal{J}_i^i \setminus \{n\}} \mu_{n:1}\mu_{m:1} = 0 .
\]
Accordingly,
\[
\zeta_{ii}(0, s) = \alpha_i(s) = 1 - s\mu_{i:1} + \frac{1}{2}s^2\mu_{i:2} + o(s^2) .
\]
On the other hand, the right-hand side of (A.4) is equal to \(1 - s\mu_{i:1} + \frac{1}{2}s^2\mu_{i:2} + o(s^2)\). We assume (A.4). Then
\[
\zeta_{i,j+1}(0, s) = \zeta_{ij}(0, s) \cdot \alpha_{j+1}(s) .
\]
From (A.3) and (A.4),

\[
(A.5) \quad \zeta_{i,j+1}(0,s) = \{1 - sS_{ij+1} + \frac{1}{2}s^2S_{ij} + o(s^2)\}\{1 - s\mu_{j+1:1} + \frac{1}{2}s^2\mu_{j+1:2} + o(s^2)\}
\]

\[
= 1 - s(S_{ij+1} + \mu_{j+1:1}) + \frac{1}{2}s^2(\mu_{j+1:2} + 2S_{ij+1} + S_{ij}) + o(s^2).
\]

Here, we note that

\[
(A.6) \quad S_{ij+1} + \mu_{j+1:1} = S_{i,j+1:1}
\]

and

\[
(A.7) \quad \mu_{j+1:1} + 2S_{ij+1} + S_{ij} = \mu_{j+1:2} + 2\mu_{j+1:1} \sum_{n=i}^{j} \mu_{n:1} + \sum_{n=i}^{j} \mu_{n:2} + \sum_{n=i}^{j} \sum_{m \in \mathcal{J}_i^1 \setminus \{n\}} \mu_{n:1} \mu_{m:1}
\]

\[
= \sum_{n=i}^{j+1} \mu_{n:2} + \sum_{n=i}^{j+1} \sum_{m \in \mathcal{J}_i^{j+1} \setminus \{n\}} \mu_{n:1} \mu_{m:1} = S_{i,j+1}.
\]

By substituting (A.6) and (A.7) into (A.5), it can seen that

\[
\zeta_{i,j+1}(0,s) = 1 - sS_{i,j+1:1} + \frac{1}{2}s^2S_{i,j+1:1} + o(s^2).
\]

□

**Theorem A.4** For \(\zeta_{ij}(w,s)\) defined in (3.13), one has

\[
(A.8) \quad \left. \frac{\partial}{\partial w} \zeta_{ij}(w,s) \right|_{w=0} = C_{0:ij} + sC_{1:ij} + o(s)
\]

where

\[
C_{0:ij} \stackrel{\text{def}}{=} - \sum_{n=i}^{j} \{\rho(n)\mu_{n:1} + E[D_n]\},
\]

\[
C_{1:ij} \stackrel{\text{def}}{=} \sum_{n=i}^{j} \left[E[D_n]S_{i,j+1} + \rho(n)\{\mu_{n:1}(S_{i,n-1:1} + S_{n+1,j:1}) + \mu_{n:2}\}\right].
\]
Proof Differentiating (3.13) with respect to $w$ and setting $w = 0$ yields that

\begin{align}
(A.9) & \left. \frac{\partial}{\partial w} \zeta_{ij}(w, s) \right|_{w=0} \\
& = \left\{ \left. \frac{\partial}{\partial w} \alpha_i(s + \rho(i)w) \right|_{w=0} \right\} \alpha_{i+1}(s)\alpha_{i+2}(s) \cdots \alpha_j(s) \\
& + \left. \alpha_i(s) \left( \frac{\partial}{\partial w} \beta_i(w) \right) \right|_{w=0} \alpha_{i+1}(s)\alpha_{i+2}(s) \cdots \alpha_j(s) \\
& + \left. \alpha_i(s) \alpha_{i+1}(s) \left( \frac{\partial}{\partial w} \beta_{i+1}(w) \right) \right|_{w=0} \alpha_{i+2}(s) \cdots \alpha_j(s) + \cdots \\
& = \sum_{n=i}^{j} \left[ \left. \frac{\partial}{\partial w} \alpha_n(s + \rho(n)w) \right|_{w=0} + \alpha_n(s) \left( \frac{\partial}{\partial w} \beta_n(w) \right) \right|_{w=0} \right] \prod_{m \in J_i \setminus \{n\}} \alpha_m(s) .
\end{align}

Here, we note that

\begin{align}
(A.10) & \left. \frac{\partial}{\partial w} \alpha_n(s + \rho(n)w) \right|_{w=0} = \rho(n) \left\{ \frac{d}{ds} \alpha_n(s) \right\} .
\end{align}

Substituting (A.3) into (A.10),

\begin{align}
(A.11) & \left. \frac{\partial}{\partial w} \alpha_n(s + \rho(n)w) \right|_{w=0} = \rho(n) \cdot \frac{d}{ds} \left\{ 1 - s\mu_{n:1} + \frac{1}{2} s^2 \mu_{n:2} + o(s^2) \right\} \\
& = \rho(n) \left\{ -\mu_{n:1} + s\mu_{n:2} + o(s) \right\} .
\end{align}

where we recall

\begin{align}
(A.12) & \left. \frac{\partial}{\partial w} \beta_n(w) \right|_{w=0} = -E[D_n] .
\end{align}

By substituting (A.11) and (A.12) into (A.9), it then follows that

\begin{align}
& \left. \frac{\partial}{\partial w} \zeta_{ij}(w, s) \right|_{w=0} = \sum_{n=i}^{j} \left[ \rho(n) \left\{ -\mu_{n:1} + s\mu_{n:2} + o(s) \right\} - E[D_n]\alpha_n(s) \right] \prod_{m \in J_i \setminus \{n\}} \alpha_m(s) .
\end{align}

we can rewrite $\prod_{m \in J_i \setminus \{n\}} \alpha_m(s)$ in the above expression as

$$\prod_{m \in J_i \setminus \{n\}} \alpha_m(s) = \zeta_{i,n-1}(0, s)\zeta_{n+1,j}(0, s) .$$
Then one sees that

\[ (A.13) \quad \left. \frac{\partial}{\partial w} \zeta_{ij}(w, s) \right|_{w=0} \]

\[ = \sum_{n=1}^{j} \left[ \rho(n) \left\{ -\mu_{n:1} + s\mu_{n:2} + o(s) \right\} \zeta_{i,n-1}(0, s) \zeta_{n+1,j}(0, s) - E[D_n] \zeta_{ij}(0, s) \right]. \]

Substitution of (A.4) into (A.13) leads to

\[ (A.14) \quad \left. \frac{\partial}{\partial w} \zeta_{ij}(w, s) \right|_{w=0} \]

\[ = \sum_{n=1}^{j} \left[ \rho(n) \left\{ -\mu_{n:1} + s\mu_{n:2} + o(s) \right\} \left\{ 1 - sS_{i,n-1:1} + \frac{1}{2}s^2S_{i,n-1} + o(s^2) \right\} \right] \cdot \left\{ 1 - sS_{n+1,j:1} + \frac{1}{2}s^2S_{n+1,j} + o(s^2) \right\} - E[D_n] \left\{ 1 - sS_{ij:1} + \frac{1}{2}s^2S_{ij} + o(s^2) \right\}. \]

As for (A.14),

\[ (A.15) \quad \left\{ -\mu_{n:1} + s\mu_{n:2} + o(s) \right\} \left\{ 1 - sS_{i,n-1:1} + \frac{1}{2}s^2S_{i,n-1} + o(s^2) \right\} \]

\[ \cdot \left\{ 1 - sS_{n+1,j:1} + \frac{1}{2}s^2S_{n+1,j} + o(s^2) \right\} \]

\[ = \left\{ -\mu_{n:1} + s(\mu_{n:1}S_{i,n-1:1} + \mu_{n:2}) + o(s) \right\} \left\{ 1 - sS_{n+1,j:1} + \frac{1}{2}s^2S_{n+1,j} + o(s^2) \right\} \]

\[ = -\mu_{n:1} + s\left\{ \mu_{n:1}(S_{i,n-1:1} + S_{n+1,j:1}) + \mu_{n:2} \right\} + o(s). \]

Substituting (A.15) into (A.14), one has

\[ \left. \frac{\partial}{\partial w} \zeta_{ij}(w, s) \right|_{w=0} = \sum_{n=1}^{j} \left[ \rho(n) \left\{ -\mu_{n:1} + s\left\{ \mu_{n:1}(S_{i,n-1:1} + S_{n+1,j:1}) + \mu_{n:2} \right\} + o(s) \right\} \right] \]

\[ - E[D_n] \left\{ 1 - sS_{ij:1} + \frac{1}{2}s^2S_{ij} + o(s^2) \right\} \]

\[ = -\sum_{n=1}^{j} \{ \rho(n)\mu_{n:1} + E[D_n] \} \]

\[ + s \sum_{n=1}^{j} \left[ E[D_n]S_{ij:1} + \rho(n) \left\{ \mu_{n:1}(S_{i,n-1:1} + S_{n+1,j:1}) + \mu_{n:2} \right\} \right] + o(s) \]

\[ = C_{0:ij} + sC_{1:ij} + o(s). \]

\[ \square \]

**Theorem A.5** For the \( j \)-th diagonal element of (3.12), it follows that

\[ \frac{\partial}{\partial w} \left( 1 - \alpha_j(s + \rho(j)w) \right) \left|_{w=0} \right. = -\frac{1}{2}\rho(j)\mu_{j:2} + o(1). \]
Proof \ By differentiating $\frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w}$ and setting $w = 0$, one sees that

\begin{align}
\partial w \left\{ \frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w} \right\} \bigg|_{w=0} & = \left[ \frac{\partial}{\partial w} \{1 - \alpha_j(s + \rho(j)w)\} \right] \frac{s + \rho(j)w}{(s + \rho(j)w)^2} \bigg|_{w=0} \\
& = -\frac{\partial}{\partial w} \alpha_j(s + \rho(j)w) \bigg|_{w=0} \cdot \frac{s - \{1 - \alpha_j(s)\} \cdot \rho(j)}{s^2}.
\end{align}

Substitution of (A.11) and (A.3) into (A.16) leads to

\begin{align}
\partial w \left\{ \frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w} \right\} \bigg|_{w=0} & = -\rho(j) \left\{ \frac{-\mu_{j:1} + s\mu_{j:2} + o(s)}{s} \right\} \frac{s + 1 - \{1 - s\mu_{j:1} + \frac{1}{2}s^2\mu_{j:2} + o(s^2)\}}{s^2} \\
& = -\frac{1}{2}\rho(j)\mu_{j:2} + o(1).
\end{align}

$\square$