# On The Duality of Gaussian Multiple-Access and Broadcast Channels

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*Abstract* — **We show that the Gaussian multiple-access channel (MAC) and broadcast channel (BC) are duals. The dual channels we consider have the same channel gains and the same noise power at all receivers. We find an expression for the capacity region of the BC in terms of the capacity region of the dual MAC, and vice versa. Duality applies to many different channel models and capacity definitions.**

## I. System Model

We consider two different multi-user channels as shown in Fig. 1: an $M$-user broadcast channel (downlink), where a single transmitter sends independent information to $M$ different receivers, and an $M$-user multiple-access channel (uplink), where $M$ independent transmitters send information to a single receiver. The received signal at user $j$ in the BC is $y_j = h_j X + n_j$ and the received signal in the MAC is $y = \sum_{j=1}^{M} h_j X_j + n$, where $n_j \sim \mathcal{N}(0, \sigma^2)$ for all $j$ and $n \sim \mathcal{N}(0, \sigma^2)$ and all quantities are scalars (the vector case is analyzed in [2]). Notice that the channel gains on the uplink and downlink are the same and that the noise power at every receiver is $\sigma^2$. We call this BC the *dual* of the MAC, and vice versa. We use $\mathcal{C}_{BC}(\overline{P}; \boldsymbol{h})$ and $\mathcal{C}_{MAC}(\overline{\boldsymbol{P}}; \boldsymbol{h})$ to refer to the capacity region of the BC (power constraint $\overline{P}$) and MAC (vector power constraint $\overline{\boldsymbol{P}}$) for channel $\boldsymbol{h} = (h_1, \dots, h_M)$.

## II. Results

We first relate the capacity regions of the dual BC and MAC:

**Lemma 1** *The capacity region of the MAC with power constraints* $\boldsymbol{P}$ *is smaller than the capacity region of the dual BC with power constraint* $\boldsymbol{1} \cdot \boldsymbol{P}$, *or* $\mathcal{C}_{MAC}(\boldsymbol{P}; \boldsymbol{h}) \subseteq \mathcal{C}_{BC}(\boldsymbol{1} \cdot \boldsymbol{P}; \boldsymbol{h})$.

In fact, we can show a stronger result that characterizes the BC in terms of the dual MAC:

**Theorem 1** *The capacity region of a Gaussian BC with power* $\overline{P}$ *is equal to the union of capacity regions of the dual MAC with power* $(P_1, \dots, P_M)$ *such that* $\sum_{j=1}^{M} P_j = \overline{P}$:

$$\mathcal{C}_{BC}(\overline{P}; \boldsymbol{h}) = \bigcup_{\{\boldsymbol{P}:\ \boldsymbol{1} \cdot \boldsymbol{P} = \overline{P}\}} \mathcal{C}_{MAC}(\boldsymbol{P}; \boldsymbol{h}). \tag{1}$$

Proofs of all results are given in [1]. Theorem 1 indicates that the dual BC and MAC are equivalent when the MAC is given a *sum power constraint* instead of individual power constraints. Additionally, we find that points on the boundary of the BC region are achievable in the dual MAC using successive decoding with the strongest (i.e. largest channel gain) user decoded first, the *opposite* of the optimal decoding order used in the BC. Theorem 1 is illustrated in Fig. 2(a), where we see a number of MAC regions, each of which touches the boundary of the dual BC capacity region at exactly one point.

In order to characterize the capacity region of the MAC in terms of the dual BC, we use the concept of channel scaling

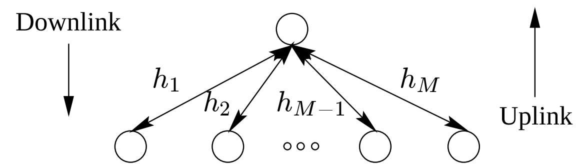
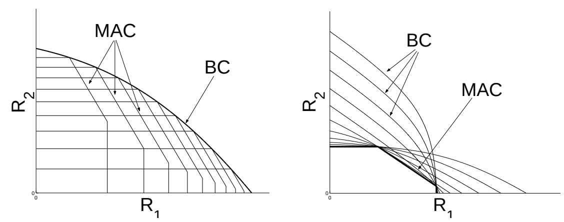



Figure 1: System Model




(a) MAC to BC (b) BC to MAC

Figure 2: Duality of the MAC and BC

in the MAC in which the gain of transmitter $j$ is multiplied by some constant $\sqrt{\alpha_j}$ and power constraint $P_j$ is divided by $\alpha_j$. Channel scaling does not affect MAC capacity (i.e. $\mathcal{C}_{MAC}(\boldsymbol{P}; \boldsymbol{h}) = \mathcal{C}_{MAC}(\frac{\boldsymbol{P}}{\boldsymbol{\alpha}}; \sqrt{\boldsymbol{\alpha}}\boldsymbol{h})$), but the capacity region of the dual BC is different for different values of $\boldsymbol{\alpha}$.

**Theorem 2** *The capacity region of a Gaussian MAC is equal to the intersection of the capacity regions of the scaled dual BC over all possible channel scalings:*

$$\mathcal{C}_{MAC}(\overline{\boldsymbol{P}}; \boldsymbol{h}) = \bigcap_{\boldsymbol{\alpha} > 0} \mathcal{C}_{BC}(\boldsymbol{1} \cdot \frac{\overline{\boldsymbol{P}}}{\boldsymbol{\alpha}}; \sqrt{\boldsymbol{\alpha}}\boldsymbol{h}). \tag{2}$$

In Fig. 2(b) we see a number of scaled BC capacity regions whose intersection is equal to the dual MAC capacity region.

Duality holds for fading channels as well [1]. For flat-fading channels duality holds for a number of different capacity definitions: ergodic capacity, outage capacity, and minimum rate capacity. In addition, a duality has been shown to exist between the capacity region of the MIMO MAC and an achievable region of the MIMO BC [2]. Using duality, results known for only one of the two channels can often be extended to the dual channel as well.

## References

[1] N. Jindal, S. Vishwanath, and A. Goldsmith, "On the Duality of Multiple-Access and Broadcast Channels," *Proc. Allerton Conf. on Commun., Computing, and Control*, Oct. 2001. Full paper in preparation.

[2] S. Vishwanath, N. Jindal, and A. Goldsmith, "On the Capacity of Multiple Input Multiple Output Broadcast Channels", *Proc. Int. Conf. Commun.*, April 2002.