

# An Improved Bound for Joints in Arrangements of Lines in Space\*

Sharona Feldman<sup>†</sup>      Micha Sharir<sup>‡</sup>

June 23, 2003

## Abstract

Let  $L$  be a set of  $n$  lines in space. A *joint* of  $L$  is a point in  $\mathbb{R}^3$  where at least three non-coplanar lines meet. We show that the number of joints of  $L$  is  $O(n^{112/69} \log^{6/23} n) = O(n^{1.6232})$ , improving the previous bound  $O(n^{1.643})$  of Sharir [11].

## 1 Introduction

Let  $L$  be a set of  $n$  lines in space. A *joint* of  $L$  is a point in  $\mathbb{R}^3$  where at least three non-coplanar lines  $\ell, \ell', \ell''$  of  $L$  meet. We denote the joint by the triple  $(\ell, \ell', \ell'')$  (observing that the same joint may be encoded by more than one such triple).

Let  $\mathcal{J}_L$  denote the set of joints of  $L$ , and put  $J(n) = \max |\mathcal{J}_L|$ , taken over all sets  $L$  of  $n$  lines in space. A trivial upper bound on  $J(n)$  is  $O(n^2)$ , but it was shown in [11], following a weaker subquadratic bound in [5], that  $J(n)$  is only  $O(n^{23/14} \text{polylog}(n)) = O(n^{1.643})$ . An easy construction, based on lines forming an  $n^{1/2} \times n^{1/2} \times n^{1/2}$  portion of the integer grid, shows that  $|\mathcal{J}_L|$  can be  $\Omega(n^{3/2})$  (see Figure 1 and [5]). The goal of this paper is to narrow the gap between these upper and lower bounds.

One of the main motivations for studying joints of a set  $L$  of lines in space is their connection to *elementary cycles* of  $L$ . An elementary cycle is a subset  $L'$  of at least three lines of  $L$  with the following properties: (i) The  $xy$ -projections of the lines in  $L'$  all bound a common face in the arrangement of the  $xy$ -projections of the lines in  $L$ . (ii) As we go around the boundary of the common face, we always pass from the projection of one line  $\ell$  to the projection of another line  $\ell'$  such that  $\ell'$  passes above  $\ell$  in 3-space. See Figure 2.

A major open problem in the study of visibility in three dimensions is to obtain a subquadratic bound on the number of elementary cycles of a set of lines in  $\mathbb{R}^3$ . Joints can

---

\*Work on this paper has been supported by a grant from the U.S.–Israel Binational Science Foundation, by NSF Grants CCR-97-32101 and CCR-00-98246. by a grant from the Israeli Academy of Sciences for a Center of Excellence in Geometric Computing at Tel Aviv University, and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University.

<sup>†</sup>School of Computer Science, Tel-Aviv University, Tel-Aviv, 69978, Israel; *feldman@post.tau.ac.il*. Work on this paper was done while the first author was a Ph.D. student under the supervision of the second author.

<sup>‡</sup>School of Computer Science, Tel-Aviv University, Tel-Aviv, 69978, Israel, and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA; *michas@post.tau.ac.il*.

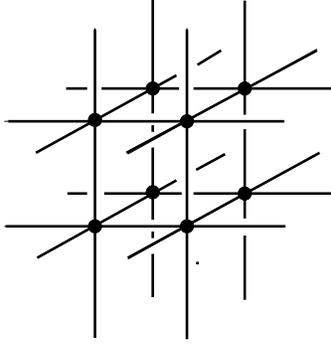


Figure 1: The lower bound construction for joints.

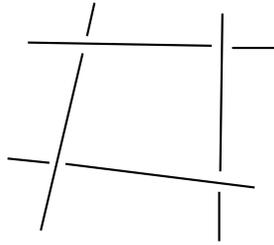


Figure 2: An elementary cycle of lines in space.

be regarded as a degenerate case of elementary cycles. In fact, a slight random perturbation of the lines in  $L$  turns any joint incident to  $O(1)$  lines into an elementary cycle with some constant probability, implying that the number of joints is strongly related to the number of elementary cycles.

Unfortunately, very little is known about the number of elementary cycles. Chazelle et al. [5] obtained a bound of  $O(n^{9/5})$  for the special case of line segments (rather than lines) whose  $xy$ -projections form a (distorted) grid. Recently, Aronov et al. [1] obtained a bound of  $O(n^{2-1/69+\varepsilon})$ , for any  $\varepsilon > 0$ , on the number of *triangular elementary cycles* (i.e., cycles formed by only three lines) for general line arrangements. Solan [13] and Har-Peled and Sharir [9] have given algorithms that eliminate all (not necessarily elementary) cycles of a set of lines in space, by cutting the lines at appropriate points. These algorithms run in subquadratic time, and cut the lines in a subquadratic number of points, provided that a subquadratic bound on the number of elementary cycles is known.

The problem of joints is considerably simpler, as witnessed by the much sharper upper bound of [11], mentioned above. Still, it is a rather challenging problem, open for 10 years, to tighten the gap between the upper and lower bounds. It is our hope that better insights into the joints problem would lead to tools that could also be used to obtain subquadratic bounds for elementary cycles, and for many other problems that involve lines in space. Recently, Sharir and Welzl [12] have shown that the number of incidences between the points in  $\mathcal{J}_L$  and the lines in  $L$  is  $O(n^{5/3})$ .

In this paper we improve the upper bound on  $J(n)$  to  $O(n^{112/69} \log^{6/23} n) = O(n^{1.6232})$ . The proof proceeds by mapping the lines of  $L$  into points and/or hyperplanes in projective

5-space, using *Plücker coordinates* [6]. We then apply a two-stage decomposition process, which partitions the problem into subproblems, using *cuttings* of arrangements of appropriate subsets of the Plücker hyperplanes. We estimate the number of joints within each subproblem, and sum up the resulting bounds to obtain the bound asserted above. The proof adapts and applies some of the tools used by Sharir and Welzl [12] and recently enhanced by Aronov and Sharir [3], related mainly to the connection between joints and *reguli* spanned by the lines of  $L$ ; see below for more details.

## 2 The Upper Bound

### 2.1 The toolbox

We begin by recalling and developing some of the tools we need for our proof.

**Szemerédi-Trotter bound [15].** Given a set  $L$  of  $n$  lines and a set  $P$  of  $m$  points, both in a common (2-dimensional) plane, we have

$$I(P, L) = O(n^{2/3}m^{2/3} + n + m) . \quad (1)$$

(This bound is tight in the worst case.) We use this bound to prove

**Lemma 2.1.** *Let  $L$  be a set of  $n$  lines in space. The number of planes that contain at least  $k$  lines of  $L$  is*

$$O\left(\frac{n^2}{k^3} + \frac{n}{k}\right),$$

and the number of containments between the lines of  $L$  and these planes is

$$O\left(\frac{n^2}{k^2} + n\right).$$

**Proof:** Let  $H$  be a set of  $t$  planes and  $L$  a set of  $n$  lines in  $\mathbb{R}^3$ . Draw a generic plane  $\pi$  in  $\mathbb{R}^3$  that meets every plane in  $H$  at a line and meets every line in  $L$  at a point. The number  $I(L, H)$  of containments between lines of  $L$  and planes of  $H$  is at most the number of incidences between the resulting  $n$  points and  $t$  lines in  $\pi$ . Applying the Szemerédi-Trotter bound (1), the number of these incidences, and thus  $I(L, H)$ , is

$$O(n^{2/3}t^{2/3} + t + n). \quad (2)$$

Now let  $H = H_{\geq k}$  be the set of all planes that contain at least  $k$  lines of  $L$ , and put  $t = |H_{\geq k}|$ . We clearly have  $I(L, H_{\geq k}) \geq tk$ . Combining this with (2) yields

$$tk = O(n^{2/3}t^{2/3} + t + n),$$

or

$$t = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right).$$

Substituting this bound in (2), we obtain

$$I(L, H_{\geq k}) = O\left(\left(\frac{n^2}{k^3} + \frac{n}{k}\right)^{2/3} n^{2/3} + \frac{n^2}{k^3} + \frac{n}{k} + n\right) = O\left(\frac{n^2}{k^2} + \frac{n^{4/3}}{k^{2/3}} + n\right),$$

and the middle term is always dominated by the first or the third term, as is easily verified.  $\square$

**Reguli (see [14]).** Two lines in  $\mathbb{R}^3$  that are disjoint and not parallel are called *skew*. Given three pairwise skew lines  $\ell_1, \ell_2, \ell_3$ , the set  $\sigma = \sigma(\ell_1, \ell_2, \ell_3)$  of lines intersecting all three lines is called a *regulus*. All lines in  $\sigma$  are pairwise skew. If  $\ell'_1, \ell'_2, \ell'_3$  are in  $\sigma$ , then  $\sigma^\perp = \sigma(\ell'_1, \ell'_2, \ell'_3)$  constitutes another regulus, that is independent of the choice of the three lines in  $\sigma$ . (Note that the three generating lines  $\ell_1, \ell_2, \ell_3$  of  $\sigma$  *do not* belong to  $\sigma$ , but rather to  $\sigma^\perp$ .)

$\bigcup_{\ell \in \sigma} \ell = \bigcup_{\ell \in \sigma^\perp} \ell$  is a *ruled surface* (which is a quadric—a hyperboloid of one sheet or a hyperbolic paraboloid) in  $\mathbb{R}^3$ , denoted by  $\sigma^* = \sigma^*(\ell_1, \ell_2, \ell_3)$ ;  $\sigma$  and  $\sigma^\perp$  are called the *generating families of  $\sigma^*$*  and we say that  $\sigma^\perp$  is the *complementary regulus of  $\sigma$* , and vice versa:  $(\sigma^\perp)^\perp = \sigma$ . Every point in  $\sigma^*$  is contained in exactly one line from  $\sigma$  and in exactly one line from  $\sigma^\perp$ . For any line  $\ell$  in  $\mathbb{R}^3$ , either  $\ell \in \sigma \cup \sigma^\perp$  (i.e.,  $\ell \subseteq \sigma^*$ ), or  $\ell$  intersects  $\sigma^*$  in at most two points.

It follows that the number of joints in  $L$  that lie on the surface of any regulus  $\sigma$  is at most

$$\min\{|L \cap \sigma| \cdot |L \cap \sigma^\perp|, 2|L|\}.$$

This follows from the observation that at most two of the lines that form such a joint can lie in  $\sigma^*$ , and the third line must cross  $\sigma^*$ . This allows us to apply the following pruning procedure. We fix a parameter  $s$ , whose value will be determined later. As long as there exists a regulus  $\sigma$  that contains more than  $s$  lines of  $L$ , we remove all these lines from  $L$ , and lose in this process at most  $2n$  joints. Repeating this step at most  $n/s$  times, we get rid of all “heavy” reguli and lose at most  $O(n^2/s)$  joints.

A similar pruning process can be applied to planes that contain more than  $s$  lines of  $L$ . Here we use the fact that any plane can contain at most  $n$  joints, because any such joint must be incident to at least one line that is not contained in the plane, and thus meets it in a single point.

To recap, we may (and will) assume in what follows that no plane or regulus contains more than  $s$  lines of  $L$ , and will add  $O(n^2/s)$  to the overall bound for the number of joints.

**Incidences between lines and reguli [3].** Given a set  $L$  of  $m$  lines and a set  $R$  of  $n$  reguli in 3-space, the number  $I(L, R)$  of incidences between the lines of  $L$  and the reguli of  $R$  (recall that we regard a regulus as a set of lines and not as the surface that they span) satisfies

$$I(L, R) = O(m^{4/7} n^{17/21} + m^{2/3} n^{2/3} + n + m). \quad (3)$$

This has recently been shown by Aronov and Sharir [3]. It extends and improves a weaker bound of  $O(m^{3/5} n^{4/5} + m + n)$  proved in [12] for a special case.

We use this to prove:

**Lemma 2.2.** *Let  $L$  be a set of  $n$  lines in space. The number of reguli that contain at least  $k$  lines of  $L$  is*

$$O\left(\frac{n^3}{k^{21/4}} + \frac{n^2}{k^3} + \frac{n}{k}\right),$$

and the number of incidences between the lines of  $L$  and these reguli is

$$O\left(\frac{n^3}{k^{17/4}} + \frac{n^2}{k^2} + n\right).$$

**Proof:** Let  $R_{\geq k}$  denote the set of these reguli, and put  $t = |R_{\geq k}|$ . The bound (3) implies that

$$tk \leq I(L, R_{\geq k}) = O(n^{4/7}t^{17/21} + n^{2/3}t^{2/3} + n + t),$$

and the rest of the analysis proceeds in complete analogy with the proof of Lemma 2.1.  $\square$

**Mapping into Plücker space.** Let  $L$  be a set of  $n$  lines in  $\mathbb{R}^3$ . We may assume, without loss of generality, that no pair of lines in  $L$  are parallel. This can be enforced by an appropriate projective transformation that maps  $L$  to another set of lines that does not have parallel pairs, without changing the incidence structure between the lines and their joints.

We start by replicating the set of lines  $L$  into two sets, color one set as blue, and the other as red. We bound the number of points at which a red line and two blue lines, not in the same plane, meet.<sup>1</sup>

We map each blue line  $\ell$  to its *Plücker hyperplane*  $\pi_\ell$ , and map each red line  $\ell$  into its *Plücker point*  $p_\ell$ . Both points and hyperplanes lie in projective 5-space, and the points all lie in a 4-dimensional quadric surface  $\Pi$  known as the *Plücker surface*. Two lines  $\ell, \ell' \in L$  meet each other if and only if  $p_\ell$  lies on  $\pi_{\ell'}$  (and  $p_{\ell'}$  lies on  $\pi_\ell$ ). See [6] for more details on this transformation.

**Cuttings.** Let  $\Gamma$  be a set of  $n$  algebraic arcs or curves in the plane, of constant maximum degree, and let  $1 \leq r \leq n$  be a parameter. A  $(1/r)$ -*cutting* of the arrangement  $\mathcal{A}(\Gamma)$  of  $\Gamma$  is a partition of  $\mathbb{R}^2$  into pairwise disjoint relatively open cells<sup>2</sup> of dimensions 0,1,2, such that each cell is *crossed by* (i.e., intersected by, but not contained in) at most  $n/r$  curves of  $\gamma$ . The *size* of the cutting is the number of its cells. It has been shown (see [4, 8]) that there always exists a  $(1/r)$ -cutting of size  $O(r^2)$ , which is asymptotically optimal.

The notion of cuttings can be extended in an obvious manner to arrangements of surfaces in higher dimensions. In general, however, optimal or near-optimal bounds for the size of the cuttings are harder to derive, and in most cases are not yet known. Still, in the case of hyperplanes in  $\mathbb{R}^d$ , there exist  $(1/r)$ -cuttings, whose cells are simplices, of optimal size  $O(r^d)$  [4]. In our analysis, we repeatedly rely on a variant of this result, in which we need to construct  $(1/r)$ -cuttings for a four-dimensional cross-section (within the Plücker surface) of an arrangement of hyperplanes in projective 5-space; see below for more details.

## 2.2 The primal partitioning stage

We construct a  $(1/r)$ -*cutting*  $\Xi$  of the arrangement of the set  $H$  of the Plücker hyperplanes, or, more precisely, of its cross section within the Plücker surface  $\Pi$ . The cutting is obtained

---

<sup>1</sup>In the first decomposition stage the colors play no significant role, but they will be more meaningful in the second decomposition stage, where each subproblem will involve two different subsets of  $L$ .

<sup>2</sup>In the standard definition of a cutting, the cells are required to have *constant descriptive complexity*, meaning that each of them is defined by a constant number of polynomial equalities and inequalities, involving polynomials of constant maximum degree. In our applications, though, this additional property is not needed.

by taking a random sample  $R$  of  $r$  hyperplanes of  $H$ , by triangulating each cell of  $\mathcal{A}(R)$ , and by taking the cross sections within  $\Pi$  of the resulting simplices. The actual construction is somewhat more involved, and follows the technique of Chazelle and Friedman [4], which uses additional samplings within some of the cells constructed above.<sup>3</sup> Omitting the routine details, we end up with a larger sample, which we still denote by  $R$ , consisting of  $O(r)$  hyperplanes, and yielding a cutting that consists of  $O(r^4 \log r)$  cells of constant descriptive complexity (each cell is the intersection of some  $j$ -simplex, for  $1 \leq j \leq 5$ , with  $\Pi$ ), so that each cell is crossed by at most  $n/r$  blue Plücker hyperplanes. (The size of the cutting is a consequence of the Zone theorem of Aronov et al. [2], which implies that the complexity of the zone of  $\Pi$  in  $\mathcal{A}(R)$  is  $O(r^4 \log r)$ , from which it follows that the cells of  $\mathcal{A}(R)$  that are crossed by  $\Pi$  can be triangulated into  $O(r^4 \log r)$  simplices.) Moreover, by splitting cells into subcells, if necessary, we may also assume that each cell contains at most  $n/(r^4 \log r)$  red Plücker points. (Recall that lower-dimensional cells may be *contained* in many more blue hyperplanes, but each is *crossed* by at most  $n/r$  of them.)

We now bound the number of red-blue-blue joints by applying a case analysis on the location, within the cutting  $\Xi$ , of the Plücker point of the red line in the joint.

**Vertices of  $\Xi$ .** Consider a joint  $(\ell_1, \ell_2, \ell_3)$ , for  $\ell_1, \ell_2, \ell_3 \in L$ , such that  $p_{\ell_1}$  is a vertex of  $\Xi$ . The number of such joints is at most the sum  $\sum_v d_v$ , where the sum is over the vertices  $v$  of  $\Xi$  and  $d_v$  is the number of lines  $\ell \in L$  such that  $\pi_\ell$  passes through  $v$ . We denote the set of these lines as  $L^{(v)}$ . We may assume that  $v$  is a vertex formed as the transversal intersection of  $\Pi$  with four hyperplanes of  $R$ . Any other vertex of  $\Xi$  will not coincide with a Plücker point  $p_\ell$ , for  $\ell \in L$ , provided that the triangulation is performed in a sufficiently generic manner.<sup>4</sup>

We fix a hyperplane  $\pi_\ell$ , for  $\ell \in L^{(v)}$ , and intersect it with all hyperplanes of  $R$  and with  $\Pi$ . Since the four hyperplanes of  $R$  that form the vertex  $v$  intersect there transversally, their cross sections within  $\pi_\ell \cap \Pi$  also intersect transversally at  $v$ , so this point is a vertex of the 3-dimensional arrangement of these cross sections. The number of such vertices, within  $\pi_\ell \cap \Pi$ , is at most  $O(r^3)$ , for a total bound of  $O(nr^3)$  on the number of joints at vertices of  $\Xi$ .

**Edges of  $\Xi$ .** Let  $\gamma$  be an intersection curve of three hyperplanes of  $R$  with  $\Pi$ . (As in the case of vertices, only edges of  $\Xi$  contained in such curves are of interest, if the triangulation is sufficiently generic. Note also that we consider here full intersection curves, each consisting of many edges of  $\Xi$ .) Let  $\ell_1, \ell_2, \ell_3$  be the three corresponding lines of  $L$ . Suppose first that these lines are pairwise skew and thus form a regulus  $\sigma$ . Let  $\ell \in L$  be such that  $p_\ell \in \gamma$ . Then  $\ell$  lies in  $\sigma^*$  (and belongs to  $\sigma$ ). Let  $(\ell, \ell', \ell'')$  be a joint that involves  $\ell$ . It is impossible that both  $\pi_{\ell'}, \pi_{\ell''}$  fully contain  $\gamma$ , because then  $\ell', \ell''$  would belong to  $\sigma^\perp$  and thus would not meet at all. Hence, say,  $\pi_{\ell'}$  crosses  $\gamma$ , and  $\ell'$  crosses  $\sigma^*$ , in at most two points. In other words, we can charge the joint under consideration to one of these crossing points of  $\ell'$  with  $\sigma^*$ . The number of such crossings is at most  $2n$  for each regulus  $\sigma$ , for a total of  $O(nr^3)$  joints.

---

<sup>3</sup>It might be simpler to digest the following analysis by ignoring the Chazelle-Friedman refinement. This will only affect the polylogarithmic factor appearing in the overall bound.

<sup>4</sup>E.g., each cell can be triangulated into simplices, all emanating from some common generic point in the relative interior of the cell.

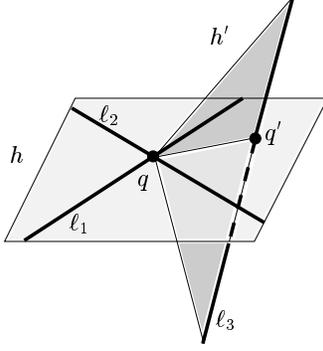


Figure 3: The pair of planes corresponding to an edge of  $\Xi$ .

Suppose next that two of the lines, say  $\ell_1, \ell_2$ , meet each other. Thus they define a common plane  $h$  and a common point  $q$ . If the third line  $\ell_3$  lies in  $h$  or passes through  $q$  then the intersection  $\pi_{\ell_1} \cap \pi_{\ell_2} \cap \pi_{\ell_3}$  is two-dimensional, as is easily seen, so these three lines do not define an *edge* of  $\Xi$ . Hence  $\ell_3$  meets  $h$  at a single point  $q' \neq q$ . It follows that any line  $\ell$  with  $p_\ell \in \gamma$  either lies in  $h$  and passes through  $q'$ , or passes through  $q$  and through  $\ell_3$ , and thus lies in the plane  $h'$  spanned by  $q$  and  $\ell_3$ . See Figure 3. In other words, any joint on  $\ell$  lies in  $h \cup h'$ , and at least one of the three lines forming the joint must cross  $h$  or  $h'$  at the joint. There are at most  $2n$  such crossing points, so the number of joints in this case is at most  $2n$ , for a total of  $O(nr^3)$  joints.

**2-Faces of  $\Xi$ .** Let  $\varphi$  be an intersection 2-surface of two hyperplanes of  $R$  with  $\Pi$  (again, only 2-faces of  $\Xi$  that lie in such 2-surfaces are of interest, and we consider full intersection 2-faces rather than individual 2-faces), and let  $\ell_1, \ell_2$  be the two corresponding lines of  $L$ . Suppose first that  $\ell_1, \ell_2$  pass through a common point  $q$ , and thus lie in a common plane  $h$ . Then any line  $\lambda$  with  $p_\lambda \in \varphi$  either lies in  $h$  or passes through  $q$ . We can thus view  $\varphi$  as the union of two sub-surfaces  $\varphi_q, \varphi_h$ , where  $\varphi_q$  (resp.,  $\varphi_h$ ) is the locus of all (points representing) lines passing through  $q$  (resp., lying in  $h$ ).

Let  $(\ell, \ell', \ell'')$  be a joint where  $p_\ell \in \varphi_q$ . We may assume that  $p_\ell$  does not lie on any edge of  $\Xi$  that is contained in  $\varphi$ , because such points have already been accounted for. If  $\pi_{\ell'}$ , say, fully contains  $\varphi_q$  then  $\ell'$  must pass through  $q$  (since it touches every line that passes through  $q$ ), and thus the joint in question must be the point  $q$  itself. The overall number of such joints is only  $O(r^2)$ . We may thus assume that both  $\pi_{\ell'}$  and  $\pi_{\ell''}$  cross  $\varphi_q$ .

Similarly, let  $(\ell, \ell', \ell'')$  be a joint where  $p_\ell \in \varphi_h$ . If  $\pi_{\ell'}$ , say, fully contains  $\varphi_h$  then  $\ell'$  must lie in  $h$ . In this case, the joint must lie in  $h$ . As we have already noted,  $h$  contains at most  $n$  joints, so the overall number of joints of this kind is at most  $O(nr^2)$ . We may thus assume that both  $\pi_{\ell'}$  and  $\pi_{\ell''}$  cross  $\varphi_h$ .

Thus, in either case, we are left with subproblems, each associated with a 2-face  $\tau$  of  $\Xi$  (the surface  $\varphi$  is now decomposed back into its constituent 2-faces), such that  $\tau$  contains at most  $n/(r^4 \log r)$  red Plücker points and is crossed by at most  $n/r$  blue Plücker hyperplanes; the problem associated with  $\tau$  considers red-blue-blue joints where the red point lies in  $\tau$  and both blue hyperplanes cross  $\tau$ . The number of subproblems is  $O(r^4 \log r)$ . We will handle these subproblems in the second dual stage of the analysis—see below.

Finally, suppose that  $\ell_1$  and  $\ell_2$  are skew. Consider a joint  $(\ell, \ell', \ell'')$ , where  $p_\ell \in \varphi$ . Neither of the hyperplanes  $\pi_{\ell'}$ ,  $\pi_{\ell''}$  can fully contain  $\varphi$ , because then the corresponding line would have to be incident to every line that meets  $\ell_1$  and  $\ell_2$ , which is clearly impossible. Hence, in this case we obtain, as above, a collection of subproblems, each associated with a 2-face  $\tau$  of  $\Xi$  (a subface of  $\varphi$ ), such that  $\tau$  contains at most  $n/(r^4 \log r)$  red Plücker points and is crossed by at most  $n/r$  blue Plücker hyperplanes. As above, the number of subproblems is  $O(r^4 \log r)$ , and they are all handled in the second dual stage of the analysis.

**3-Faces of  $\Xi$ .** Let  $p_\ell$  be a point in the relative interior of some 3-face of  $\Xi$ , contained in the intersection of  $\Pi$  with some hyperplane  $\pi_{\ell_1}$  in  $R$  (only such 3-faces are of interest). Any hyperplane incident to  $p_\ell$ , with the exception of  $\pi_{\ell_1}$ , crosses each of the two adjacent cells of  $\Xi$ . We can thus assign  $p_\ell$  to either of these cells, and count the joints on  $\ell$  as part of the subproblem associated with that cell (losing in the reduction a total of at most  $n$  joints). Thus no special treatment is needed for points on 3-faces of  $\Xi$ . Alternatively, we can regard each 3-face  $\tau$  as yielding a subproblem of its own, involving the (at most  $n/(r^4 \log r)$ ) red points that it contains and the (at most  $n/r$ ) blue hyperplanes that cross it. The number of subproblems is  $O(r^4 \log r)$  and they are handled in the subsequent dual stage.

**Cells of  $\Xi$ .** As in the case of 2-faces and 3-faces, each cell  $\tau$  of  $\Xi$  generates a subproblem involving the at most  $n/(r^4 \log r)$  red Plücker points in  $\tau$  and the at most  $n/r$  blue Plücker hyperplanes that cross  $\tau$ . There are  $O(r^4 \log r)$  subproblems of this kind.

### 2.3 The dual partitioning stage

Let  $\tau$  be a cell of  $\Xi$ ; we include here also the cases where  $\tau$  is a 2-face or a 3-face of  $\Xi$ , and only hyperplanes that cross  $\tau$  are considered. Let  $L_\tau$  be the set of all lines  $\ell \in L$  such that  $p_\ell \in \tau$ , and let  $L'_\tau$  be the set of all lines  $\ell \in L$  such that  $\pi_\ell$  crosses  $\tau$ ; we have  $|L_\tau| \leq n/(r^4 \log r)$  and  $|L'_\tau| \leq n/r$ . We “dualize” the problem, by mapping the lines of  $L_\tau$  to (red) Plücker hyperplanes and lines of  $L'_\tau$  to (blue) Plücker points in projective 5-space. Recall that we consider here joints  $(\ell_1, \ell_2, \ell_3)$  where  $\ell_1 \in L_\tau$ ,  $\ell_2, \ell_3 \in L'_\tau$ . Since both  $\ell_2, \ell_3$  are mapped to (distinct) points, the triple interaction of  $\ell_1, \ell_2, \ell_3$  is not localized at any point of this dual parametric 5-space. We therefore do not consider at all any triple interaction at this stage. Instead, we charge the joint in question simply to the incidence between  $p_{\ell_2}$  and  $\pi_{\ell_1}$ , or to the incidence between  $p_{\ell_3}$  and  $\pi_{\ell_1}$ . Clearly, this count is a (probably gross) overestimate of the number of joints under consideration.<sup>5</sup>

We construct a  $(1/r)$ -cutting  $\Xi'_\tau$  of the cross section within  $\Pi$  of the hyperplanes  $\pi_\ell$ , for  $\ell \in L_\tau$ , using, as above, a generic triangulation of the arrangement  $\mathcal{A}(R_\tau)$ , for an appropriate sample  $R_\tau$  of  $O(r)$  of these hyperplanes. As above, the size of  $\Xi'_\tau$  is  $O(r^4 \log r)$ , and we may assume that each of its cells  $\tau'$  contains at most  $(n/r)/(r^4 \log r) = n/(r^5 \log r)$  blue Plücker points  $p_\ell$ , for  $\ell \in L'_\tau$ , and is crossed by at most  $(n/(r^4 \log r))/r = n/(r^5 \log r)$  red Plücker hyperplanes  $\pi_\ell$ , for  $\ell \in L_\tau$ .

We proceed to bound the number of incident pairs  $(p_{\ell_2}, \pi_{\ell_1})$ , for  $\ell_2 \in L'_\tau$ ,  $\ell_1 \in L_\tau$ , applying a case analysis on the location of  $p_{\ell_2}$  in  $\Xi'_\tau$ .

---

<sup>5</sup>Arguably, this is one of the weak spots of our analysis. Any method of ‘preserving’ the triple interactions at joints would likely lead to an improved bound on  $J(n)$ .

**Vertices of  $\Xi'_\tau$ .** Consider a joint  $(\ell_1, \ell_2, \ell_3)$  where  $\ell_1 \in L_\tau$ ,  $\ell_2, \ell_3 \in L'_\tau$ , such that  $p_{\ell_2}$ , say, is a vertex of  $\Xi'_\tau$ . As in the primal stage, the number of such joints is at most the sum  $\sum_v d_v$ , taken over the vertices  $v$  of  $\Xi'_\tau$ , where  $d_v$  is the number of red lines  $\ell \in L_\tau$  such that  $\pi_\ell$  passes through  $v$ . We denote the set of these lines as  $L_\tau^{(v)}$ . As in the primal stage, only vertices  $v$  incident to four hyperplanes of  $R_\tau$  that meet there transversally need to be considered.

We fix a hyperplane  $\pi_\ell$  for  $\ell \in L_\tau^{(v)}$  and intersect it with all hyperplanes of  $R_\tau$  and with  $\Pi$ . Since the four hyperplanes of  $R_\tau$  that form the vertex  $v$  intersect there transversally, their cross sections within  $\pi_\ell \cap \Pi$  also intersect transversally at  $v$ , so that this point is a vertex of the 3-dimensional arrangement of these cross sections. The number of such vertices, within  $\pi_\ell \cap \Pi$ , is at most  $O(r^3)$ , for a total of  $O(r^3 \cdot \frac{n}{r^4 \log r})$ , which, multiplied by the number of cells  $\tau$ , yields a bound of  $O(nr^3)$  on the number of joints at vertices of the cuttings  $\Xi'_\tau$ .

**Regulus edges of  $\Xi'_\tau$ .** This is the most intricate part of our analysis. Let  $\gamma$  be an intersection curve of three hyperplanes of  $R_\tau$  with  $\Pi$ , representing three respective lines  $\ell_1, \ell_2, \ell_3$  (again, only such curves are of interest). Suppose first that these lines are pairwise skew, so that they form a regulus  $\sigma$ . Let  $M_\sigma$  (resp.,  $M'_\sigma$ ) denote the number of lines  $\ell$  of  $L_\tau$  (resp., of  $L'_\tau$ ) that are contained in  $\sigma^\perp$  (resp., in  $\sigma$ ); in 5-space these are lines for which  $\pi_\ell$  contains  $\gamma$  (resp.,  $p_\ell$  lies in  $\gamma$ ). We need to bound the number of incident pairs of lines  $(\ell, \ell') \in L_\tau \times L'_\tau$ , such that  $p_{\ell'} \in \gamma$ . We do not include in this count lines  $\ell' \in L'_\tau$  whose points  $p_{\ell'}$  are vertices of  $\Xi'_\tau$ , since they have already been accounted for. We distinguish between the case where  $\pi_\ell$  contains  $\gamma$  and the case where  $\pi_\ell$  crosses  $\gamma$ .

Consider first the case where  $\pi_\ell$  contains  $\gamma$ , so  $\ell \in \sigma^\perp$ . A trivial upper bound on the number of joints under consideration (or, rather, the number of incident pairs  $(\ell, \ell')$ , as above) is  $M_\sigma \cdot M'_\sigma$ . Our next steps proceed by case analysis on the values of  $M_\sigma$  and  $M'_\sigma$ , which uses two threshold values  $s, t$  that we will specify later, where  $s$  is the parameter used in the process of pruning away heavy reguli and planes, applied at the beginning of the analysis.

**(a)  $M_\sigma \leq t$ :** In this case we bound the number of joints by  $t \sum_\sigma M'_\sigma$ , where the sum extends over all reguli  $\sigma$  with this property. Since, in 5-space,  $M'_\sigma$  counts points that lie on the curves representing the reguli and each point is counted only once (since we exclude vertices of the cutting), the above sum is at most  $tn/r$ . Summed over all cells  $\tau$ , this yields an overall bound of  $O(nr^3 t \log r)$  joints (which already dominates the bounds  $O(nr^3)$  obtained for the vertices of the dual cuttings, as well as for the vertices and edges of the primal cutting).<sup>6</sup>

**(b)  $M_\sigma > t$ :** By the initial pruning process, we may assume that  $M'_\sigma \leq s$ . In this case we use Lemma 2.2 to conclude that the number of reguli  $\sigma$  for which  $M_\sigma > t$  (for the fixed cell  $\tau$ ) is at most

$$O \left( \left( \frac{\left( \frac{n}{r^4 \log r} \right)^3}{t^{21/4}} + \frac{\left( \frac{n}{r^4 \log r} \right)^2}{t^3} + \frac{\frac{n}{r^4 \log r}}{t} \right) = O \left( \frac{n^3}{r^{12} t^{21/4} \log^3 r} + \frac{n^2}{r^8 t^3 \log^2 r} + \frac{n}{r^4 t \log r} \right),$$

---

<sup>6</sup>This is one of the two ‘weak spots’ in our analysis—see the discussion at the end of the paper.

and the sum  $\sum_{\sigma} M_{\sigma}$ , over these reguli  $\sigma$ , is

$$O\left(\frac{n^3}{r^{12}t^{17/4}\log^3 r} + \frac{n^2}{r^8t^2\log^2 r} + \frac{n}{r^4\log r}\right).$$

Multiplying by  $s$  and by the number of cells  $\tau$ , we obtain the bound

$$O\left(\frac{n^3s}{r^8t^{17/4}\log^2 r} + \frac{n^2s}{r^4t^2\log r} + ns\right)$$

on the number of joints under consideration.

Consider next the case where  $\pi_{\ell}$  crosses  $\gamma$ . We split  $\gamma$  into the edges of the cutting that it is comprised of, and repeat this for all curves  $\gamma$  that represent reguli. This yields a collection of  $O(r^4 \log r)$  subproblems, each associated with an edge  $\tau'$  of  $\Xi'_{\tau}$ , such that  $\tau'$  contains at most  $n/(r^5 \log r)$  blue Plücker points of  $L'_{\tau}$  and is crossed by at most  $n/(r^5 \log r)$  red Plücker hyperplanes of  $L_{\tau}$ . Any joint under consideration is an intersection point of two lines, one mapped into one of these Plücker points and the other into one of these Plücker hyperplanes. Hence the number of these joints is at most  $O((n/(r^5 \log r)) \cdot (n/(r^5 \log r))) = O(n^2/(r^{10} \log^2 r))$ . The overall bound on the number of joints of this kind, summed over all such edges  $\tau'$  of  $\Xi'_{\tau}$ , and over all cells  $\tau$  of the primal cutting, is

$$O\left((r^4 \log r)^2 \cdot \frac{n^2}{r^{10} \log^2 r}\right) = O\left(\frac{n^2}{r^2}\right).$$

**Non-regulus edges of  $\Xi'_{\tau}$ .** Suppose next that two of the lines that define the intersection curve, say  $\ell_1, \ell_2$ , meet each other. Thus they define a common plane  $h$  and a common point  $q$ . If the third line  $\ell_3 \in L_{\tau}$  lies in  $h$  or passes through  $q$  then the intersection  $\pi_{\ell_1} \cap \pi_{\ell_2} \cap \pi_{\ell_3} \cap \Pi$  is two-dimensional, so these three lines do not define an *edge* of  $\Xi'_{\tau}$ . Hence  $\ell_3$  meets  $h$  at a single point  $q' \neq q$ . It follows (cf. Figure 3) that any line  $\ell$  with  $p_{\ell} \in \gamma$  either lies in  $h$  and passes through  $q'$ , or it passes through  $q$  and through  $\ell_3$ , and thus lies in the plane  $h'$  spanned by  $q$  and  $\ell_3$ . In other words, any joint on  $\ell$  lies in  $h \cup h'$ . We can decompose  $\gamma$  into two subcurves  $\gamma_h, \gamma_{h'}$ , where  $\gamma_h$  (resp.,  $\gamma_{h'}$ ) consists of all points  $p_{\ell}$  for which  $\ell$  lies in  $h$  and passes through  $q'$  (resp., lies in  $h'$  and passes through  $q$ ).

We next repeat the preceding analysis, handling planes instead of reguli, which makes it somewhat simpler.<sup>7</sup> Let then  $\gamma = \gamma_h \cup \gamma_{h'}$  be an intersection curve of three hyperplanes of  $R_{\tau}$ , representing lines  $\ell_1, \ell_2, \ell_3$  that form a pair of planes  $h, h'$ , as above. We focus on one of the subcurves, say  $\gamma_h$ . Let  $M_h$  (resp.,  $M'_h$ ) denote the number of lines  $\ell$  of  $L_{\tau}$  (resp., of  $L'_{\tau}$ ) that are contained in  $h$ ; in 5-space these are lines for which  $\pi_{\ell}$  contains  $\gamma_h$  (resp.,  $p_{\ell}$  lies in  $\gamma_h$ ). We need to bound the number of incident pairs of lines  $(\ell, \ell') \in L_{\tau} \times L'_{\tau}$ , for which  $p_{\ell'} \in \gamma_h$ . We do not include in this count lines  $\ell' \in L'_{\tau}$  whose points  $p_{\ell'}$  are vertices of  $\Xi'_{\tau}$ , since they have already been accounted for. As in the case of reguli, we distinguish between the case where  $\pi_{\ell}$  contains  $\gamma_h$  and the case where  $\pi_{\ell}$  crosses  $\gamma_h$ .

Consider first the case where  $\pi_{\ell}$  contains  $\gamma_h$ . A trivial upper bound on the number of joints under consideration is  $M_h \cdot M'_h$ . Our next steps proceed by case analysis on the values of  $M_h$  and  $M'_h$ , which uses the same two threshold values  $s, t$  as for the case of reguli.

---

<sup>7</sup>It also yields smaller bounds, as we shall see, so this part of the analysis does not really affect the final overall bound.

(a)  $M_h \leq t$ : In this case we bound the number of joints by  $t \sum_h M'_h$ , where the sum extends over all planes  $h$  with this property. Since, in 5-space,  $M'_h$  counts blue points (representing lines in  $L'_\tau$ ) that lie on the corresponding curves  $\gamma_h$ , and each point is counted only once (since we exclude vertices of the cutting), the above sum is at most  $tn/r$ . Summed over all cells  $\tau$ , this yields an overall bound of  $O(nr^3t \log r)$ .

(b)  $M_h > t$ : The pruning process allows us to assume that  $M'_h \leq s$ . In this case we use Lemma 2.1 to conclude that the number of planes  $h$  for which  $M_h > t$  (for the fixed cell  $\tau$ ) is at most

$$O\left(\frac{\left(\frac{n}{r^4 \log r}\right)^2}{t^3} + \frac{\frac{n}{r^4 \log r}}{t}\right) = O\left(\frac{n^2}{r^8 t^3 \log^2 r} + \frac{n}{r^4 t \log r}\right),$$

and the sum  $\sum_h M_h$ , over these planes  $h$ , is

$$O\left(\frac{n^2}{r^8 t^2 \log^2 r} + \frac{n}{r^4 \log r}\right).$$

Multiplying by  $s$  and by the number of cells  $\tau$ , we obtain the bound

$$O\left(\frac{n^2 s}{r^4 t^2 \log r} + ns\right) \tag{4}$$

on the number of joints under consideration.

Consider next the case where  $\pi_\ell$  crosses  $\gamma_h$ . As in the case of reguli, we split  $\gamma_h$  into the edges of the cutting that it is comprised of, and repeat this for all curves  $\gamma_h$ . This yields a collection of  $O(r^4 \log r)$  subproblems, each involving at most  $n/(r^5 \log r)$  blue Plücker points of  $L'_\tau$  and at most  $n/(r^5 \log r)$  red Plücker hyperplanes of  $L_\tau$ . Arguing as above, the overall number of joints under consideration is at most  $O(n^2/r^2)$ .

**2-Faces of  $\Xi'_\tau$ .** The analysis follows closely that for the 2-faces of the primal cutting  $\Xi$ . Specifically, let  $\varphi$  be an intersection 2-surface of two hyperplanes of  $R_\tau$  with  $\Pi$ , and let  $\ell_1, \ell_2$  be the two corresponding lines of  $L_\tau$ . Suppose first that  $\ell_1, \ell_2$  pass through a common point  $q$ , and thus lie in a common plane  $h$ . Then any line  $\ell$  with  $p_\ell \in \varphi$  either lies in  $h$  or passes through  $q$ . We can thus view  $\varphi$  as the union of two surfaces  $\varphi_q, \varphi_h$ , where  $\varphi_q$  (resp.,  $\varphi_h$ ) is the locus of all (points representing) lines passing through  $q$  (resp., lying in  $h$ ).

Let  $(\ell, \ell', \ell'')$  be a joint where  $\ell \in L_\tau$ ,  $\ell', \ell'' \in L'_\tau$ , and, say,  $p_{\ell'} \in \varphi_q$ . We may assume that  $p_{\ell'}$  does not lie on any edge of  $\Xi'_\tau$  that is contained in  $\varphi$ , because such points have already been taken care of. If  $\pi_\ell$  fully contains  $\varphi_q$  then  $\ell$  must pass through  $q$ , and thus the joint in question must be the point  $q$  itself. The overall number of such joints is only  $O(r^2)$ , for an overall bound of  $O(r^6 \log r)$ . We may thus assume that  $\ell$  does not pass through  $q$ , and that  $\pi_\ell$  crosses  $\varphi_q$ .

Similarly, let  $(\ell, \ell', \ell'')$  be a joint as above, where  $p_{\ell'} \in \varphi_h$ . If  $\pi_\ell$  fully contains  $\varphi_h$  then  $\ell$  must lie in  $h$ . In this case, the joint must lie in  $h$ . We then proceed exactly as in the analysis of non-regulus edges of  $\Xi'_\tau$ . (In case (a) of the analysis, the sum  $\sum_h M'_h$  is at most  $n/r$ , since it counts lines of  $L'_\tau$  without multiplicity, as we ignore the corresponding Plücker points that lie on edges of  $\Xi'_\tau$ .) This yields the same bounds as in cases (a) and (b) of the non-regulus edges, i.e., a total bound of  $O\left(\frac{n^2 s}{r^4 t^2 \log r} + ns + nr^3 t \log r\right)$  for the number of joints of this kind. We may thus assume that  $\pi_\ell$  crosses  $\varphi_h$ .

Thus, in either case, we are left with subproblems, each associated with a 2-face  $\tau'$  of  $\Xi'_\tau$  (the surface  $\varphi$  is now decomposed back into its constituent 2-faces), such that  $\tau'$  contains at most  $n/(r^5 \log r)$  blue Plücker points of  $L'_\tau$  and is crossed by at most  $n/(r^5 \log r)$  red Plücker hyperplanes of  $L_\tau$ . Arguing as in the case of edges, we obtain  $O(r^4 \log r)$  subproblems of this kind, implying, as above, that the overall number of joints under consideration is at most  $O(n^2/r^2)$ .

Finally, suppose that  $\ell_1$  and  $\ell_2$  are skew. Consider a joint  $(\ell, \ell', \ell'')$ , where, say,  $p_{\ell'} \in \varphi$ . The hyperplane  $\pi_\ell$  cannot fully contain  $\varphi$ , because then the line  $\ell$  would have to be incident to every line that meets  $\ell_1$  and  $\ell_2$ , which is clearly impossible. Hence, in this case we obtain, as above, a collection of  $O(r^4 \log r)$  subproblems, each associated with a 2-face  $\tau'$  of  $\Xi'_\tau$  (a subface of  $\varphi$ ), such that  $\tau'$  contains at most  $n/(r^5 \log r)$  blue Plücker points and is crossed by at most  $n/(r^5 \log r)$  red Plücker hyperplanes. As above, the overall number of joints under consideration is  $O(n^2/r^2)$ .

**Cells of  $\Xi'_\tau$ .** Each cell  $\tau'$  of  $\Xi'_\tau$  is crossed by at most  $n/(r^5 \log r)$  red Plücker hyperplanes  $\pi_\ell$ , for  $\ell \in L_\tau$ , and contains at most  $n/(r^5 \log r)$  blue Plücker points  $p_\ell$ , for  $\ell \in L'_\tau$ . Hence, similar to the analysis of edges and 2-faces, the number of joints that involve lines  $\ell$  with  $p_\ell \in \tau'$  is at most  $n^2/(r^{10} \log^2 r)$ . Summing these bounds over all cells  $\tau'$  and  $\tau$ , we obtain an overall number of  $O(n^2/r^2)$  joints.

**3-Faces of  $\Xi'_\tau$ .** We argue here in much the same way as in the primal decomposition. Let  $p_{\ell'}$ , where  $\ell' \in L'_\tau$ , be a blue point in the relative interior of some 3-face of  $\Xi'_\tau$ , contained in  $\pi_\ell \cap \Pi$ , for some  $\ell \in L_\tau$ . Any hyperplane incident to  $p_{\ell'}$ , with the exception of  $\pi_\ell$ , crosses each of the two adjacent cells of  $\Xi'_\tau$ . We can thus assign  $p_{\ell'}$  to either of these cells, and count the joints on  $\ell'$  of the type we seek as part of the subproblem associated with that cell. This excludes the joint formed (if at all) by  $\ell'$  and  $\ell$ . The overall number of such joints is  $O(r^4 \log r \cdot \frac{n}{r}) = O(nr^3 \log r)$ . (Alternatively, we may proceed as in the cases of edges, 2-faces and cells, and obtain directly the same bound of  $O(n^2/r^2)$  joints.)

**Putting it all together.** Adding the bounds obtained in the preceding analysis steps, we obtain a grand total of

$$O\left(\frac{n^2}{s} + nr^3 t \log r + \frac{n^3 s}{r^8 t^{17/4} \log^2 r} + \frac{n^2}{r^2} + \frac{n^2 s}{r^4 t^2 \log r} + ns + r^6 \log r\right)$$

joints. We now choose

$$r = \frac{n^{13/69}}{\log^{3/23} n}, \quad s = r^2 = \frac{n^{26/69}}{\log^{6/23} n}, \quad t = \frac{n}{r^5 \log n} = \frac{n^{4/69}}{\log^{8/23} n},$$

to obtain that the overall number of joints is  $O(n^{112/69} \log^{6/23} n)$ . (This choice of parameters equalizes the first four terms in the above bound; the last three terms are dominated by the first four.)

We thus obtain the main result of this paper:

**Theorem 2.3.** *The number of joints of a set of  $n$  lines in 3-space is  $O(n^{112/69} \log^{6/23} n) = O(n^{1.6232})$ .*

## 2.4 Discussion

There are two natural conjectures concerning  $J(n)$ . The first (in view of the best known lower bound) is that  $J(n) = \Theta(n^{3/2})$ . The second, and somewhat weaker conjecture, is that  $J(n) \approx O(n^{8/5})$ . There are several informal reasons for the second conjecture. For example, observe that the two stages of decomposition end up with about  $r^8$  subproblems, each involving about  $n/r^5$  lines, which leads to a recurrence relation, whose basic solution is about  $n^{8/5}$ . (Of course, the subproblems are different from the original one, since joints are ‘lost’ there. Still, the general characteristics of the decomposition suggest this bound.)

We strongly believe that at least the second conjecture is true. There are two weak spots in our analysis. The first is the handling of regulus-edges of the dual cuttings. We can handle well reguli that contain many lines of  $L'_\tau$ , but it seems that we handle the ‘lighter’ reguli in a suboptimal manner. At any rate, the term that the analysis of these light reguli yields, namely  $O(nr^3 t \log r)$  is one of the causes for our bound to be weaker than  $O(n^{8/5})$ . The second cause is the way we handle the subproblems at the second partitioning stage: We bound there the number of relevant joints simply by the product of the sizes of the two corresponding sets of lines. We suspect that this is a gross overestimate, and that sharper bounds can be obtained using a more careful analysis.

## Acknowledgments

The authors wish to thank Boris Aronov for many helpful discussions on the problem. In particular, the present proof of Lemma 2.1 was suggested by him. The companion note [3] arose from these discussions.

## References

- [1] B. Aronov, V. Koltun and M. Sharir, Cutting triangular cycles of lines in space, *Proc. 35th Annu. ACM Sympos. Theory Comput.* (2003), 547–555.
- [2] B. Aronov, M. Pellegrini and M. Sharir, On the zone of a surface in a hyperplane arrangement, *Discrete Comput. Geom.* 9 (1993), 177–186.
- [3] B. Aronov and M. Sharir, A note on incidences in higher dimensions, manuscript, 2003.
- [4] B. Chazelle and J. Friedman, A deterministic view of random sampling and its use in geometry, *Combinatorica* 10 (1990), 229–249.
- [5] B. Chazelle, H. Edelsbrunner, L. Guibas, R. Pollack, R. Seidel, M. Sharir and J. Snoeyink, Counting and cutting cycles of lines and rods in space, *Comput. Geom. Theory Appl.* 1 (1992), 305–323.
- [6] B. Chazelle, H. Edelsbrunner, L.J. Guibas, M. Sharir, and J. Stolfi, Lines in space: Combinatorics and algorithms, *Algorithmica* 5 (1996), 428–447.
- [7] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* 5 (1990), 99–160.

- [8] S. Har-Peled, Constructing cuttings in theory and practice, *SIAM J. Comput.* 29 (2000), 2016–2039.
- [9] S. Har-Peled and M. Sharir, On-line point location in planar arrangements and its applications, *Discrete Comput. Geom.* 26 (2001), 19–40.
- [10] J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York 1995.
- [11] M. Sharir, On joints in arrangements of lines in space and related problems, *J. Combin. Theory, Ser. A* 67 (1994), 89–99.
- [12] M. Sharir and E. Welzl, Point-line incidences in space, *Proc. 18th Annu. ACM Sympos. Comput. Geom.*, 2002, 107–115.
- [13] A. Solan, Cutting cycles of rods in space, *Proc. 14th Annu. ACM Sympos. Comput. Geom.*, 1998, 135–142.
- [14] D.M.Y. Sommerville, *Analytic Geometry of Three Dimensions*, Cambridge University Press, Cambridge, 1934.
- [15] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* 3 (1983), 381–392.