Variance of Quadrature over Scrambled Unions of Nets

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Abstract

This article studies the variance of quadrature over a scrambled union of two nets, $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m-d, s)$-net in base b. For every integrand in $\mathcal{L}^2[0,1)^s$, a variance formula is given. Under some mild conditions on integrand, the rates of convergence are found. For smooth integrands, if $d/m \to c$ with $c < 1$ and $d - cm$ is bounded as $m \to \infty$, then the variance is of order $n^{-3+c}(\log n)^{s-1}$ as $n = \lambda_0 b^m + \lambda_1 b^{m-d} \to \infty$.

Key words and phrases: Integration, multiresolution, Quasi-Monte Carlo

1 Introduction

We consider the problem of approximating integral $I = \int_{[0,1)^s} f(\boldsymbol{x})d\boldsymbol{x}$ for $s \geq 1$ by the sample mean $\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_i)$, where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are n points in $[0,1)^s$. We assume that f is square integrable, that is, $f \in \mathcal{L}^2[0,1)^s$.

For high dimensional case, Monte Carlo methods and equidistribution or Quasi-Monte Carlo methods are most widely used. Under Monte Carlo methods, n points $\boldsymbol{x}_i$ are drawn from the uniform distribution on $[0,1)^s$, and so the estimator $\hat{I}$ is a random variable with mean I and variance σ^2/n where $\sigma^2 = \int_{[0,1)^s}[f(\boldsymbol{x}) - I]^2 d\boldsymbol{x}$. Thus the error in $\hat{I}$ is of order $n^{-1/2}$ in probability. Quasi-Monte Carlo methods use deterministic lists of n points $\boldsymbol{x}_i$, such as good lattice points, (t, m, s)-nets and (t, s)-sequences, that are constructed to avoid gaps and clusters among the $\boldsymbol{x}_i$. If f is of bounded variation in the sense of Hardy and Krause, then it is possible to construct a sequence of equidistribution estimators along which $|\hat{I} - I| = O(n^{-1}(\log n)^{s-1})$. For details on Quasi-Monte Carlo methods see Hua and Wang (1981), Niederreiter (1992), Sloan and Joe (1994), and its applications in statistics see Fang and Wang (1994). Owen (1995, 1997a) proposed a hybrid of these two techniques based on scrambling the digits in a (t, m, s)-net or (t, s)-sequence in base b. The resulting method provides unbiased estimates of I having a variance that is $o(n^{-1})$ along the sequence $n = \lambda b^m$, $1 \le \lambda < b$, $0 \le m$. Owen (1997b) shows that under mild smoothness conditions on f, the scrambled net variance is of order $n^{-3}(\log n)^{s-1}$ as $n = \lambda b^m \to \infty$, for nets with $t = 0$.

The problem studied here is to find expressions for variance when n, the number of points used, is not equal to λb^m. Specifically, we consider the variance of $\hat{I}$ based on scrambling a union of two nets in base b. Section 2 reviews some background material which will be used in this article. Section 3 gives expressions for the variance along scrambled sequences with $n = \lambda_0 b^m + \lambda_1 b^{m-d}$, $1 \le \lambda_0 < b$, $1 \le \lambda_1 \le b - \lambda_0$, and $0 \le d \le m$. Section 4 considers the order of variance under some mild conditions on f. Section 5 gives an example integrand for which it is possible to compute both the scrambled net variance and the Monte Carlo variance. Finally Section 6 gives some discussions.

2 Preliminaries

This section introduces the notation and definitions used. A point in the unit cube $[0,1)^s$ is denoted by $\boldsymbol{x} = (x^1, \ldots, x^s)$ or $\boldsymbol{x}_i = (x_i^1, \ldots, x_i^s)$. The set $\mathcal{A} = \{1, \cdots, s\}$ denotes the coordinate axes of $[0,1)^s$. The letter u denotes a subset of $\mathcal{A}$ and $|u|$ is the cardinality of u. Let $[0,1)^u$ denote the $|u|$-dimensional unit cube involving the coordinates in u. Let $\boldsymbol{x}^u$ denote the coordinate projection of $\boldsymbol{x}$ onto $[0,1)^u$ and $d\boldsymbol{x}^u = \prod_{r \in u} dx^r$.

The integer $b \ge 2$ is used throughout as a base for representing points in $[0,1)$. Thus $x_i^r = \sum_{k=1}^{\infty} x_{irk} b^{-k}$ where the x_{irk} are integers with $0 \le x_{irk} < b$.

2.1 Equidistribution and its randomization

Here we briefly introduce equidistribution methods known as (t, m, s)-nets and (t, s)-sequences. Equidistribution methods produce sequences $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in [0,1)^s$ such that the discrete uniform distribution on $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ closely approximates the continuous uniform distribution on $[0,1)^s$.

An *elementary interval of* $[0,1)^s$ *in base* b is a set of the form

$$E = \prod_{r=1}^{s} \Big[\frac{t_r}{b^{k_r}}, \frac{t_r + 1}{b^{k_r}} \Big)$$

with integers $k_r \ge 0$, $0 \le t_r < b^{k_r}$ for $1 \le r \le s$. Let t and m be nonnegative integers. A finite sequence $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in [0,1)^s$ with $n = b^m$ is a (t, m, s)*-net in base* b if every elementary interval in base b of volume b^{t-m} contains exactly b^t points of the sequence. Clearly, smaller values of t imply better equidistribution properties of the net. For the particular case $t = 0$, every elementary interval of volume $1/n$ has one of the n points in the sequence.

An infinite sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \in [0,1)^s$ is a (t,s)*-net in base* b if for all integers $k \geq 0$ and $m \geq t$ the finite sequence $\boldsymbol{x}_{kb^m+1}, \ldots, \boldsymbol{x}_{(k+1)b^m}$ is a (t,m,s)-net in base b. An advantage of using nets taken from (t,s)-sequences is that one can increase n through a sequence of values $n = \lambda b^m$, $1 \leq \lambda < b$, and find that all of the points used in $\hat{I}_{\lambda b^m}$ are also used in $\hat{I}_{(\lambda+1)b^m}$. The initial λb^m points of a (t,s)-sequence are well equidistributed but are not ordinarily a (t,m,s)-net. Owen (1997a) introduces the following definition to describe such point sets.

Let s, m, t, b, λ be integers with $s \geq 1$, $m \geq 0$, $0 \leq t \leq m$, $b \geq 2$ and $1 \leq \lambda < b$. A sequence $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in [0,1)^s$ with $n = \lambda b^m$ is called a (λ, t, m, s)*-net in base* b if every elementary interval in base b of volume b^{t-m} contains λb^t points of the sequence and no elementary interval in base b of volume b^{t-m-1} contains more than b^t points of the sequence.

Numerical integration by averaging over the points of a (t,m,s)-net has an error of order $n^{-1}(\log n)^{s-1}$, for functions of bounded variation in the sense of Hardy and Krause. For a fuller discussion of equidistribution methods and related ones see Niederreiter (1992) and Hickernell (1996, 1997).

Owen (1995) proposes a randomization of (t,m,s)-nets that preserves their net properties. Suppose that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is a (t,m,s)-net in base b or a (t,s)-sequence in base b. Write $a_i^r = \sum_{k=1}^{\infty} a_{irk} b^{-k}$. Let $\boldsymbol{x}_i = (x_i^1, \ldots, x_i^s)$ with $x_i^r = \sum_{k=1}^{\infty} x_{irk} b^{-k}$. where x_{irk} is a random permutation applied to a_{irk}. The permutation applied to a_{irk} depends on the values of a_{irh} for $h < k$. Specifically $x_{ir1} = \pi_r(a_{ir1})$, $x_{ir2} = \pi_{ra_{ir1}}(a_{ir2})$, $x_{ir3} = \pi_{ra_{ir1}a_{ir2}}(a_{ir3})$, and in general $x_{ir3} = \pi_{ra_{ir1}a_{ir2}\cdots a_{ir\,k-1}}(a_{irk})$ where $\pi_{ra_{ir1}a_{ir2}\cdots a_{ir\,k-1}}$ is a random permutation of $\{0, 1, \cdots, b-1\}$. The permutations are mutually independent and each is uniformly distributed over its $b!$ possible values. Owen (1995, 1997a) proves the following two propositions.

Proposition 1 *If* $\{\boldsymbol{a}_i\}$ *is a* (λ, t, m, s)*-net in base* b *then* $\{\boldsymbol{x}_i\}$ *is a* (λ, t, m, s)*-net in base* b *with probability 1.*

Proposition 2 *Let* $\boldsymbol{a}$ *be a point in* $[0,1)^s$ *and let* $\boldsymbol{x}$ *be the scrambled version of* $\boldsymbol{a}$ *as described above. Then* $\boldsymbol{x}$ *has the uniform distribution on* $[0,1)^s$.

2.2 ANOVA and Haar-like decomposition of $\mathcal{L}^2[0,1)^s$

The ANOVA decomposition approach has been widely used in statistics and quadrature since it was introduced in Efron and Stein (1981), such as Wahba (1990), Owen (1992), and Hickernell (1996). For $f \in \mathcal{L}^2[0,1)^s$, define $\alpha_\emptyset = \int_{[0,1)^s} f(\boldsymbol{x})d\boldsymbol{x} = I$ and

$$\alpha_u(\boldsymbol{x}) = \int_{[0,1)^s} \Big[f(\boldsymbol{x}) - \sum_{v \subset u} \alpha_v(\boldsymbol{x})\Big] d\boldsymbol{x}^{\mathcal{A}-u}, \quad u \subseteq \mathcal{A},$$

where the sum is over strict subsets $v \neq u$. The crossed ANOVA decomposition of f is

$$f(\boldsymbol{x}) = \sum_{u \subseteq \mathcal{A}} \alpha_u(\boldsymbol{x})$$

where the sum is over all 2^s subsets of coordinates of $[0,1)^s$. The following properties are well known:

$$\int_{[0,1)} \alpha_u dx^r = 0, \quad \forall r \in u; \qquad \int_{[0,1)^s} \alpha_u \alpha_v d\boldsymbol{x} = 0, \quad \forall u \neq v.$$

It follows that

$$\sigma^2 = \int_{[0,1)^s} (f - I)^2 d\boldsymbol{x} = \sum_{|u|>0} \int_{[0,1)^s} \alpha_u^2 d\boldsymbol{x} \equiv \sum_{|u|>0} \sigma_u^2,$$

which is the usual ANOVA decomposition.

Another kind of ANOVA decomposition, called as a nested ANOVA decomposition, is also used to derive a formula for variance over one-dimensional scrambled nets. The terms of the nested ANOVA are as follows: $\beta_0 = \int_{[0,1)} f(x)dx = I$, and

$$\beta_k(x) = b^k \int_{\lfloor b^k z \rfloor = \lfloor b^k x \rfloor} \Big[f(x) - \sum_{0 \le h < k} \beta_h(z)\Big] dz, \quad k \ge 1,$$

where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. The equality $\lfloor b^k z \rfloor = \lfloor b^k x \rfloor$ means that z and x agree to $k \ge 0$ places post the decimal point in base $b \ge 2$. Each β_k is constant on intervals of form $[tb^{-k}, (t+1)b^{-k})$ for integers $0 \le t < b^k$, that is, $\beta_k(x) = \beta_k(\lfloor b^k x \rfloor b^{-k})$. For $k \ge 1$, $\beta_k(x)$ may be expressed as

$$\beta_k(x) = b^k \int_{\lfloor b^k z \rfloor = \lfloor b^k x \rfloor} f(z)dz - b^{k-1} \int_{\lfloor b^{k-1} z \rfloor = \lfloor b^{k-1} x \rfloor} f(z)dz$$

with properties

$$\sum_{c=0}^{b-1} \beta_k((bt+c)b^{-k}) = 0$$

for $0 \le t < b^{k-1}$ and

$$\sum_{k=0}^{K} \beta_k(x) = b^K \int_{\lfloor b^k z \rfloor = \lfloor b^k x \rfloor} f(z)dz.$$

Furthermore f may be expressed as

$$f(x) = \sum_{k=0}^{\infty} \beta_k(x) \tag{2.1}$$

and then

$$\sigma^2 = \int_{[0,1)} [f(x) - \beta_0]^2 dx = \sum_{k=1}^{\infty} \int_{[0,1)} [\beta_k(x)]^2 dx \equiv \sum_{k=1}^{\infty} \sigma_k^2. \tag{2.2}$$

As to multidimensional case, Owen (1997a) develops a multivariate base b Harr-like multiresolution of $\mathcal{L}^2[0,1)^s$ using ideas from Jawerth and Sweldens (1994), Daubechies (1992) and Madych (1992). For $u \subseteq \mathcal{A}$, let κ be a vector of $|u|$ nonnegative integers k_r, $r \in u$, and let $|\kappa|$ denote $\sum_{r\in u} k_r$. Then there are $b^{|u|+|\kappa|}$ elementary intervals

$$E_{u,\kappa,\tau} = \prod_{r\in u} \Big[\frac{t_r}{b^{k_r+1}}, \frac{t_r+1}{b^{k_r+1}}\Big)$$

where τ is a $|u|$ vector of nonnegative integers $t_r < b^{k_r+1}$. Define

$$\nu_{\emptyset,0} = I, \qquad \nu_{u,\kappa}(\boldsymbol{x}) = \sum_{\tau(u,\kappa)} \sum_{\gamma(u)} \langle f, \psi_{u\kappa\tau\gamma}\rangle \psi_{u\kappa\tau\gamma}(\boldsymbol{x}), \tag{2.3}$$

where $\psi_{u\kappa\tau\gamma}$ is a product over $r \in u$ of scaled and translated wavelets, that is

$$\psi_{u\kappa\tau\gamma}(\boldsymbol{x}) = \prod_{r\in u} \Big(b^{(k_r+1)/2} 1_{\lfloor b^{k_r+1} x^r \rfloor = bt_r + c_r} - b^{(k_r-1)/2} 1_{\lfloor b^{k_r} x^r \rfloor = t_r}\Big),$$

$\langle \cdot, \cdot \rangle$ is the $\mathcal{L}^2$-inner product, and $\gamma(u)$ is a $|u|$ vector of nonnegative integers $0 \le c_r < b$. Each $\nu_{u,\kappa}$ is constant within each of $b^{|u|+|\kappa|}$ elementary intervals $E_{u,\kappa,\tau}$. Moreover, the $\nu_{u,\kappa}$ are mutually orthogonal. The multiresolution decomposition of $f \in \mathcal{L}^2[0,1)^s$ is

$$f(\boldsymbol{x}) = I + \sum_{|u|>0} \sum_{\kappa} \nu_{u,\kappa}(\boldsymbol{x}). \tag{2.4}$$

The ANOVA decomposition in terms of the $\nu_{u,\kappa}$ is

$$\sigma^2 = \int_{[0,1)^s} [f(\boldsymbol{x}) - I]^2 d\boldsymbol{x} = \sum_{|u|>0} \sum_{\kappa} \int_{[0,1)^s} [\nu_{u,\kappa}(\boldsymbol{x})]^2 d\boldsymbol{x} \equiv \sum_{|u|>0} \sum_{\kappa} \sigma_{u,\kappa}^2. \tag{2.5}$$

3 Variance over a scrambled union of nets

Suppose that f is in $\mathcal{L}^2[0,1)^s$. Let P_0 and P_1 be two sets of n_0 and n_1 points in $[0,1)^s$, respectively, and $P = P_0 \cup P_1$. Now let Q be the scrambled version of P, obtained by proceeding the uniform random permutation scheme as described in the previous section. Suppose that Q_0 and Q_1 are the corresponding version of P_0 and P_1 and denote $Q_0 = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_0}\}$, $Q_1 = \{\tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_{n_1}\}$ and $n = n_0 + n_1$. Then $Q = Q_0 \cup Q_1$. The case $Q_0 = Q_1$ is well discussed in Owen (1997a,b), so we assume throughout that Q_0 and Q_1 are two different sequences. Specifically, we will require some equidistribution properties described as follows:

Assumption 1 *P_0 is a $(\lambda_0, 0, m, s)$-net in base b and P_1 is a $(\lambda_1, 0, m-d, s)$-net in base b with integers $1 \le \lambda_0 < b$, $1 \le \lambda_1 \le b - \lambda_0$ and $0 \le d \le m$.*

Assumption 2 *No elementary interval of volume b^{-m-1} contains more than one points of the union $P = P_0 \cup P_1$.*

Denote the estimates of I based on Q_0, Q_1 and Q by $\hat{I}_{Q_0}$, $\hat{I}_{Q_1}$ $\hat{I}_Q$, respectively, that is,

$$\hat{I}_{Q_0} = \frac{1}{n_0}\sum_{i=1}^{n_0} f(\boldsymbol{x}_i), \qquad \hat{I}_{Q_1} = \frac{1}{n_1}\sum_{j=1}^{n_1} f(\tilde{\boldsymbol{x}}_j) \qquad \hat{I}_Q = \frac{1}{n}\Big[n_0\hat{I}_{Q_0} + n_1\hat{I}_{Q_1}\Big]. \tag{3.1}$$

Proposition 2 implies that these estimates are all unbiased. Now we compute the variance of $\hat{I}_Q$.

3.1 One-dimensional case

Suppose that $P_0 = \{a_1, \ldots, a_{n_0}\}$ and $P_1 = \{\tilde{a}_1, \ldots, \tilde{a}_{n_1}\}$ are two sequences in $[0, 1)$. Write $a_i = \sum_{k=0}^{\infty} a_{ik} b^{-k}$ and $\tilde{a}_j = \sum_{k=0}^{\infty} \tilde{a}_{jk} b^{-k}$ for $i = 1, \ldots, n_0$ and $j = 1, \ldots, n_1$. At first, we do not assume that both P_0 and P_1 have any nontrivial equidistribution properties. Let $Q_0 = \{x_1, \ldots, x_{n_0}\}$ and $Q_1 = \{\tilde{x}_1, \ldots, \tilde{x}_{n_1}\}$ be the corresponding scrambled version of P_0 and P_1.

Define for $k \ge 0$

$$M_k = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1} 1_{\lfloor b^k x_i\rfloor = \lfloor b^k \tilde{x}_j\rfloor} = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1} 1_{\lfloor b^k a_i\rfloor = \lfloor b^k \tilde{a}_j\rfloor}. \tag{3.2}$$

Here, the following fact is used

$$\lfloor b^k x_i\rfloor = \lfloor b^k \tilde{x}_j\rfloor \quad \text{if and only if} \quad \lfloor b^k a_i\rfloor = \lfloor b^k \tilde{a}_j\rfloor.$$

M_k counts the number of times that an $x_i \in Q_0$ and an $\tilde{x}_j \in Q_1$ share the same first k (or more) digits, which reflects how Q_0 is close to Q_1 and vice versa.

Lemma 1 *Suppose that f is in $\mathcal{L}^2[0, 1)$. Let Q_0, Q_1, $\hat{I}_{Q_0}$, $\hat{I}_{Q_1}$ and M_k be as described above. Then*

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = \frac{1}{n_0 n_1}\sum_{k\ge 1}\frac{bM_k - M_{k-1}}{b-1}\sigma_k^2, \tag{3.3}$$

where σ_k^2 is defined in (2.2).

Proof Using the nested ANOVA of f described in (2.1) and Proposition 2,

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = \frac{1}{n_0 n_1}\sum_{i=1}^{n_0}\sum_{j=1}^{n_1}\sum_{k\ge 1}\sum_{h\ge 1} E[\beta_k(x_i)\beta_h(\tilde{x}_j)].$$

From Lemmas 1 and 2 of Owen (1997a), we get

$$E[\beta_k(x_i)\beta_h(\tilde{x}_j)] = 0 \qquad \forall k \ne h$$

and

$$E[\beta_h(\tilde{x}_j)|x_i] = \beta_k(x_i)\Big(\frac{b}{b-1}1_{\lfloor b^k a_i\rfloor=\lfloor b^k\tilde{a}_j\rfloor} - \frac{1}{b-1}1_{\lfloor b^{k-1} a_i\rfloor=\lfloor b^{k-1}\tilde{a}_j\rfloor}\Big)$$

for all $x_i \in Q_0$ and $\tilde{x}_j \in Q_1$. Noting that $E[\beta_k(x_i)]^2 = \sigma_k^2$ and using the definition of M_k,

$$\begin{aligned}\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) &= \frac{1}{n_0 n_1}\sum_{k\geq 1}\sum_{i=1}^{n_0}\sum_{j=1}^{n_1} E\{\beta_k(x_i)E[\beta_h(\tilde{x}_j)|x_i]\} \\ &= \frac{1}{n_0 n_1}\sum_{k\geq 1}\frac{bM_k - M_{k-1}}{b-1}\sigma_k^2. \qquad \square\end{aligned}$$

Now suppose that P_0 and P_1 satisfy Assumptions 1 and 2 with $s = 1$. Letting $0 \leq k \leq m$, then from Assumption 1 for each $\tilde{a}_j \in P_1$ there are $\lambda_0 b^{m-k}$ points $a_i \in P_0$ with $\lfloor b^k a_i\rfloor = \lfloor b^k \tilde{a}_j\rfloor$. It follows from (3.2) that $M_k = n_1\lambda_0 b^{m-k}$ for $0 \leq k \leq m$. For $k \geq m+1$, for each $\tilde{a}_j \in P_1$ no point in P_0 shares the same first k digits with $\tilde{a}_j \in P_1$ because of Assumption 2. Therefore, we get

$$\frac{bM_k - M_{k-1}}{b-1} = \begin{cases} -\dfrac{\lambda_0 n_1}{b-1}, & k = m+1 \\ 0, & \text{otherwise} \end{cases}$$

and (3.3) becomes

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = -\frac{\sigma_{m+1}^2}{b^m(b-1)} \equiv C_1(m). \tag{3.4}$$

Remark 1. Equation (3.4) shows that for any two different nets, $(\lambda_0, 0, m, 1)$-net and $(\lambda_1, 0, m, 1)$-net in base b, satisfying Assumption 2, the corresponding corvariance is negative and only depends on m but not on the values of λ_0, λ_1 and d.

Combining above result with Corollary 2 of Owen (1997a) yields the following theorem:

Theorem 1 *Under Assumptions 1 and 2, the variance of $\hat{I}_Q$ can be expressed as*

$$\mathrm{Var}(\hat{I}_Q) = \frac{1}{n^2}\Big[n_0^2 V_1(\lambda_0, m) + n_1^2 V_1(\lambda_1, m-d) + 2n_0 n_1 C_1(m)\Big], \tag{3.5}$$

where $C_1(m)$ is given by (3.4), and $V_1(\lambda, m)$ is the variance based on a $(\lambda, 0, m, 1)$-net in base b, that is

$$V_1(\lambda, m) = \frac{1}{\lambda b^m}\Big(\frac{b-\lambda}{b-1}\sigma_{m+1}^2 + \sum_{k\geq m+2}\sigma_k^2\Big).$$

As shown in Owen (1997a), for $f \in \mathcal{L}^2[0,1)$, $n_0 V_1(\lambda_0, m) \to 0$ as $m \to \infty$, and $n_1 V_1(\lambda_1, m-d) \to 0$ if $m - d \to \infty$ as $m \to \infty$ else $n_1 V_1(\lambda_1, m-d) \leq \sigma^2$. Therefore, if $d = d(m)$ is bounded or tends to ∞ as $m \to \infty$, then we have

$$n\mathrm{Var}(\hat{I}_Q) \to 0 \quad \text{as} \quad n = \lambda_0 b^m + \lambda_1 b^{m-d} \to \infty.$$

It turns out that for any nonconstant integrand f the ratio of the scrambled net-union variance to the ordinary Monte Carlo variance tends to zero as $n \to \infty$.

3.2 Multidimensional case

Suppose that f is in $\mathcal{L}^2[0,1)^s$. For two sequences in $[0,1)^s$, $P_0 = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n_0}\}$ and $P_1 = \{\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_{n_1}\}$, write $a_i^r = \sum_{k=1}^{\infty} a_{irk} b^{-k}$ and $\tilde{a}_j^r = \sum_{k=1}^{\infty} \tilde{a}_{jrk} b^{-k}$ for $r = 1, \dots, s$. Denote the scrambled version of P by $Q = Q_0 \cup Q_1 = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n_0}\} \cup \{\tilde{\boldsymbol{x}}_1, \dots, \tilde{\boldsymbol{x}}_{n_0}\}$. The estimates $\hat{I}_{Q_0}$, $\hat{I}_{Q_1}$ and $\hat{I}_Q$ are as described in (3.1). At first, we do not assume that both P_0 and P_1 have any nontrivial equidistribution properties.

For each $\boldsymbol{a}_i \in P_0$ and each $\tilde{\boldsymbol{a}}_j \in P_1$, define

$$N_{ijrk_r} = 1_{\lfloor b^{k_r+1} a_i^r \rfloor = \lfloor b^{k_r+1} \tilde{a}_j^r \rfloor}, \qquad W_{ijrk_r} = 1_{\lfloor b^{k_r} a_i^r \rfloor = \lfloor b^{k_r} \tilde{a}_j^r \rfloor}.$$

These are indicator functions designating "narrow" and "wide" matches respectively between the components a_i^r and $\tilde{a}_j^r$. For each $u \subseteq \mathcal{A}$ with $|u| > 0$ and $\kappa = \kappa(u)$, define

$$M_{u,\kappa} = \frac{1}{(b-1)^{|u|}} \sum_{i=1}^{n_0} \sum_{j=1}^{n_1} \prod_{r \in u} (b N_{ijrk_r} - W_{ijrk_r}). \tag{3.6}$$

After some straightforward calculations, $M_{u,\kappa}$ can be expressed as

$$M_{u,\kappa} = \frac{1}{(b-1)^{|u|}} \sum_{v \subseteq u} (-1)^{|u|-|v|} b^{|v|} \sum_{i=1}^{n_0} \sum_{j=1}^{n_1} \Big(\prod_{r \in v} N_{ijrk_r} \Big) \Big(\prod_{r \in u-v} W_{ijrk_r} \Big). \tag{3.7}$$

Lemma 2 *Let Q_0, Q_1 and $M_{u,\kappa}$ be as described above, and $\sigma_{u,\kappa}^2$ as in (2.5). Then*

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = \frac{1}{n_0 n_1} \sum_{|u|>0} \sum_{\kappa(u)} M_{u,\kappa} \sigma_{u,\kappa}^2. \tag{3.8}$$

Proof By using multiresolution of f, (2.4)

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = \frac{1}{n_0 n_1} \sum_{i=1}^{n_0} \sum_{j=1}^{n_1} \sum_{|u|>0} \sum_{\kappa(u)} \sum_{|u'|>0} \sum_{\kappa'(u')} E[\nu_{u,\kappa}(\boldsymbol{x}_i) \nu_{u',\kappa'}(\tilde{\boldsymbol{x}}_j)],$$

where $\nu_{u,\kappa}$ and $\nu_{u',\kappa'}$ are as in (2.3). From Lemmas 4 and 5 of Owen (1997a), we get

$$E[\nu_{u,\kappa}(\boldsymbol{x}_i) \nu_{u',\kappa'}(\tilde{\boldsymbol{x}}_j)] = 0$$

if $u \neq u'$ or $\kappa \neq \kappa'$ or $\tau \neq \tau'$, and

$$E[\nu_{u,\kappa}(\tilde{\boldsymbol{x}}_j) | \boldsymbol{x}_i] = \nu_{u,\kappa}(\boldsymbol{x}_i) \prod_{r \in u} \Big(\frac{b}{b-1} N_{ijrk_r} - \frac{1}{b-1} W_{ijrk_r} \Big).$$

It follows from the definition (3.6) that

$$\begin{aligned}\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) &= \frac{1}{n_0 n_1} \sum_{i=1}^{n_0} \sum_{j=1}^{n_1} \sum_{|u|>0} \sum_{\kappa(u)} E\{\nu_{u,\kappa}(\boldsymbol{x}_i) E[\nu_{u,\kappa}(\tilde{\boldsymbol{x}}_j) | \boldsymbol{x}_i]\} \\ &= \frac{1}{n_0 n_1} \sum_{|u|>0} \sum_{\kappa(u)} M_{u,\kappa} E[\nu_{u,\kappa}(\boldsymbol{x}_i)]^2 \\ &= \frac{1}{n_0 n_1} \sum_{|u|>0} \sum_{\kappa(u)} M_{u,\kappa} \sigma^2_{u,\kappa}. \qquad \square\end{aligned}$$

Now suppose that P_0 and P_1 satisfy Asssumptions 1 and 2. From the equidistribution properties, if $|\kappa| + |v| \leq m$, for each $\tilde{\boldsymbol{a}}_j \in P_1$, there are $\lambda_0 b^{m-|\kappa|-|v|}$ points $\boldsymbol{a}_i \in P_0$ such that $\lfloor b^{k_r+1} a_i^r \rfloor = \lfloor b^{k_r+1} \tilde{a}_j^r \rfloor$ for all $r \in v$ and $\lfloor b^{k_r} a_i^r \rfloor = \lfloor b^{k_r} \tilde{a}_j^r \rfloor$ for all $r \in u - v$, else no point in P_0 meets these conditions. then expression (3.7) becomes

$$\begin{aligned}M_{u,\kappa} &= \frac{1}{(b-1)^{|u|}} \sum_{v \subseteq u} (-1)^{|u|-|v|} b^{|v|} n_1 \lambda_0 b^{m-|\kappa|-|v|} 1_{|\kappa|+|v| \leq m} \\ &= \frac{(-1)^{|u|}}{(b-1)^{|u|}} \lambda_0 n_1 b^{m-|\kappa|} \sum_{l=0}^{|u|} \binom{|u|}{l} (-1)^l 1_{|\kappa|+|v| \leq m}.\end{aligned}$$

Therefore, for $|\kappa| \leq m - |u|$ or $|\kappa| \geq m+1$ we have $M_{u,\kappa} = 0$, and for $m - |u| < |\kappa| \leq m$ we have

$$\begin{aligned}M_{u,\kappa} &= \frac{(-1)^{|u|}}{(b-1)^{|u|}} \lambda_0 n_1 b^{m-|\kappa|} \sum_{l=0}^{m-|\kappa|} \binom{|u|}{l} (-1)^l \\ &= \frac{(-1)^{m+|u|-|\kappa|}}{(b-1)^{|u|}} \lambda_0 n_1 b^{m-|\kappa|} \binom{|u|-1}{m-|\kappa|}\end{aligned}$$

and (3.8) becomes

$$\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1}) = \frac{1}{b^m} \sum_{|u|>0} \sum_{m-|u|<|\kappa| \leq m} \frac{(-1)^{m+|u|-|\kappa|}}{(b-1)^{|u|}} b^{m-|\kappa|} \binom{|u|-1}{m-|\kappa|} \sigma^2_{u,\kappa} \equiv C_s(m). \quad (3.9)$$

Remark 2 As in the case $s = 1$, under Assumptions 1 and 2, the covariance of two different nets in base b only depends on m but not on λ_0, λ_1 and d. Moreover, since the terms

$$b^{m-|\kappa|} \binom{|u|-1}{m-|\kappa|}$$

are nondecreasing as $m - |\kappa|$ increasing from 0 to $|u| - 1$, then for large m the most significant term in the sum in (3.9) occurs at $|\kappa| = m - |u| + 1$. It follows that

$$C_s(m) \sim -\frac{1}{b^m} \sum_{|u|>0} \sum_{|\kappa|=m-|u|+1} \left(\frac{b}{b-1}\right)^{|u|} \sigma^2_{u,\kappa} \qquad \text{as} \quad m \to \infty, \quad (3.10)$$

that is, $C_s(m)$ is negative asymptotically.

Combining (3.9) with Corollary 4 of Owen (1997a), we get

Theorem 2 *Let $P = P_0 \cup P_1$ satisfy Assumptions 1 and 2. Then*

$$\mathrm{Var}(\hat{I}_Q) = \frac{1}{n^2}\Big[n_0^2 V_s(\lambda_0, m) + n_1^2 V_s(\lambda_1, m-d) + 2n_0 n_1 C_s(m)\Big], \tag{3.11}$$

where $C_s(m)$ is given in (3.5), and $V_s(\lambda, m)$ is the variance based on a scrambled $(\lambda, 0, m, s)$-net in base b, that is

$$V_s(\lambda, m) = \frac{1}{\lambda b^m}\sum_{|u|>0}\Big[\sum_{m-|u|<|\kappa|\le m}\Gamma_{|u|,|\kappa|}(\lambda, m)\sigma_{u,\kappa}^2 + \sum_{|\kappa|\ge m+1}\sigma_{u,\kappa}^2\Big]$$

where

$$\Gamma_{|u|,|\kappa|}(\lambda, m) = 1 + \frac{(-1)^{|u|}}{(b-1)^{|u|}}\Big[\lambda b^{m-|\kappa|}\binom{|u|-1}{m-|\kappa|}(-1)^{m-|\kappa|} - \sum_{l=0}^{m-|\kappa|}\binom{|u|}{l}(-b)^l\Big].$$

The constants $\Gamma_{|u|,|\kappa|}$ are interpreted as "gains" that multiply the variance contribution of $\nu_{u,\kappa}$. See Owen (1997a,b) for the full discussion about these gain factors. From Theorem 1 of Owen (1997b), it is easy to show that

$$\mathrm{Var}(\hat{I}_Q) \le \frac{\sigma^2}{n}(1+e)$$

for a scrambled union Q of $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m-d, s)$-net in base b. Thus for $n = \lambda_0 b^m + \lambda_1 b^{m-d}$ with $1 \le \lambda_0 < b$, $1 \le \lambda_1 \le b - \lambda_0$ and $0 \le d \le m$, the scrambled variance is never more than a constant ($1 + e \approx 3.718$) times the Monte Carlo variance for any f in any dimension.

4 Variance as $n = n_0 + n_1 \to \infty$

In this section, we consider the variance of quadrature based on a scrambled union of two nets satisfying Assumptions 1 and 2, as $n = n_0 + n_1 \to \infty$. For one-dimensional case we assume that f satisfies a Lipschitz condition on $[0, 1)$, while for multidimensional case we assume that f is smooth on $[0, 1)^s$ as in Owen (1997b).

4.1 One-dimensional case

Here we take $\lambda_0 = \lambda_1 = 1$ for simplicity, that is, we suppose that $Q = Q_0 \cup Q_1$ is a scrambled union of $(0, m, 1)$-net and $(0, m-d, 1)$-net in base b.

Owen (1997b) shows that for smooth integrand f in $[0,1)$, the variance based on a scrambled $(0,m,1)$-net in base b is of order n^{-3} as $n=b^m\to\infty$. This result is also true under weaker condition. We give the following lemma:

Lemma 3 *Suppose that f satisfies a Lipschitz condition on $[0,1)$, that is*

$$|f(x')-f(x'')|\le B|x'-x''|$$

for a finite $B\ge 0$ and any $x',x''\in[0,1)$. then the variance based on a scrambled $(0,m,1)$-net in base b is of order n^{-3} as $n=b^m\to\infty$.

Proof Let $\{x_1,\ldots,x_n\}$ be a scrambled $(0,m,1)$-net in base b. Then from the definition, each interval of the form $[tb^{-m},\ (t+1)b^{-m})$, $0\le t<b^m$, contains exactly one of the x_i, denote it by z_t, and the z_t are independent random variables having the uniform distribution over $[tb^{-m},\ (t+1)b^{-m})$. Put $\bar{x}_t=(t+0.5)/n$ and

$$\hat{I}=\frac{1}{n}\sum_{i=1}^{n}f(x_i)=\frac{1}{n}\sum_{t=0}^{n-1}f(z_t)=\frac{1}{n}\sum_{t=0}^{n-1}f(\bar{x}_t)+\frac{1}{n}\sum_{t=0}^{n-1}[f(z_t)-f(\bar{x}_t)].$$

It follows that

$$\begin{aligned}\mathrm{Var}(\hat{I}) &= \frac{1}{n^2}\sum_{t=0}^{n-1}\mathrm{Var}(f(z_t)-f(\bar{x}_t))\\ &\le \frac{1}{n^2}\sum_{t=0}^{n-1}E|f(z_t)-f(\bar{x}_t)|^2\\ &\le \frac{B^2}{n^2}E|z_t-\bar{x}_t|^2=\frac{B^2}{12n^3}. \qquad \square\end{aligned}$$

Now let f satisfy a Lipschitz condition on $[0,1)$. Then for a scrambled union of $(0,m,1)$-net and $(0,m-d,1)$-net in base b satisfying Assumption 2, from (3.5) we have

$$\mathrm{Var}(\hat{I}_Q)\le\frac{n_0^2}{n^2}\mathrm{Var}(\hat{I}_{Q_0})+\frac{n_1^2}{n^2}\mathrm{Var}(\hat{I}_{Q_1})=O(n^{-3+c})$$

if $d/m\to c$ with $0\le c\le 1$ and $d-cm$ is bounded as $m\to\infty$. In particular, if d is bounded as $m\to\infty$, then $\mathrm{Var}(\hat{I}_Q)=O(n^{-3})$. However, if $m-d$ is bounded then we may only have $\mathrm{Var}(\hat{I}_Q)=O(n^{-2})$.

4.2 Multidimensional case

For multidimensional case, Owen (1997b) shows that if f is smooth in $[0,1)^s$, that is, there exists finite $B\ge 0$ and $\beta\in(0,1]$ such that

$$\Big|\frac{\partial^s}{\partial\boldsymbol{x}}f(\boldsymbol{x}')-\frac{\partial^s}{\partial\boldsymbol{x}}f(\boldsymbol{x}'')\Big|\le B\|\boldsymbol{x}'-\boldsymbol{x}''\|^{\beta}$$

for any $\boldsymbol{x}'$, $\boldsymbol{x}'' \in [0,1)^s$, where $\|\cdot\|$ is the Euclidean norm, then the variance based on a scrambled $(\lambda, 0, m, s)$-net in base b is of order $n^{-3}(\log n)^{s-1}$ as $n = \lambda b^m \to \infty$. Hight powers of $\log n$ might not be negligible until n is very large, and this raises the possibility that the scrambled net variance might be worse than the Monte Carlo variance for finite n.

Now under the same smoothness condition, for a scrambled union of $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m-d, s)$-net in base b satisfying Assumption 2, we have the following theorem:

Theorem 3 *Suppose that f is smooth in $[0,1)^s$ and the sequences used satisfy Assumptions 1 and 2. If $m-d$ is bounded as $m \to \infty$ then*

$$\mathrm{Var}(\hat{I}_Q) = O(n^{-2});$$

If $d/m \to c$ with $0 \le c < 1$ and $d - cm$ is bounded as $m \to \infty$ then

$$\mathrm{Var}(\hat{I}_Q) = O(n^{-3+c}(\log n)^{s-1}).$$

In particular, if d is bounded as $m \to \infty$ then

$$\mathrm{Var}(\hat{I}_Q) = O(n^{-3}(\log n)^{s-1}).$$

Proof Consider each of three terms in expression (3.11). Firstly, from Theorem 2 of Owen (1997b)

$$V_s(\lambda_0, m) = \mathrm{Var}(\hat{I}_{Q_0}) = O(n_0^{-3}(\log n_0)^{s-1}) = O(n^{-3}(\log n)^{s-1})$$

as $m \to \infty$. Secondly, from (3.9)

$$\begin{aligned} |C_s(m)| &= |\mathrm{Cov}(\hat{I}_{Q_0}, \hat{I}_{Q_1})| \le \frac{1}{b^m} \sum_{|u|>0} \sum_{m-|u|<|\kappa|\le m} \frac{1}{(b-1)^{|u|}} b^{m-|\kappa|} \binom{|u|-1}{m-|\kappa|} \sigma^2_{u,\kappa} \\ &\le \frac{1}{b^m} \sum_{|u|>0} \sum_{m-|u|<|\kappa|\le m} \frac{b^{|u|-1}}{(b-1)^{|u|}} \sigma^2_{u,\kappa}. \end{aligned}$$

Using Lemma 2 of Owen (1997b)

$$\sigma^2_{u,\kappa} = b^{-2|\kappa|} \Big(\frac{b^2-1}{12b^2}\Big)^{|u|} \Big\| \frac{\partial^{|u|}\alpha_u}{\partial \boldsymbol{x}^u} \Big\|^2 \Big(1 + O\Big(\sum_{r\in u} b^{-(1+\beta)k_r}\Big)\Big),$$

we have

$$\begin{aligned} \sum_{m-|u|<|\kappa|\le m} \sigma^2_{u,\kappa} &= b^{-2m} \Big\| \frac{\partial^{|u|}\alpha_u}{\partial \boldsymbol{x}^u} \Big\|^2 \Big(\frac{b^2-1}{12}\Big)^{|u|} \sum_{r=1}^{|u|} b^{-2r} \binom{|u|-1}{m-|\kappa|} \\ &\sim b^{-2m} \Big\| \frac{\partial^{|u|}\alpha_u}{\partial \boldsymbol{x}^u} \Big\|^2 \Big(\frac{b^2-1}{12}\Big)^{|u|} \frac{m^{|u|-1}}{(|u|-1)!} \frac{1-b^{-2|u|}}{b^2-1} \quad \text{as } m \to \infty. \end{aligned}$$

Therefore

$$|C_s(m)| = O(n_0^{-3}(\log n_0)^{s-1}) = O(n^{-3}(\log n)^{s-1}).$$

Finally, if $m-d$ is bounded as $m \to \infty$, then n_1 is finite and so

$$\frac{n_1^2}{n^2}V_s(\lambda_1, m-d) = \frac{n_1^2}{n^2}\mathrm{Var}(\hat{I}_{Q_1}) = O(n^{-2}).$$

If $d/m \to c$ and $d - cm$ is bounded with $c < 1$, then $m - d \to \infty$ as $m \to \infty$, and by the similar arguments in Theorem 2 of Owen (1997b)

$$\begin{aligned}
\frac{n_1^2}{n^2}V_s(\lambda_1, m-d) &= O\Big(\frac{1}{n^2}\frac{\lambda_1}{n_1}\sum_{|u|>0}\Big\|\frac{\partial^{|u|}\alpha_u}{\partial \boldsymbol{x}^u}\Big\|^2\Big(\frac{b^2-1}{12}\Big)^{|u|}\frac{(m-d)^{|u|-1}}{(|u|-1)!}\frac{1}{b^2-1}\Big) \\
&= O\Big(\frac{1}{n^2}\frac{(m-d)^{s-1}}{n_1}\Big) = O\Big(\frac{m^{s-1}}{n^{3-c}}\frac{n^{1-c}}{n_1}\Big(1-\frac{d}{m}\Big)^{s-1}\Big) \\
&= O\Big(\frac{m^{s-1}}{n^{3-c}}b^{d-cm}\Big) = O(n^{-3+c}(\log n)^{s-1}). \qquad \square
\end{aligned}$$

5 An example

We consider the following multilinear integrand

$$f(\boldsymbol{x}) = 12^{s/2}\prod_{r=1}^{s}\Big(x^r - 0.5\Big) \tag{5.1}$$

which has integral $I = 0$ and variance $\sigma^2 = 1$ for any $s \geq 1$. Owen (1997b) shows that it has only s dimensional structure, and

$$\sigma^2_{u,\kappa} = 1_{|u|=s}b^{-2|\kappa|}\Big(\frac{b^2-1}{b^2}\Big)^s.$$

The variance formula given by Theorem 2 has been evaluated numerically for $s = 1, \ldots, 10$, b equal to all prime powers between s and 11 inclusive and all $n = \lambda_0 b^m + \lambda_1 b^{m-d}$ from $n = 1$ to the smallest such n greater than or equal to 10^7. Figures 1 and 2 show some of these calculation results. In each plot, the horizontal axis displays sample size n and the vertical axis displays $[\mathrm{Var}(\hat{I}_Q)]^{1/2}$. Two reference lines are also given, one is $n^{-1/2}$ corresponding to the simple Monte Carlo rate for this function, and the other is $n^{-3/2}$ corresponding to the asymptotic rate for this function when $s = 1$ and $n = \lambda b^m$. The dots in each plot correspond to the square root of variance along $n = \lambda b^m$.

For comparing sequences, we define efficiency of Q with respect to Q_0 as follows

$$E_{n_0,n} = \frac{[\mathrm{Var}(\hat{I}_{Q_0})]^{1/2}}{[\mathrm{Var}(\hat{I}_Q)]^{1/2}}. \tag{5.2}$$

It enables us to study the effects of adding $n_1 = \lambda_1 b^{m-d}$ points to Q_0 on estimator of I. The values of $E_{n_0,n}$ for various cases are calculated. Figure 3 shows the values of $E_{n_0,n}$ for $n_0 = b^m$ and $n = n_0 + \lambda_1 b^{m-d}$ with $1 \le \lambda_1 < b$, $1 \le d \le m$, using $b = 7$ and 11, $m = 5$, 6 and 7 respectively. Table 1 lists the "worst" and the "best" efficiencies among sequences of the form $(0, m, s) \cup (\lambda_1, 0, m, s)$-net in base b with $1 \le \lambda_1 < b$, $0 \le d \le m$, where if $d = 0$ then $\lambda_1 = 1$.

These results show that among sequences used with the number ranged from $\lambda_0 b^m$ to $(\lambda_0 + 1)b^m$ of points, the one with $(\lambda_0 + 1)b^m$ points, that is $(\lambda_0 + 1, 0, m, s)$-net, has the smallest variance. For lower dimension and mild or larger size of n_0, increasing sample size may cause augment of variance unless $n = (\lambda_0 + 1)b^m$. This is different from the case of simple Monte Carlo methods.

6 Discussion

This article has considered the variance of quadrature based on a randomized sequence with $n = \lambda_0 b^m + \lambda_1 b^{m-d}$ points. This sequence may be obtained by scrambling a union of nets, $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m - d, s)$-net in base b. It turns out that if d is bounded as $m \to \infty$ the variance is of order $o(n^{-1})$ for any square integrable integrand, $O(n^{-3})$ for univariate Lipschitz integrand and $O(n^{-3}(\log n)^{s-1})$ for smooth multivariate integrand. On the other hand, for any fixed n_0, increasing sample size n through $n = n_0 + \lambda_1 b^{m-d}$ can not guarantee to reduce the variance of quadrature unless $d = 0$. In some cases, for instance, in lower dimension case, adding some points may cause a large augment of variance.

The nets involved in this article are of the form (λ, t, m, s)-nets in base b with $t = 0$. It would be desirable to deal with the case with $t > 0$. Much less is known about randomized versions of the nets with $t > 0$. Resently, Owen (1997c) studies the variance in this case. Another further work is to study the question what the order of variance is under weaker conditions on integrand.

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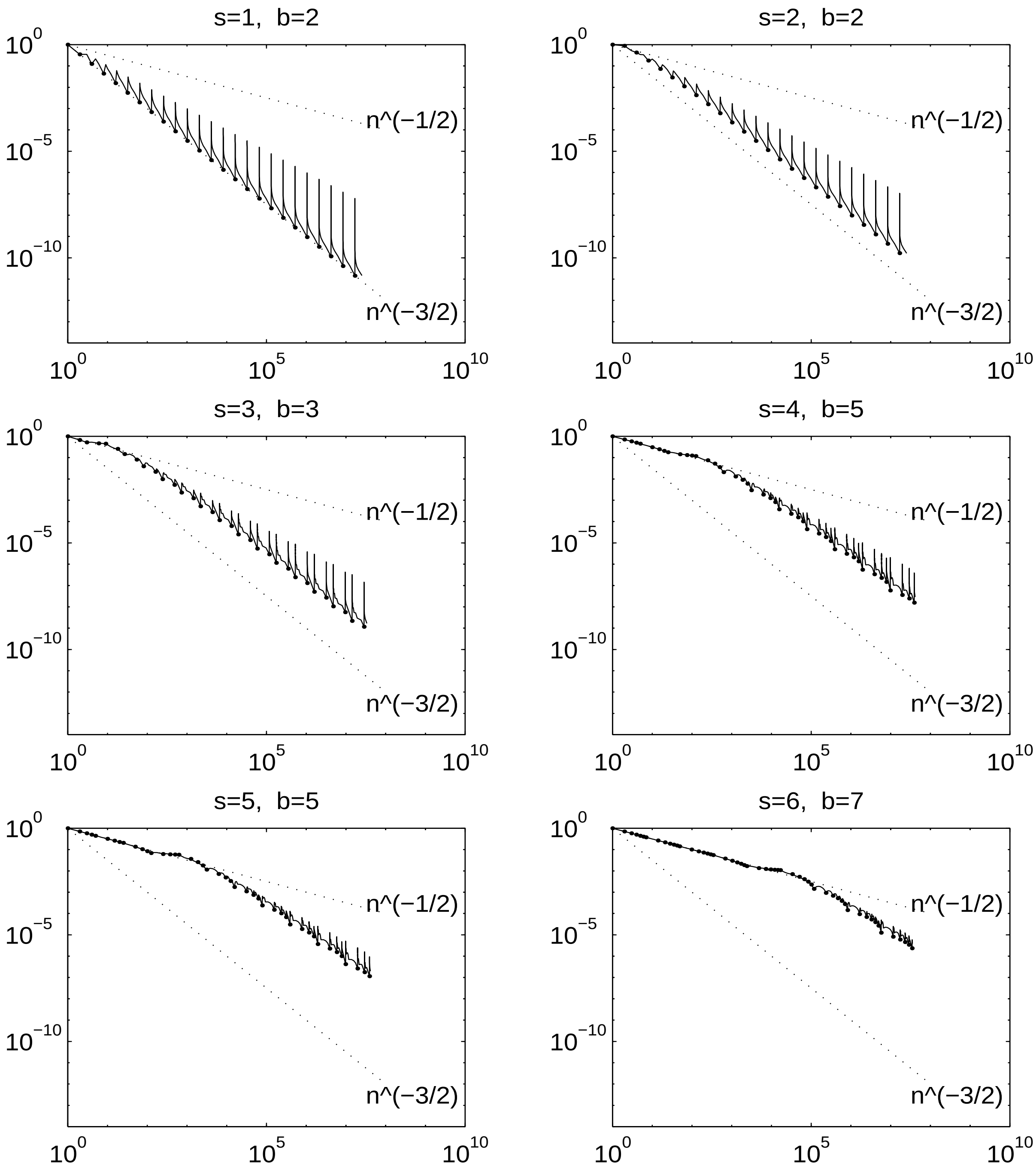

Fig. 1. These plots show the square roots of the variance for integration of f given by (5.1) versus the sample size n. For each dimension $s = 1, \ldots, 6$, a curve is ploted of the exact square root for a scrambled union of two nets, $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m-d, s)$-net in base b, where b is the smallest prime power for which $b \geq s$. The sample size n are all integers of the form $\lambda_0 b^m + \lambda_1 b^{m-d}$. The dots correspond to the square root of variance along $n = \lambda b^m$ with $1 \leq \lambda < b$.

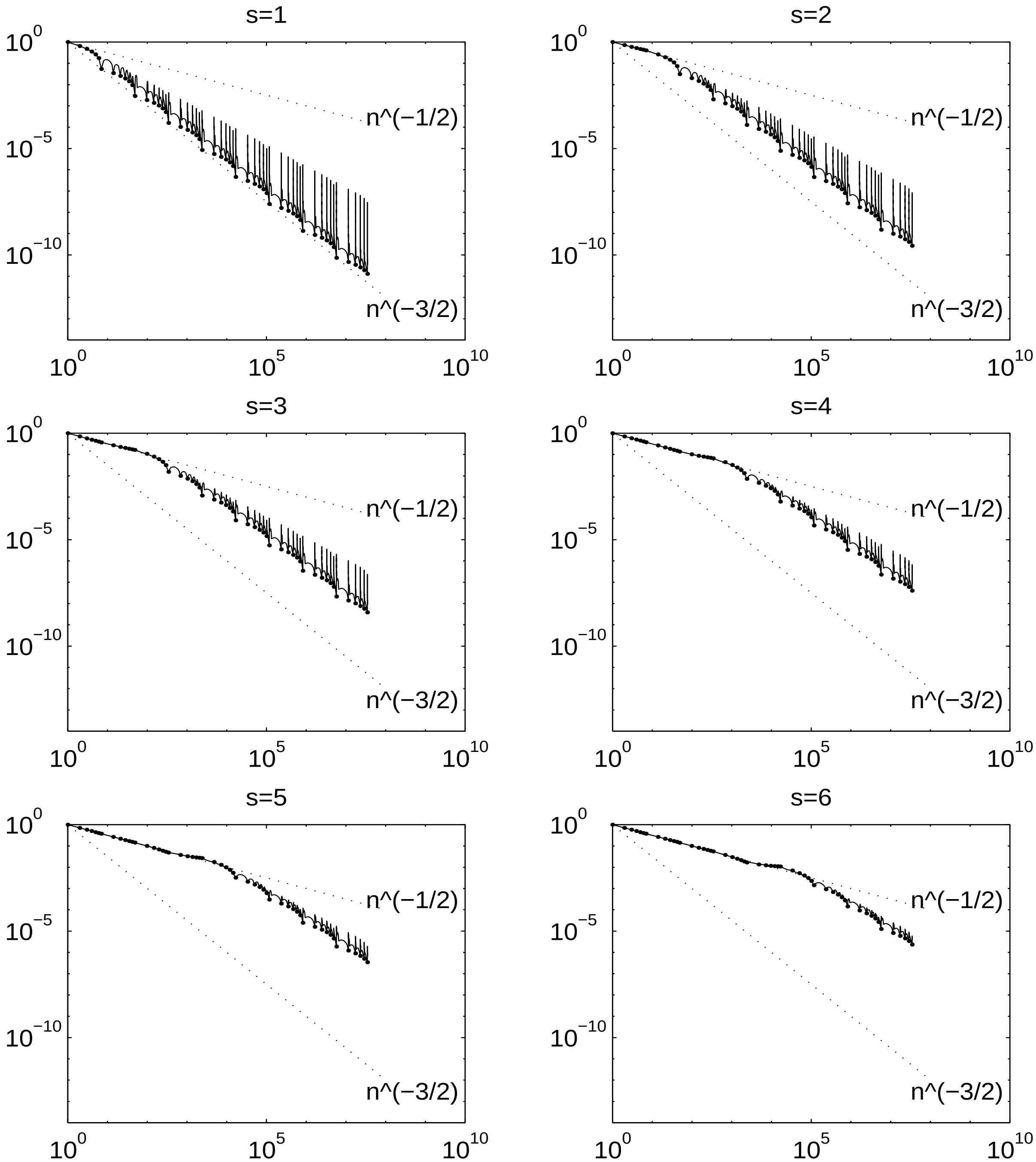

Fig. 2. These plots show the square roots of the variance for integration of f given by (5.1) versus the sample size n. For each dimension $s = 1, \ldots, 6$, a curve is ploted of the exact square root for a scrambled union of two nets, $(\lambda_0, 0, m, s)$-net and $(\lambda_1, 0, m - d, s)$-net in base $b = 7$. The sample size n are all integers of the form $\lambda_0 7^m + \lambda_1 7^{m-d}$. The dots correspond to square root of variance along $n = \lambda 7^m$ with $1 \leq \lambda < 7$.

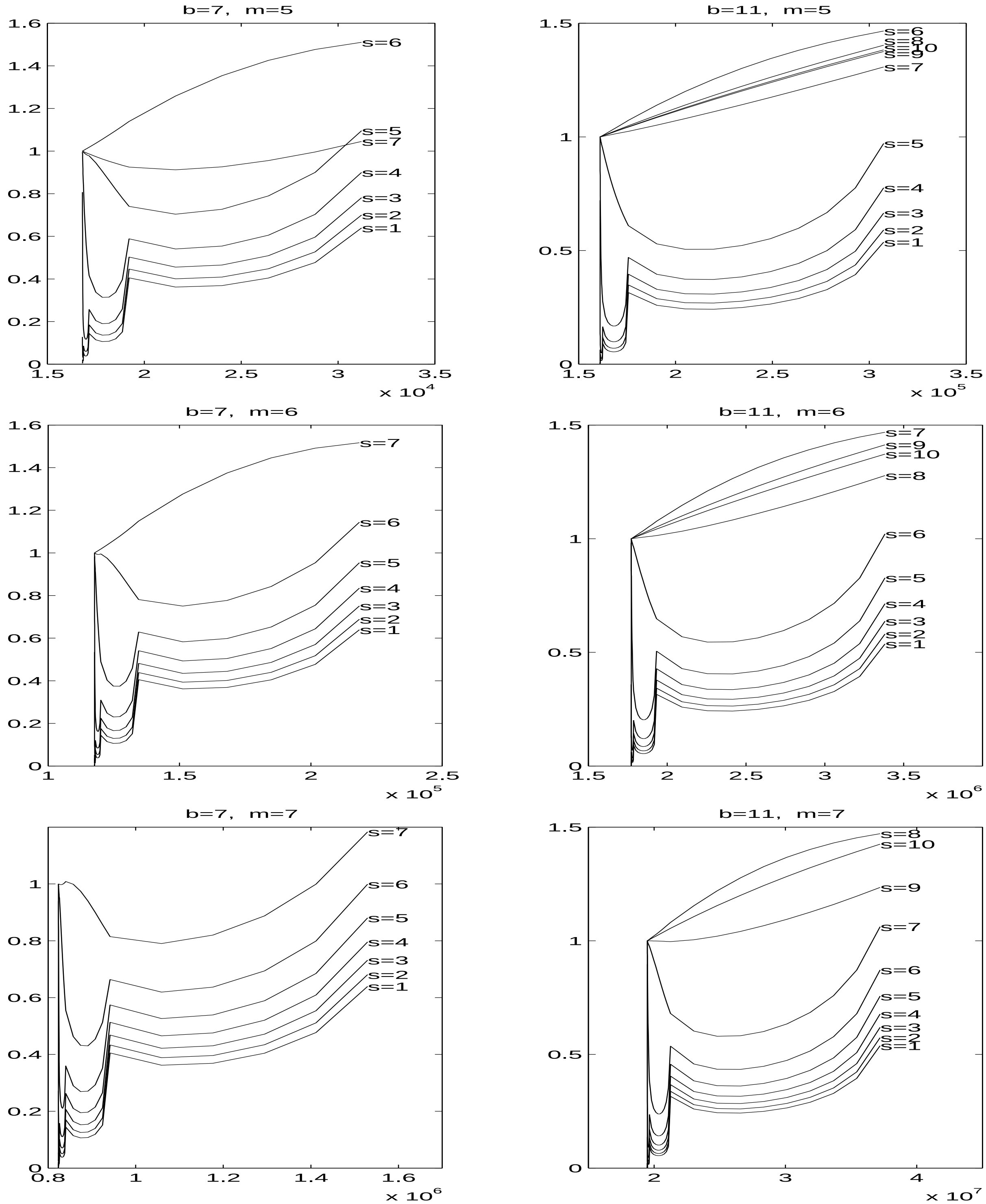

Fig. 3. These plots show the values of $E_{n_0,n}$ defined by (5.2) for $n_0 = b^m$ and $n = n_0 + \lambda_1 b^{m-d}$ with $1 \leq \lambda_1 < b$, $1 \leq d \leq m$, using $b = 7$ and 11, $m = 5$, 6 and 7 respectively.

Table 1 The worst and the best efficiencies, that is, $E_{n_0,n_0+n_1^*}$ and $E_{n_0,2n_0}$, among the sequences obtained by scrambling unions of $(0, m, s)$-net and $(\lambda_1, 0, m-d, s)$-net in base b with $1 \le \lambda_1 < b$, $0 \le d \le m$, where if $d = 1$ then $\lambda_1 = 1$.

		$b=7$			$b=11$		
s	m	n_1^*	$E_{n_0,n_0+n_1^*}$	$E_{n_0,2n_0}$	n_1^*	$E_{n_0,n_0+n_1^*}$	$E_{n_0,2n_0}$
1	5	$4 \times b^0$	0.0054	1.5460	$6 \times b^0$	0.0014	1.4900
	6	$4 \times b^0$	0.0020	1.5460	$6 \times b^0$	0.0004	1.4900
	7	$4 \times b^0$	0.0008	1.5460	$6 \times b^0$	0.0001	1.4897
2	5	$4 \times b^1$	0.0314	1.5454	$6 \times b^1$	0.0106	1.4899
	6	$4 \times b^1$	0.0130	1.5455	$6 \times b^1$	0.0035	1.4899
	7	$4 \times b^1$	0.0053	1.5456	$6 \times b^1$	0.0011	1.4899
3	5	$4 \times b^2$	0.1165	1.5445	$6 \times b^2$	0.0490	1.4897
	6	$4 \times b^2$	0.0536	1.5448	$6 \times b^2$	0.0182	1.4898
	7	$4 \times b^2$	0.0240	1.5450	$6 \times b^2$	0.0065	1.4898
4	5	$3 \times b^3$	0.3140	1.5429	$5 \times b^3$	0.1677	1.4893
	6	$4 \times b^3$	0.1619	1.5437	$6 \times b^3$	0.0693	1.4895
	7	$4 \times b^3$	0.0808	1.5442	$6 \times b^3$	0.0276	1.4896
5	5	$2 \times b^4$	0.7040	1.5399	$3 \times b^4$	0.5055	1.4887
	6	$4 \times b^4$	0.3745	1.5419	$5 \times b^4$	0.2034	1.4891
	7	$4 \times b^4$	0.2102	1.5429	$6 \times b^4$	0.0911	1.4893

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