A note on improper Kurzweil-Henstock integral in Riesz spaces

A. Boccuto–B. Riečan *

ABSTRACT. Recently a connection has been found between the improper Kurzweil-Henstock integral on the real line and the integral over a compact space. In the paper the mentioned results are generalized for the case of Riesz space valued functions.


KEY WORDS: Riesz spaces, compact topological spaces, Henstock-Kurzweil integral.

1 Introduction.

In [9] two possibilities are mentioned of defining the improper Kurzweil-Henstock integral on the real line. Their coincidence has been proved in [4]. On the other hand in [1] and [10] the Kurzweil-Henstock integral has been studied for real functions defined on a compact space. In [3] there was shown a natural connection between these two situations: the improper integral on the real line and the integral on a compact space. Here we shall show that the relation holds even in the case that the considered functions take valued in a type of Riesz spaces.

*Authors’ Address: A. Boccuto: Dipartimento di Matematica e Informatica, via Vanvitelli,1 I-06123 PERUGIA (ITALY)
B. Riečan: Matematický Ústav, Slovenská Akadémia Vied, Štefánikova 49, SK-81473 BRATISLAVA (SLOVAKIA)

E-mail: boccuto@dipmat.unipg.it, riecan@mat.savba.sk
2 Preliminaries.

Let $\mathbb{N}$ be the set of all strictly positive integers, $\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ be the set of all strictly positive real numbers, $\overline{\mathbb{R}}$ the set of all extended real numbers. We begin with some preliminary definitions and results.

**Definition 2.1** A Riesz space $R$ is said to be **Dedekind complete** if every nonempty subset of $R$, bounded from above, has supremum in $R$.

**Definition 2.2** Given a sequence $(r_n)$ in $R$, we say that $(r_n)$ **$(D)$-converges** to an element $r \in R$ if there exists a bounded double sequence $(a_{i,j})_{i,j}$ in $R$, such that, for each $i \in \mathbb{N}$, $a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1} \forall j \in \mathbb{N}$ and $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$ (such a sequence will be called a **regulator** or $(D)$-sequence from now on), and satisfying the following condition:

$$\forall \text{ mapping } \varphi : \mathbb{N} \to \mathbb{N}, \text{there exists an integer } n_0 \text{ such that }$$

$$|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \geq n_0$. In this case, we write $(D) \lim_{n \to +\infty} r_n = r$.

Analogously, given $l \in R$, a function $f : A \to R$, where $\emptyset \neq A \subset \overline{\mathbb{R}}$, and a limit point $x_0$ for $A$, we will say that $(D) \lim_{x \to x_0} f(x) = l$ if there exists a $(D)$-sequence $(a_{i,j})_{i,j}$ in $R$ such that, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a neighborhood $\mathcal{U}$ of $x_0$ such that for all $x \in \mathcal{U} \cap A \setminus \{x_0\}$ we get

$$|f(x) - l| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

**Definition 2.3** We say that $R$ is **weakly $\sigma$-distributive** if for every $(D)$-sequence $(a_{i,j})$ one has:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

It is easy to check that the usual order convergence implies $(D)$-convergence, while the converse is true in weakly $\sigma$-distributive spaces (see also [5]).

From now on, we shall always assume that $R$ is a Dedekind complete weakly $\sigma$-distributive Riesz space.

The following lemma will be useful in the sequel (see [6], [12]).
Lemma 2.4 Let \((a_{i,j}^n)_{i,j} : n \in \mathbb{N}\) be any countable family of regulators. Then for each fixed element \(u \in \mathbb{R}, u \geq 0\), there exists a regulator \((a_{i,j})_{i,j}\) such that for every \(\varphi \in \mathbb{N}^\mathbb{N}\) one has
\[
u \land \sum_{n=1}^{\infty} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^n \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.
\]

3 The Kurzweil-Henstock integral.

Let \(X\) be a Hausdorff compact topological space. If \(A \subset X\), then the interior of the set \(A\) is denoted by \(\text{int} A\).

We shall work with a family \(\mathcal{F}\) of compact subsets of \(X\) such that \(X \in \mathcal{F}\) and closed under the intersection and finite union, and a monotone and additive mapping \(\lambda : \mathcal{F} \to [0, +\infty]\). The additivity means that
\[
\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B)
\]
whenever \(A, B, A \cup B \in \mathcal{F}\).

By a partition (detailed, \((\mathcal{F}, \lambda)-\text{partition}\)) of a set \(A \in \mathcal{F}\) we mean a finite collection \(\{(\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k)\}\) such that
(i) \(\mathcal{U}_1, \ldots, \mathcal{U}_k \in \mathcal{F}\),
(ii) \(\bigcup_{i=1}^{k} \mathcal{U}_i = A\),
(iii) \(\lambda(\mathcal{U}_i \cap \mathcal{U}_j) = 0\) whenever \(i \neq j\),
(iv) \(t_i \in \mathcal{U}_i (i = 1, \ldots, k)\).

A finite collection \(\{(\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k)\}\) of subsets of \(A \in \mathcal{F}\), satisfying conditions (i), (iii) and (iv), but not necessarily (ii), is said to be decomposition of \(A\). We shall assume that \(\mathcal{F}\) separates points in the following way: to any \(A \in \mathcal{F}\) there exists a sequence \((\mathcal{A}_n)_{n}\) of partitions of \(A\) such that
(i) \(\mathcal{A}_{n+1}\) is a refinement of \(\mathcal{A}_n\),
(ii) to any \(x, y \in A, x \neq y\), there exist \(n \in \mathbb{N}\) and \(B \in \mathcal{A}_n\) such that \(x \in B\) and \(y \notin B\).
We note that this assumption is fulfilled if the topological space \(X\) is metrizable or it satisfies the second axiom of countability (see [10]).

A *gauge* on a set \(A \subset X\) is a mapping \(\delta\) assigning to every point \(x \in A\) a neighborhood \(\delta(x)\) of \(x\). If \(D = \{(U_1, t_1), \ldots, (U_k, t_k)\}\) is a decomposition of \(A\) and \(\delta\) is a gauge on \(A\), then we say that \(D\) is \(\delta\)-fine if \(U_i \subset \delta(t_i)\) for any \(i \in \{1, 2, \ldots, k\}\).

We obtain a simple example putting \(X = [a, b] \subset \mathbb{R}\) with the usual topology, \(\mathcal{F}\) = the family of all finite unions of closed subintervals of \(X\), \(\lambda([\alpha, \beta]) = \beta - \alpha\), \(a \leq \alpha < \beta \leq b\). Any gauge can be represented by a real function \(d : [a, b] \to \mathbb{R}^+\), if we put \(\delta(x) = (x - d(x), x + d(x))\).

Another example is the unbounded interval \([a, +\infty) = [a, +\infty) \cup \{+\infty\}\) considered as the one-point compactification of the locally compact space \([a, +\infty)\). The base of open sets consists on open subsets of \([a, +\infty)\) and the sets of the type \((b, +\infty) \cup \{+\infty\}, a \leq b < +\infty\). Any gauge in \([a, +\infty)\) has the form \(\delta(x) = (x - d(x), x + d(x))\), if \(x \in [a, +\infty] \cap \mathbb{R}\), and \(\delta(+\infty) = (b, +\infty) = (b, +\infty) \cup \{+\infty\}\), where \(d\) denotes a positive real-valued function defined on \([a, +\infty)\), and \(b\) denotes a real number.

Let us return to the definition of Kurzweil-Henstock integral (KH-integral) on \(X\). If \(D = \{(U_1, t_1), \ldots, (U_k, t_k)\}\) is a decomposition of a set \(A\), and \(f : X \to \mathbb{R}\), then we define the Riemann sum as follows:

\[
S(f, D) = \sum_{i=1}^{k} f(t_i) \lambda(U_i),
\]

if the sum exists in \(\mathbb{R}\), with the convention \(0 \cdot (+\infty) = 0 \cdot (-\infty) = 0\).

We note that the fact that \(\mathcal{F}\) separates points guarantees the existence of at least one \(\delta\)-fine partition \(D\) such that \(S(f, D)\) is well-defined for any gauge \(\delta\) (see [10]).

**Definition 3.1** A function \(f : X \to \mathbb{R}\) is said to be *integrable* on a set \(A\) if there exist \(I \in \mathbb{R}\) and a \((D)\)-sequence \((b_{i,j})_{i,j}\) such that \(\forall \varphi \in \mathbb{N}^\mathbb{N}\) there exists a gauge \(\delta\) on \(A\) such that \(S(f, D)\) exists, and

\[
|S(f, D) - I| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}
\]  \hspace{1cm} (2)

whenever \(D\) is a \(\delta\)-fine partition of \(A\) such that \(S(f, D)\) exists in \(\mathbb{R}\). We denote

\[
I = \int_A f
\]
On the Henstock-Kurzweil integral

We now state the Cauchy criterion for KH-integrability.

**Theorem 3.2** A map \( f : X \to R \) is integrable on \( A \) if and only if there exists a \((D)\)-sequence \((b_{i,j})_{i,j}\) such that, \( \forall \varphi \in \mathbb{N}^N, \exists \) a gauge \( \delta \) such that for every \( \delta \)-fine partition \( D_1, D_2 \) of \( A \) we have

\[
|S(f, D_1) - S(f, D_2)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.
\]

**Proof:** The proof is similar to the one of Theorem 5.2.9, p. 77, of [12]. \( \square \)

We now prove a result about HK-integrability on subsets.

**Theorem 3.3** If \( f : X \to R \) is integrable on \( A \in \mathcal{F} \) and \( B \in \mathcal{F}, B \subset A \), then \( f \) is integrable on \( B \) too. Moreover, if \((b_{i,j})_{i,j}\) is a regulator, satisfying condition (2) of integrability relatively to \( A \), then \((b_{i,j})_{i,j}\) satisfies condition (2) of integrability with respect to \( B \) too.

**Proof:** The proof is similar to the one of Proposition 5.2.10, p. 79, of [12], and to the one of Lemma 1.10, p. 157, of [10]. \( \square \)

Similarly as in [10], Lemma 2.1., pp. 158-159, and in [12], Theorem 5.3.1, pp. 82-83, it is possible to prove the following version of Henstock’s Lemma:

**Theorem 3.4** Let \( f : X \to R \) be integrable on \( A \), and \((b_{i,j})_{i,j}\) be a regulator, such that \( \forall \varphi \in \mathbb{N}^N \) there exists a gauge \( \delta \) such that

\[
\left| S(f, D) - \int_A f \right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}
\]

for all \( \delta \)-fine partitions \( D \) of \( A \).

Then, if \( \delta \) is a gauge and \( D = \{(J_i, \xi_i) : i = 1, \ldots, k\} \) is a \( \delta \)-fine decomposition of \( A \), we have

\[
\sum_{i=1}^{k} \left| f(\xi_i) \lambda(J_i) - \int_{J_i} f \right| \leq 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.
\]
4 The convergence theorem.

We now prove the following theorem:

**Theorem 4.1** Let $X = X_0 \cup \{x_0\}$ be the one-point compactification of a locally compact space $X_0$. Let $f : X \to R$ be a function such that $f(x_0) = 0$. Let $(A_n)_n$ be a sequence of sets, such that $A_n \in \mathcal{F}$, $A_n \subset \text{int } A_{n+1}$, $A_{n+1} \setminus \text{int } A_n \in \mathcal{F}$, $\lambda(A_n \setminus \text{int } A_n) = 0$ ($n \in \mathbb{N}$), $\bigcup_{n=1}^{\infty} A_n = X_0$. Let $f$ be integrable on $A$ for every $A \in \mathcal{F}$, with $A \subset X_0$, and let there exist a regulator $(a_{i,j})_{i,j}$ and an element $I \in R$ such that, $\forall \varphi \in \mathbb{N}^R$, there exists an integer $n_0$ such that

$$\left| \int_A f - I \right| \leq \sum_{i=1}^{\infty} a_{i,\varphi(i)} \quad \forall A \in \mathcal{F}, X_0 \supset A \supset A_{n_0}. \quad (3)$$

Moreover, suppose that

4.1.1) there exist $u \in R$, $u \geq 0$, and a gauge $\delta_0$, such that for every $\delta_0$-fine partition $\mathcal{D}$ of $X$, $\mathcal{D} = \{(U_1,t_1), \ldots, (U_k,t_k)\}$, we have:

$$\left| S(f,\mathcal{D}) - \int_{U_{i_1 \ldots i_k \neq x_0}} U_i f \right| \leq u.$$ 

Then $f$ is integrable on $X$ and $\int_X f = I$.

**Proof:** Let $(a_{i,j})_{i,j}$ be as in the hypotheses of the theorem, choose arbitrarily an element $\varphi \in \mathbb{N}^R$ and, in correspondence with $\varphi$, let $n_0$ be as in (3). Put $A_0 = \emptyset$, $B_0 = A_1$, $B_n = A_{n+1} \setminus \text{int } A_n$ ($n \in \mathbb{N}$). For all $n \in \mathbb{N}$ there exists a $(D)$-sequence $(b_{i,j}^{(n)})_{i,j}$ such that $\forall \varphi \in \mathbb{N}^R$ there exists a gauge $\delta_n$ on $B_n$ such that

$$\left| \int_{B_n} f - S(f,\mathcal{D}_n) \right| \leq \sum_{i=1}^{\infty} b_{i,\varphi(i+n)}^{(n)} \quad (4)$$

for any $\delta_n$-fine partition $\mathcal{D}_n$ of $B_n$. From (4) and the Henstock Lemma it follows that

$$\left| \int_{U_{i_{k-1}i_1 \ldots i_k}} V_i f - S(f,\mathcal{E}_n) \right| \leq 2 \sum_{i=1}^{\infty} b_{i,\varphi(i+n)}^{(n)} \quad (5)$$
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for each \( \delta_n \)-fine decomposition \( \mathcal{E}_n = \{(\mathcal{V}_1, t_1), \ldots, (\mathcal{V}_h, t_h)\} \) of \( B_n \). Let now \((c_{i,j})_{i,j}\) be a \( (D) \)-sequence such that

\[
\left( \sum_{n=1}^{\infty} \left( \bigvee_{i=1}^{\infty} b_{i,\varphi(i+n)}^{(n)} \right) \right) \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}, \quad \forall \varphi \in \mathbb{N}^N,
\]

where \( u \) is as in 4.1.1): such a sequence does exist, by virtue of the Fremlin Lemma (see [12]). Evidently we have:

\[
B_n \cap B_{n-1} = A_n \setminus \text{int } A_n \quad \forall n \in \mathbb{N}.
\]

Therefore

\[
B_n = (B_n \cap B_{n-1}) \cup (\text{int } B_n) \cup (B_n \cap B_{n+1}).
\]

Moreover, it is easy to check that

\[
B_j \cap B_l = \emptyset \quad \text{whenever } |j - l| \geq 2 \tag{6}
\]

and that

\[
(\text{int } B_n) \cap (\text{int } B_{n+1}) = \emptyset \quad \forall n \in \mathbb{N}. \tag{7}
\]

Now define a gauge \( \delta \) on \( X \) by the following formula:

\[
\delta(x) = \begin{cases} 
\delta_n(x) \cap (\text{int } B_n) & \text{if } x \in \text{int } B_n, \\
\delta_n(x) \cap \delta_{n+1}(x) \cap (\text{int } A_{n+1}) & \text{if } x \in B_n \cap B_{n+1}, \quad (n = 1, 2, \ldots) \\
(X_0 \setminus A_{i_0}) \cup \{x_0\} & \text{if } x = x_0.
\end{cases}
\]

Let \( \mathcal{D} = \{(\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k)\} \) be a \( \delta \)-fine partition of \( X \). There exists \((\mathcal{U}_{i_0}, t_{i_0}) \in \mathcal{D}, \) with \( i_0 \in \{1, 2, \ldots, k\} \), such that \( x_0 \in \mathcal{U}_{i_0} \). We shall prove that \( t_{i_0} = x_0 \). Namely, in the opposite case,

\[
x_0 \in \mathcal{U}_{i_0} \subset \delta(t_{i_0}) \subset \delta_n(t_{i_0})
\]

for some \( n \). But \( \delta_n(t) \subset X_0 \) for \( t \neq x_0 \). We have obtained \( x_0 \in X_0 \), that is a contradiction.
Since $f(x_0) = 0$, the Riemann sum $S(f, \mathcal{D})$ has the form
\[
\sum_{i=1, \ldots, k; i \neq i_0} f(t_i) \lambda(U_i),
\]
and $t_i \in X_0$ ($i = 1, \ldots, k; i \neq i_0$). Let
\[
A = \bigcup_{i=1, \ldots, k; i \neq i_0} U_i
\]
and
\[
T = \{n \in \mathbb{N} : \exists i \in \{1, \ldots, k\}, i \neq i_0 : B_n \cap U_i \neq \emptyset\}. \tag{8}
\]
By (8), and since $\mathcal{D}$ is a $\delta$-fine partition of $X$, we get
\[
X_0 \supset A \supset A_{i_0}. \tag{9}
\]
By hypothesis we have
\[
\left| \int_A f - I \right| \leq \sup_{i=1}^{\infty} a_{i, \nu(i)}. \tag{10}
\]
We claim that, if $U_i, i \neq i_0$, has nonempty intersection with at least two of the $\text{int } B_n$’s, then necessarily there exists $n \in \mathbb{N}$ such that the point $t_i$ corresponding to $U_i$ belongs to $B_n \cap B_{n+1}$. Indeed, if $t_i \in \text{int } B_n$ for some $n$, then, from (8) and the fact that $\mathcal{D}$ is a $\delta$-fine partition of $X$, we’d have
\[
U_i \subset \delta(t_i) \subset \text{int } B_n:
\]
this is impossible, by virtue of (6) and (7). From this and since
\[
(B_{n-1} \cap B_n) \cap (B_n \cap B_{n+1}) = \emptyset \quad \forall n,
\]
it follows that, for every $i = 1, 2, \ldots, k, i \neq i_0$, the $B_n$’s having nonempty intersection with $U_i$ are at most two, while the $B_n$’s which have nonempty intersection with $U_{i_0}$ can be infinitely many (even all the $B_n$’s). Thus we proved that the set $T$ in (8) is finite.

For $n \in T$ define a decomposition $\mathcal{E}_n$ of $B_n$ in the following way:
\[
\mathcal{E}_n = \{(U_i, t_i) : t_i \in \text{int } B_n\}
\]
\[
\cup \{(U_i \cap B_n, t_i) : t_i \in B_n \cap B_{n-1}\}
\]
\[
\cup \{(U_i \cap B_n, t_i) : t_i \in B_n \cap B_{n+1}\}.
\]
Then, by construction, we have:

\[ S(f, D) = \sum_{n \in T} S(f, E_n). \]  

(11)

by additivity of \( \lambda \) and since \( A_n \setminus \text{int} A_n = B_n \cap B_{n+1} \subset \text{int} A_{n+1} \) and \( \lambda(A_n \setminus \text{int} A_n) = 0 \ \forall \ n \in \mathbb{N} \). Moreover, we get:

\[ \sum_{n \in T} \int_{\bigcup_{l_i \subset \text{int} B_n, i \neq i_0 U_i} f = \int_A f. \]  

(12)

Since \( D_n \) is \( \delta_n \)-fine, we have (4). From 4.1.1), (4), (10), (11), (12) and (9) we obtain:

\[
|S(f, D) - I| = \left| \sum_{n \in T} S(f, E_n) - I \right|
+ \sum_{n \in T} \left| \int_{\bigcup_{l_i \subset \text{int} B_n, i \neq i_0 U_i} f - I \right|
\leq \sum_{n \in T} \left| S(f, E_n) - \int_{\bigcup_{l_i \subset \text{int} B_n, i \neq i_0 U_i} f \right| + \left| \int_A f - I \right|
\leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)},
\]

where \( d_{i,j} = 2(c_{i,j} + a_{i,j}) \ \forall \ i, j \in \mathbb{N} \). From this the assertion follows, since the double sequence \( (d_{i,j})_{i,j} \) is a \( (D) \)-sequence. \( \square \)

**Remark 4.2** First of all we observe that, in the classical cases, 4.1.1) is readily fulfilled. There are also several other situations in which 4.1.1) is satisfied: we now prove it when \( R = L^0(X, B, \mu) \), where \( (X, B, \mu) \) is a measure space, with \( \mu \) positive, \( \sigma \)-additive and \( \sigma \)-finite. By proceeding similarly as in the proof of the classical case and by using the same notations as above, by classical properties of \( L^0 \), we get that \( \forall \ n \in \mathbb{N}, \) there exists \( u_n \in R, \ u_n \geq 0, \) such that, \( \forall \varepsilon > 0, \ \exists \) a gauge \( \delta_n \) on \( B_n \) such that

\[ \left| \int_{\bigcup_{i=1}^{h} V_i} f - S(f, E_n) \right| \leq \frac{\varepsilon}{2} u_n \]  

(13)

for each \( \delta_n \)-fine decomposition \( E_n = \{(V_1, t_1), \ldots, (V_h, t_h)\} \) of \( B_n \). Since \( R \) satisfies property \( \sigma \), then, in correspondence with the sequence \( (u_n)_{n} \), there exist a sequence
(\lambda_n)_n \text{ of positive real-valued numbers and a positive element } u \in R, \text{ such that } \\
\lambda_n u_n \leq u \quad \forall n \in \mathbb{N}.

So, we note that, \( \forall n \in \mathbb{N}, \forall \varepsilon > 0, \exists \) a gauge \( \delta_n \) on \( B_n \) such that
\[
\left| \int_{\bigcup_{i=1}^k V_i} f - S(f, \mathcal{E}_n) \right| \leq \frac{\varepsilon \lambda_n}{2n+2} u_n \leq \frac{\varepsilon}{2n+2} u
\]
for every \( \delta_n \)-fine decomposition \( \mathcal{E}_n = \{(V_1, t_1), \ldots, (V_h, t_h)\} \) of \( B_n \). Let now \( \delta \) be a gauge as in (8): proceeding analogously as in the proof of Theorem 4.1.1), we get that, if \( D = \{(U_1, t_1), \ldots, (U_h, t_h)\} \) is any \( \delta \)-fine partition of \( X \), then
\[
\left| S(f, D) - \int_{\bigcup_{i=1}^k U_i \neq x_0} f \right| \leq \frac{\varepsilon}{4} u.
\]
From this, taking arbitrarily \( \varepsilon \in [0, 4] \), 4.1.1) follows. \( \square \)

5 Applications.

We now prove the following results, which are consequences of Theorem 4.1:

**Proposition 5.1** Let \( f : [a, +\infty) \to R \) be such that \( f(+\infty) = 0 \), \( f \) be integrable on \([a, b]\) for any \( b > a \), and let there exist in \( R \) the limit
\[
(D) \lim_{b \to +\infty} \int_{[a, b]} f.
\]
Moreover, suppose that

5.1.1) there exist \( u \in R, u \geq 0, \) and a map \( d_0 : [a, +\infty) \to \mathbb{R}^+ \), such that for every \( b \) with \( a < b < +\infty \) and for every \( d_0 \)-fine partition \( \Pi \) of \([a, b]\), we have:
\[
\sum_{\Pi} f_a^b - \int_a^b f \leq u
\]
(Here, the symbol \( \sum_{\Pi} f_a^b \) denotes the Riemann sum, related to the interval \([a, b]\)).

Then \( f \) is integrable on \([a, +\infty]\), and
\[
\int_{[a, +\infty]} f = (D) \lim_{b \to +\infty} \int_{[a, b]} f.
\]
Sketch of the proof: We note that Proposition 5.1 is a consequence of Theorem 4.1, where \( X = [a, +\infty] \), \( x_0 = +\infty \). Indeed, let us consider a strictly increasing sequence \((b_n)_n\) of real numbers, with \( b_1 = a \) and \( \lim_{n \to +\infty} b_n = +\infty \), and pick \( A_n = [a, b_n], \ n \in \mathbb{N} \). So, if \( b_n < b \leq b_{n+1} \), to any partition of the interval \([a, b]\) corresponds a partition \( D \) of \([0, +\infty)\), \( D = \{(U_1, t_1), \ldots, (U_h, t_h)\} \), in which the \( U_i \)'s can be taken as closed intervals of the real line, except \( U_{i_0} \), where \( i_0 \) is taken according to notations of Theorem 4.1, \( U_{i_0} = [b, +\infty] \) and \( \max[\bigcup_{i \neq i_0} U_i] = b \).

Proposition 5.2 Let \( a, b \in \mathbb{R}, \ a < b, \ f : [a, b] \to \mathbb{R} \) be integrable on \([a, x]\) for any \( a \leq x < b \), and let there exist in \( \mathbb{R} \) the limit
\[
(D) \lim_{x \to b^-} \int_{[a,x]} f.
\]
Moreover, suppose that

5.2.1) there exist \( u \in \mathbb{R}, \ u \geq 0 \), and a map \( d_0 : [a, b] \to \mathbb{R}^+ \), such that for every \( x \) with \( a \leq x < b \) and for every \( d_0 \)-fine partition \( \Pi \) of \([a, x]\), we have:
\[
\left| \sum_{\Pi} \frac{x}{a} f - \int_{a}^{x} f \right| \leq u
\]
(Here, the symbol \( \sum_{\Pi} \frac{x}{a} \) denotes the Riemann sum, related to the interval \([a, x]\)).

Then \( f \) is integrable on \([a, b]\), and
\[
\int_{[a,b]} f = (D) \lim_{x \to b^-} \int_{[a,x]} f.
\]

Proof: We observe that \([a, b] = [a, b) \cup \{b\}\) can be considered as the one-point compactification of \([a, b]\). The only difference is that we did not assume \( f(b) = 0 \). Of course, one can put \( g(x) = f(x) - f(b) \), and use Theorem 4.1 with respect to the function \( g \). Then we have
\[
\int_{[a,b]} g = (D) \lim_{x \to b^-} \int_{[a,x]} g;
\]
and hence
\[
\int_{[a,b]} f = (b - a) f(b) + \int_{[a,b]} g
= (D) \lim_{x \to b^-} [(x - a) f(b)] + (D) \lim_{x \to b^-} \int_{[a,x]} g
= (D) \lim_{x \to b^-} \int_{[a,x]} (g + f(b)) = (D) \lim_{x \to b^-} \int_{[a,x]} f.
\]
Now, it is enough to proceed analogously as in the proof of Proposition 5.1.

References


