Applying Linear Quantifier Elimination

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The linear quantifier elimination algorithm for ordered fields in [15] is applicable to a wide range of examples involving even non-linear parameters. The Skolem sets of the original algorithm are generalized to elimination sets whose size is linear in the number of atomic formulas, whereas the size of Skolem sets is quadratic in this number. Elimination sets may contain non-standard terms which enter the computation symbolically. Many cases can be treated by special methods improving further the empirical computing times.

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1. INTRODUCTION

In [15] the second author presented a quantifier elimination algorithm for "linear" formulas in ordered fields which is asymptotically worst case optimal up to a constant. It achieves a worst-case computing time bound which is polynomial in the length of the input formula, exponential in the number of quantified variables, and doubly exponential in the number of quantifier blocks only. Until recently (compare [9]), research on the implementation of quantifier elimination algorithms has been concentrated on the full elementary theory of real closed fields; there is a widespread belief [5] starting with the original work of Tarski [13] that these algorithms are of a rather academic nature and are not feasible for any "realistic" applications.

On the other hand, it was known for some time, that special cases of the general algorithm, like the simplex algorithm, perform very well in practice, and have even a polynomial average computing time. Recently Colmerauer[2] has used the simplex algorithm in the realm of logic programming with constraints and he incorporated it in the new programming language PROLOG III.

In this paper we optimize the algorithm in [15] in various respects and apply the optimized versions to a number of test problems. In sections 2 and 3 we introduce definitions and theorems that allow us to use elimination sets that are considerably smaller than the Skolem sets introduced in the original exposition. Based on a variant of Tarski's principle for linear ordered fields we introduce implicitly non-standard terms into the elimination sets and the computation. This enables us to reduce the size of the elimination sets from quadratic to linear in the length of the input. Moreover, a number of special cases can be handled even more efficiently. In section 4 we apply the modified elimination algorithms to several problems and study the empirical behavior and computing times. It turns out that in some well-known test examples the algorithms perform significantly better than the all-purpose CAD-algorithm in [3], and comparable to the improved version of this algorithm in [8]. Morover, we obtain for the first time a (huge, in principle) solution of the 3-dimensional planar transportation problem (see [14]). By way of contrast, some linear test problems (such as the recursive definition of a sequence with period 9, or the perfect rectangle problem, see[2]) that have been solved with PROLOG III rather successfully, could not be solved so far by our method due to storage limitations. This may be partly explained by the fact that the PROLOG III programs are tailored specifically to the problems, while our method is essentially a "general purpose" method for linear real problems.

The bottleneck of our method is never the time required, but instead the size of the intermediate or final results, that can be coped with only insufficiently by the simplifiers in the present implementations (in REDUCE in Passau and in ALDES/SAC-2 in Tübingen). What is needed in addition is a highly efficient simplifier for quantifier-free formulas that combines both boolean simplification and the simplification rules for linear orders and ordered fields.

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A corresponding REDUCE-implementation of the linear quantifier elimination procedure for the domain of integers in [16] has been carried out in [10].

2. ELIMINATION SETS AND SKOLEM SETS

In this section, we describe the general logical principles that underly the method of quantifier elimination by elimination sets.

We consider elementary languages L given by a finite set of constants and finitary operation and relation symbols. From these symbols together with an infinite supply V of variables $x, y, \ldots$, the equality sign "=", the logical symbols $\wedge, \vee, \neg, \exists, \forall$ and brackets (,), the terms and formulas of L are built up: **terms** $t, t', \ldots$ are formed from constants and variables by superposition of operation symbols; **atomic formulas** are Boolean constants T (true), F (false), or equations $(t = t')$ between terms, or formal relations $R(t_1, \ldots, t_n)$ between terms; arbitrary **formulas** $\varphi, \psi, \ldots$ are obtained from atomic formulas by closure under $\wedge, \vee, \neg$ and quantifiers $\exists x$ or $\forall x, \varphi \longrightarrow \psi$ and $\varphi \leftrightarrow \psi$ are abbreviations for $\neg\varphi \vee \psi$ and $(\varphi \longrightarrow \psi) \wedge (\psi \longrightarrow \varphi)$. An occurence of a variable x in a formula φ is **bound**, if it is in the scope of a quantifier $\exists x$ or $\forall x$; otherwise, it is **free**. A formula containing no quantifier is **quantifier-free**. A formula containing no variable free is a **sentence**. A **theory** T in L is a set of L-sentences. A L-**structure** A is a non-empty set, where the constants, operation symbols and relation symbols of L are interpreted as elements, operations and relations of the appropriate arity. If $\varphi(\mathbf{x})$ is an L-formula with all free variables in the string $\mathbf{x}$, A is an L-structure and $\mathbf{a}$ is a string of elements of A matching $\mathbf{x}$, then $A \models \varphi(\mathbf{a})$ means "φ **holds** in A for the parameters $\mathbf{x} = \mathbf{a}$". For a sentence φ the parameters are deleted. A is a **model** of an L-theory T, if $A \models \varphi$ for all sentences $\varphi \in T$. A sentence φ is a **consequence** of $T, T \models \varphi$, if φ holds in all models of T. Let T be a theory in L and Φ a set of L-formulas. Then a **decision procedure** for Φ, T is a procedure that decides for any sentence $\varphi \in \Phi$ whether $T \models \varphi$ or not. A **quantifier elimination** for Φ, T is a procedure $\varphi \mapsto \varphi'$ assigning to any formula $\varphi \in \Phi$ a quantifier-free formula $\varphi' \in \Phi$ such that φ and φ' are T-equivalent. If in these definition Φ is the set $Fo(L)$ of all L-formulas, then the reference to Φ is omitted.

Let L be an elementary language and let T be a theory in L. Let $X \subset V$ be an infinite set of variables, let Z be a set consisting of L-terms and possibly some "improper" terms involving new symbols (such as e.g. ∞), let Θ be a set of atomic L-formulas, and let Φ the closure of Θ under $\wedge, \vee, \neg$ and quantifiers $\exists x, \forall x$ with $x \in X$. Assume, moreover that θ, X, Z are related as follows:

There is a modified substitution procedure assigning to a variable $x \in X$, a term $t \in Z$, and an atomic formula $\vartheta \in \Theta$ a quantifier-free formula $\vartheta(x//t) \in \Phi$ such that whenever t is a proper L-term, then $\vartheta(x//t)$ is equivalent in T to the formula $\vartheta(x/t)$ obtained from ϑ by substituting t for x in the usual sense.

For $\varphi \in \Phi$, we let $\varphi(x//t)$ denote the expression resulting from φ by replacing each occurence of an atomic subformula ϑ in φ, in which x is free, by $\vartheta(x//t)$. Then $\varphi(x//t)$ is a formula in Φ.

If t is a term, φ is a formula, then $X(t), X(\varphi)$ denote the set of variables $x \in X$ occurring in t and φ respectively.

DEFINITION 2.1. *Let φ be a quantifier-free formula in Φ, let $x \in X$, and let S be a set of terms $t \in Z$ with $x \notin X(t)$. Then we say S is an* **elimination set** *for $\exists x\varphi$ (for $\forall x\varphi$) with respect to T, if the equivalence*

$$\exists x\varphi \iff \bigvee_{t\in S} \varphi(x//t)$$

$$(\ \forall x\varphi \iff \bigwedge_{t\in S} \varphi(x//t)\ )$$

holds in T.

The following properties of elimination sets are obvious:

REMARK 2.1. (i) Let S be a set of L-terms t with $x \notin X(t)$, and assume

$$\exists x\varphi \implies \bigvee_{t\in S} \varphi(x//t)$$

$$(\ \forall x\varphi \impliedby \bigwedge_{t\in S} \varphi(x//t)\ )$$

holds in T. Then S is an elimination set for $\exists x\varphi$ (for $\forall x\varphi$).

(ii) Let S be an elimination set for $\exists x\varphi$ (for $\forall x\varphi$) and let $S' \supseteq S$ be a finite set of L-terms t with $x \notin X(t)$. Then S' is an elimination set for $\exists x\varphi$ (for $\forall x\varphi$) as well.

(iii) Let S be an elimination set for $\exists x\varphi$ (for $\forall x\varphi$). Then S is also an elimination set for $\forall x\neg\varphi$ (for $\exists x\neg\varphi$).

Let us say T **has quantifier elimination for Φ by (proper) terms**, if for every quantifier-free L-formula φ and every variable $x \in X$ one can compute an elimination set S for $\exists x\varphi$ (that consists of L-terms only).

A theory T with this property has a very strong model theoretic property: Call a substructure A of an L-structure B a Φ-**elementary substructure**, if for every formula $\varphi(x_1, \ldots, x_n) \in \Phi$ and all $a_1, \ldots, a_n \in A$, $\varphi(a_1, \ldots, a_n)$ holds in A iff it holds in B. Then we can assert:

THEOREM 2.1. *(i) If T has quantifier elimination for Φ by terms, then there exists an algorithm that computes for every formula $\varphi \in \Phi$ a quantifier-free formula $\varphi' \in \Phi$ such that φ and φ' are equivalent in T.*

(ii) If T has quantifier elimination for Φ by proper terms, then every substructure A of a T-model B is a Φ-elementary substructure of B.

Proof. (i) The algorithm is by an easy recursion on the number of quantifiers in φ.

(ii) Using part (i) of remark 2.1 in the proof of (i) above, one finds that the equivalence between φ' and φ is valid in every substructure A of some T-model B. So if $\varphi = \varphi(\mathbf{x})$ and $\mathbf{a} \in A^n$, then

$$A \models \varphi(\mathbf{a}) \iff A \models \varphi'(\mathbf{a}) \iff$$
$$B \models \varphi'(\mathbf{a}) \iff B \models \varphi(\mathbf{a}) \qquad \blacksquare$$

If Ψ is a finite subset of $\Theta, x \in X$, S is a finite set of L-terms $t \in Z$ with $x \notin X(t)$, then we say, S is **a set of Skolem terms for** x, Ψ if

$$T \models \forall x(\bigvee_{t \in S} \bigwedge_{\psi \in \Psi} (\psi \leftrightarrow \psi(x//t)).$$

We call T a **Skolem theory** with respect to X, Z, Θ, if for every $x \in X$ and every finite $\Psi \subseteq \Theta, x, \Psi$ has a set of Skolem terms.

LEMMA 2.2. *Let φ be a quantifier-free formula in Φ, let $x \in X$, let Ψ be the set of all atomic subformulas of φ containing the variable x, and let S be a set of Skolem terms for x, Ψ. Then S is an elimination set for $\exists x\varphi$ and for $\forall x\varphi$.*

Proof. Let A be a model of T, let $\varphi = \varphi(x, y_1, \ldots, y_n)$, let $a, b_1, \ldots, b_n \in A$ and assume $A \models \varphi(a, b_1, \ldots, b_n)$. Then there exists $t = t(y_1, \ldots, y_n) \in S$ such that for all $\psi \in \Psi$, $A \models \psi(a, b_1, \ldots, b_n) \iff A \models \psi(x//t)(b_1, \ldots, b_n)$. So $A \models \varphi(t(\mathbf{b}), \mathbf{b})$. By remark 2.1, this shows that S is an elimination set for $\exists x\varphi$. Since φ and $\neg\varphi$ have the same set Ψ of atomic subformulas, the same argument shows that S is also an elimination set for $\exists x(\neg\varphi)$, and hence for $\forall x\varphi$. $\blacksquare$

The converse to this lemma fails as the following simple example shows:

EXAMPLE 2.1. Let T be the theory of ordered fields in $L = \{0, 1, +, -, \cdot, =, <\}$, let $X = V$, let Z be the set of all L-terms and let Θ be the set of all atomic L-formulas. Consider the L-formula $\varphi : \quad x = 0 \vee x < 0 \vee -x < 0$. Then $S = \{0\}$ is an elimination set for $\exists x\varphi$ and for $\forall x\varphi$, but not a Skolem set for x and $\Psi = \{x = 0, x < 0, -x < 0\}$, since any such Skolem set must contain at least 3 terms.

3. ELIMINATION SETS FOR LINEAR FORMULAS IN ORDERED FIELDS

3.1. Rational Elimination sets

In this section we restrict our attention to the theory T_{OF} of ordered fields in the language $L_{OF} = \{0, 1, +, -, \cdot, ^{-1}, =, \neq, \leq, <\}$ with the convention that $0^{-1} = 0$. The reason, why we include $\neq$ and $\leq$ among the atomic relations is technical and will become apparent below. We let X be an arbitrary subset of V. Z consists of all L-terms that can be written (provably in T) in the form $a_0 + a_1x_1 + \ldots + a_nx_n$, where $x_1, \ldots, x_n$ are distinct variables in X and $a_0, \ldots, a_n$ are arbitrary L-terms with $X(a_i) = \emptyset$. Θ consists of all equations and inequalities between terms in Θ. We refer to X, Z, Θ, Φ as the set of **linear variables, linear terms, linear atomic formulas, linear formulas**, respectively (compare [15]). Modified substitution of a linear term for a linear variable in a linear atomic formula coincides with ordinary substitution. One verifies easily that this does not lead outside the set Θ of linear atomic formulas.

Let $x \in X$. Then any linear atomic formula ψ that involves x can be written equivalently in T in the form $ax = b$ or $ax < b$, where b is a linear term with $x \notin X(b)$ and a is a linear term containing no variable linear variable. If ψ is given in this form, we say ψ is *normalized with respect to x.*

The following lemma (see [15], lemma 2.4 and 2.5) shows that T_{OF} is a Skolem theory with respect to X, Z, Θ :

LEMMA 3.1. *Let $x \in X$ and let $\Psi = \{a_ix\ \rho_i\ b_i : i \in I,\ \rho_i \in \{=, \neq, \leq, <\}\}$ be a set of atomic linear formulas that are normalized with respect to the linear variable x. Then the following set S is a set of Skolem terms for x, Ψ:*

$$S = \{b_ia_i^{-1}, b_ia_i^{-1} \pm 1 : i \in I\} \cup$$
$$\{1/2(b_ia_i^{-1} + b_ja_j^{-1}) : i, j \in I, i \neq j\}$$

Moreover, if $\rho_i \in \{=, \neq\}$, then

$$S' = \{b_ia_i^{-1}, b_ia_i^{-1} \pm 1 : i \in I\}$$

is a set of Skolem terms for x, Ψ.

COROLLARY 3.2. (LINEAR TARSKI PRINCIPLE)

(i) T_{OF} has quantifier elimination by proper terms.
(ii) Let A be an ordered subfield of the ordered field B. Then A is a Φ-elementary substructure of B with respect to the set Φ of linear formulas.

For the rest of this section, our goal is to find elimination sets for $\exists\varphi$ and for $\forall x\varphi$, that are smaller than the Skolem set S for x, Ψ specified above. This will be achieved by taking into account the different atomic relations $=, \neq, \leq, <$ occuring in the linear formula φ. Further improvements are possible, if the formula φ has a special boolean structure. The resulting quantifier elimination procedures will be considerably faster in practice as will be documented in section 4. By the lower complexity bounds in [15], the asymptotic order of growth of the complexity of quantifier elimination in the worst case is doubly exponential in the number of quantifier-blocks of the (prenex) input formula. So for an arbitrary linear formula φ, this growth rate cannot be improved. We can, however, significantly reduce the multiplicative constant in the exponent of the asymptotic complexity. This is achieved by replacing the Skolem set above, whose size is quadratic in the

number $atom(\varphi)$ of atomic subformulas of φ by various elimination sets of size linear in $atom(\varphi)$.

DEFINITION 3.1. *A linear formula is* **positive**, *if it contains no negation. Let φ be a positive quantifier-free linear formula, and let $x \in X$ be a linear variable. Then $A(\varphi)$ denotes the set of all atomic subformulas of φ. $A_{lin}(\varphi)$ denotes the set of all atomic subformulas of φ containing at least one linear variable $A_1(x,\varphi)$ $[A_2(x,\varphi)$, $A_3(x,\varphi)$, $\mathbf{A}_4(x,\varphi)]$ denotes the set of all atomic subformulas ψ of φ such that $x \in X(\psi)$ and ψ is of the form $t = t'$ [of the form $t \leq t'$, of the form $t < t'$, of the form $t \neq t'$]. $A(x,\varphi) = A_1(x,\varphi) \cup A_2(x,\varphi) \cup A_3(x,\varphi) \cup A_4(x,\varphi)$.*

THEOREM 3.3. *Let φ be a positive quantifier-free linear formula, and let $x \in X$ be a linear variable. We assume without restriction that for $k = 1,2,3,4$, $A_k(x,\varphi) = \{a_i x \ \rho_k \ b_i : \ i \in I_k\}$, where $\rho_k \in \{=, \leq, <, \neq\}$. Let*

$$S = \{\frac{b_i}{a_i} : i \in I_1 \cup I_2\} \cup$$

$$\{\frac{1}{2}(\frac{b_i}{a_i} + \frac{b_j}{a_j}) : \ i,j \in I_3 \cup I_4, \ i \neq j\} \cup$$

$$\{\frac{b_i}{a_i} \pm 1 : \ i \in I_3 \cup I_4\}.$$

Then S is an elimination set for $\exists x\varphi$.

The elimination set S specified above should be compared with the corresponding Skolem set S' described above in lemma 3.1:

$$S' = \{\frac{b_i}{a_i} : i \in I\} \cup$$

$$\{\frac{1}{2}(\frac{b_i}{a_i} + \frac{b_j}{a_j}) : \ i,j \in I, \ i \neq j\} \ \cup \ \{\frac{b_i}{a_i} \pm 1 : \ i \in I\},$$

which is considerably larger, especially if $I_1 \cup I_2 \neq \emptyset$. Nevertheless, the set S is still quadratic in the size of $A_3(x,\varphi) \cup A_4(x,\varphi)$.

Proof. We assume to begin with that φ is a conjunction of atomic formulas and that $I_4 = \emptyset$. For fixed values of all a_i, b_i in some ordered field F, we consider the solution set M of φ wrt x, i.e. the set of all $c \in F$ such that $\varphi(c)$ holds in F. Suppose that $M \neq \emptyset$. Then by our hypothesis, M is a non-empty interval (possibly semiinfinite or infinite). We distinguish several cases:

Case 1. M is unbounded from above. Then M contains one of the points $\frac{b_i}{a_i} + 1$ for $i \in I_3$ or one of the points $\frac{b_i}{a_i}$ for $i \in I_2$.

Case 2. M is unbounded from below. Then M contains one of the points $\frac{b_i}{a_i} - 1$ for $i \in I_3$ or one of the points $\frac{b_i}{a_i}$ for $i \in I_2$.

Case 3. M is bounded. Then M is of one of the following types:

1. $[\frac{b_i}{a_i}, \frac{b_j}{a_j}]$ with $i,j \in I_1 \cup I_2$.
2. $[\frac{b_i}{a_i}, \frac{b_j}{a_j})$ with $i \in I_2, \ j \in I_3$.
3. $(\frac{b_i}{a_i}, \frac{b_j}{a_j}]$ with $i \in I_3, \ j \in I_2$.
4. $(\frac{b_i}{a_i}, \frac{b_j}{a_j})$ with $i,j \in I_3$.

Moreover, in each subcase, $a_i, a_j \neq 0$. In subcases 1 and 2, $\varphi(x//\frac{b_i}{a_i})$ holds in $\mathbf{R}$; in subcase 3, $\varphi(x//\frac{b_j}{a_j})$ holds in $\mathbf{R}$; finally in subcase 4, $\varphi(x//\frac{1}{2}(\frac{b_i}{a_i} + \frac{b_j}{a_j}))$ holds in $\mathbf{R}$.

If φ is an arbitrary positive quantifier-free linear formula, one can first eliminate all relations " $\neq$ " occuring in φ by replacing $c \neq d$ by $c < d \vee -c < -dc$; then the resulting formula is equivalent to a disjunction of conjunctions φ_j $(j \in J)$ of atomic formulas, where $I_4 = \emptyset$. As a consequence, $\exists x\varphi$ is equivalent to $\bigvee_{j \in J} \exists x\varphi_j$. So by the proof above and remark 2.1 (ii), S is an elimination set for each φ_i, and hence for φ. ∎

COROLLARY 3.4. *Let φ be a positive quantifier-free linear formula, and let $x \in X$ be a linear variable. We assume without restriction that for $k = 1,2,3,4$, $A_k(x,\varphi) = \{a_i x \ \rho_k \ b_i : \ i \in I_k\}$, where $\rho_k \in \{=, \leq, <, \neq\}$. Let*

$$S = \{\frac{b_i}{a_i} : i \in I_3 \cup I_4\} \cup$$

$$\{\frac{1}{2}(\frac{b_i}{a_i} + \frac{b_j}{a_j}) : \ i,j \in I_1 \cup I_2, \ i \neq j\} \cup$$

$$\{\frac{b_i}{a_i} \pm 1 : \ i \in I_1 \cup I_2\}.$$

Then S is an elimination set for $\forall x\varphi$.

Proof. Let ψ be the quantifier-free linear formula resulting from $\neg\varphi$ by "pulling negations in front of atomic subformulas" and replacing negated atomic formulas by their obvious unnegated equivalents in $\mathbf{R}$. Then $A_1(x,\psi) = \{a_i x = b_i : i \in I_4\}$, $A_2(x,\psi) = \{(-a_i)x \leq (-b_i) : \ i \in I_3\}$, $A_3(x,\psi) = \{(-a_i)x < (-b_i) : \ i \in I_2\}$, $A_4(x,\psi) = \{a_i x \neq b_i : i \in I_1\}$. So by the theorem above, S is an elimination set for $\exists x\psi$. Hence

$$\forall x\varphi \iff \neg\exists x\neg\varphi \iff$$

$$\neg\exists x\psi \iff \neg \bigvee_{t \in S} \psi(x//t) \iff$$

$$\neg \bigvee_{t \in S} \neg\varphi(x//t) \iff \bigwedge_{t \in S} \varphi(x//t)$$ ∎

3.2. Elimination Sets with $\pm\infty$

In order to further optimize the size of elimination sets, we now introduce the substitution of the dummy symbols ∞ and $-\infty$ as improper terms for a linear variable in a linear formula. For the case that the linear formula in question contains no parameters, the method is due to [6]. It suffices to define the substitution $\varphi(x//\pm\infty)$ of $\pm\infty$ for $x \in X$ in an atomic formula φ of the form $ax \ \rho \ b$, where a,b are linear terms, $\rho \in \{=, \leq, <\}$ and a contains no linear variable. If a contains no parameters,

then a can be computed as a rational number. In this case, the definition is as follows:

$$a\cdot(\pm\infty)=b \;:=\; \begin{cases} 0=b & \text{if } a=0\\ F & \text{otherwise}\end{cases}$$

$$a\cdot(\pm\infty)\le b \;:=\; \begin{cases} 0\le b & \text{if } a=0\\ F(T) & \text{if } a>0\\ T(F) & \text{if } a<0\end{cases}$$

$$a\cdot(\pm\infty)< b \;:=\; \begin{cases} 0< b & \text{if } a=0\\ F(T) & \text{if } a>0\\ T(F) & \text{if } a<0\end{cases}$$

$$a\cdot(\pm\infty)\ne b \;:=\; \begin{cases} 0\ne b & \text{if } a=0\\ T & \text{otherwise}\end{cases}$$

If a contains at least one parameter, a corresponding case distinction is formulated as a linear formula; in this the definition is as follows:

$$a\cdot(\pm\infty)=b \;:=\; (a=0\ \wedge\ 0=b)$$

$$a\cdot(\pm\infty)\le b \;:=\; (a=0\ \wedge\ 0\le b)\ \vee\ (a\overset{\le}{>}0)$$

$$a\cdot(\pm\infty)< b \;:=\; (a=0\ \wedge\ 0< b)\ \vee\ (a\overset{\le}{>}0)$$

$$a\cdot(\pm\infty)\ne b \;:=\; (a=0\ \wedge\ 0\ne b)\ \vee\ a\ne 0$$

So in each case, this definition correctly simulates the truth value of the linear formula $a\cdot x\ \rho\ b$ for sufficiently large values of x in an arbitrary ordered field. If a quantifier-free linear formula φ contains no parameters then $\varphi(x//\pm\infty)$ has the same "Boolean structure" as φ. In particular, under the substitution of $\pm\infty$, all the sets $A(x,\varphi)$, $A_k(x,\varphi)$ $(k=1,2,3,4)$ are transformed into sets of atomic linear formulas of the same number of elements. If φ does contain parameters, this is not the case. Notice, however, that the additional atomic formulas entering in $\varphi(\pm\infty)$ contain no linear variables. So the number of elements in $A_{lin}(\varphi)$ is invariant under the substitution of $\pm\infty$.

An easy inspection of the proof shows now that in lemma 3.1, theorem 3.3 and corollary 3.4 the terms $\frac{b_i}{a_i}+1$ can be replaced by ∞ and the terms $\frac{b_i}{a_i}-1$ can be replaced by $-\infty$ (compare also the proofs of theorem 3.5 and corollary 3.6 below).

3.3. Elimination Sets with Infinitesimals

An even greater reduction in size of elimination sets can be achieved by introducing **infinitesimals**. Recall that by the linear Tarski principle 3.2 the following holds: Whenever $F\subseteq F'$ are ordered fields, then$F$ is a Φ-elementary substructure of F'. Moreover, every ordered field F has an ordered extension field F' containing a element ϵ, that is positive and infinitesimal wrt F, i.e. $0<\epsilon<c$ for every positive $c\in F$.

In the following, we use ϵ as a dummy symbol for such a positive infinitesimal. We are going to formally substitute improper terms of the form $c\pm\epsilon$ (c a linear term) for a linear variable x in a quantifier-free linear formula. It suffices again to define the substitution $\varphi(x//c\pm\epsilon)$ of $c\pm\epsilon$ for $x\in X$ in an atomic formula φ of the form $ax\ \rho\ b$, where a,b are linear terms, $\rho\in\{=,\le,<,\ne\}$ and a contains no linear variable. If a contains no parameters, then a can be computed as a rational number. In this case, the definition is as follows:

$$a\cdot(c\pm\epsilon)=b \;:=\; \begin{cases} 0=b & \text{if } a=0\\ F & \text{otherwise}\end{cases}$$

$$a\cdot(c\pm\epsilon)\le b \;:=\; \begin{cases} ac<b & \text{if } a>0\ \ (\text{if } a<0)\\ ac\le b & \text{if } a\le 0\ \ (\text{if } a\ge 0)\end{cases}$$

$$a\cdot(c\pm\epsilon)< b \;:=\; \begin{cases} ac<b & \text{if } a\ge 0\ \ (\text{if } a\le 0)\\ ac\le b & \text{if } a<0\ \ (\text{if } a>0)\end{cases}$$

$$a\cdot(c\pm\epsilon)\ne b \;:=\; \begin{cases} 0\ne b & \text{if } a=0\\ T & \text{otherwise}\end{cases}$$

If a contains at least one parameter, a corresponding case distinction is formulated as a linear formula; in this case the definition is as follows:

$$a\cdot(c\pm\epsilon)=b \;:=\; (a=0\ \wedge\ 0=b)$$

$$a\cdot(c+\epsilon)\le b \;:=\; (a>0\wedge ac<b)\ \vee\ (a\le 0\wedge ac\le b)$$

$$a\cdot(c+\epsilon)< b \;:=\; (a<0\ \wedge\ ac\le b)\ \vee\ (a\ge 0\wedge ac<b)$$

$$a\cdot(c-\epsilon)\le b \;:=\; (a\ge 0\wedge ac\le b)\ \vee\ (a<0\wedge ac<b)$$

$$a\cdot(c-\epsilon)< b \;:=\; (a\le 0\ \wedge\ ac<b)\ \vee\ (a>0\wedge ac\le b)$$

$$a\cdot(c\pm\epsilon)\ne b \;:=\; 0\ne b\ \vee\ a\ne 0$$

So in each case, this definition correctly simulates the truth value of the linear formula $a\cdot x\ \rho\ b$ for a value $c\pm\epsilon$ of x in F', where $a,b,c\in F$ (F an arbitrary ordered field) and $\epsilon\in F'$ a positive infinitesimal wrt F. If a quantifier-free linear formula φ contains no parameters then $\varphi(x//c\pm\epsilon)$ has the same "boolean structure" as φ. In particular, all the sets $A(\varphi)$, $A_k(x,\varphi)$ $(k=1,2,3,4)$ are transformed into sets of atomic linear formulas of the same number of elements under the substitution of $c\pm\epsilon$. If φ does contain parameters, this is not the case. Instead, the size of $A_{lin}(\varphi)$ doubles. Notice, however, that each inequality in $A(x,\varphi)$ produces only one strict and one weak inequality involving linear variables, both of which have the same right- and left-hand sides.

Using the substitution of these improper terms, the size of elimination sets can now be decreased significantly. In the following we use the set notation $\{\infty,c_j: j\in J\}$ to denote the set $\{\infty\}\ \cup\ \{c_j: j\in J\}$ in case $J\ne\emptyset$, and to denote $\emptyset$ otherwise.

THEOREM 3.5. *Let φ be a positive quantifier-free linear formula, and let $x\in X$ be a linear variable. We assume without restriction that for $k=1,2,3,4$, $A_k(x,\varphi)=\{a_i x\ \rho_k\ b_i:\ i\in I_k\}$, where $\rho_k\in\{=,\le,<,\ne\}$. Let $S=\{\frac{b_i}{a_i}: i\in I_1\cup I_2\}\cup\{\infty,\ \frac{b_i}{a_i}-\epsilon:\ i\in I_3\cup I_4\}$. Then S is an elimination set for $\exists x\varphi$.*

Proof. As in the proof of 3.3, it suffices to consider the case that φ is a conjunction of atomic formulas. Let M be the non-empty interval defined in this proof, and let $\epsilon\in F'\supseteq F$ be positive infinitesimal wrt F. Suppose to begin with that M is unbounded from above. If $I_3\cup I_4\ne\emptyset$ then $\varphi(x//\infty)$; otherwise M contains some

point of the form $\frac{b_i}{a_i}$ with $i \in I_1 \cup I_2$, and so $\varphi(x//\frac{b_i}{a_i})$ holds in F'. holds in F'.

If M is bounded from above, two cases can occur:

Case 1: M contains its upper endpoint, which is then of the form $\frac{b_i}{a_i}$ with $i \in I_1 \cup I_2$. Then $\varphi(x//\frac{b_i}{a_i})$ holds in F'.

Case 2: M does not contain its upper endpoint, which is then of the form $\frac{b_i}{a_i}$ with $i \in I_3 \cup I_4$. Then $\frac{b_i}{a_i} - \epsilon \in M$, and so $\varphi(x//\frac{b_i}{a_i} - \epsilon)$ holds in F'. ■

Applying the proof of 3.4 to this theorem, we obtain the corresponding elimination set for a universal quantifier:

COROLLARY 3.6. *Let φ be a positive quantifier-free linear formula, and let $x \in X$ be a linear variable. We assume without restriction that for $k = 1,2,3,4$, $A_k(x,\varphi) = \{a_i x \ \rho_k \ b_i : \ i \in I_k\}$, where $\rho_k \in \{=, \leq, <, \neq\}$. Let $S = \{\frac{b_i}{a_i} : i \in I_3 \cup I_4\} \cup \{\infty, \frac{b_i}{a_i} - \epsilon : \ i \in I_1 \cup I_2\}$. Then S is an elimination set for $\forall x \varphi$.*

REMARK 3.1. The elimination sets S in theorem 3.5 and corollary 3.6 appear to be asymmetric with respect to ∞ and $-$; an inspection of the proof of 3.5 shows, however, that this is not the case: If one replaces ∞ by $-\infty$ and $\frac{b_i}{a_i} - \epsilon$ by $\frac{b_i}{a_i} + \epsilon$ in the statement of 3.3 or3.4, one obtains another elimination set for $\exists x \varphi$ or $\forall x \varphi$, respectively.

In order to compare the efficiency of the quantifier elimination methods provided by 3.3 and 3.4 versus 3.5 and the elimination by a Skolem set, we apply these methods to the elimination of a block of existential or universal quantifiers. We use the following complexity measures: $at(\varphi) = \sharp A(\varphi)$, $at_{lin}(\varphi) = \sharp A_{lin}(\varphi)$. For an n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ of linear variables, $A_k(\mathbf{x},\varphi)$ $k = 1,2,3,4$ and $A(\mathbf{x},\varphi)$ are defined similar as for a single linear variable x; so $A_1(\mathbf{x},\varphi)$, $A_2(\mathbf{x},\varphi)$, $A_3(\mathbf{x},\varphi)$ $A_4(\mathbf{x},\varphi)$ is the set of all equations, weak inequalities, strict inequalities, inequations, respectively, that contain at least one x_i. $at_k(\mathbf{x},\varphi) = \sharp A_k(\mathbf{x},\varphi)$ for $k = 1,2,3,4$, $at(\mathbf{x},\varphi) = \sharp A(\mathbf{x},\varphi)$. $at'(\varphi)$ $[at'(x,\varphi), at'_{lin}(\varphi)]$ is the number of atomic formulas in $At(\varphi)$ [in $At(x,\varphi)$, in $At_{lin}(\varphi)$], where formulas that differ only in their relation-symbols are counted only once. So $at(\varphi) \leq 4 \cdot at'(\varphi)$, $at_{lin}(\varphi) \leq 4 \cdot at'_{lin}(\varphi)$ and $at(x,\varphi) \leq 4 \cdot at'(x,\varphi)$.

THEOREM 3.7. *Let φ be a positive quantifier-free linear formula, let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-tuple of linear variables. Then quantifier elimination for the linear formula $\exists \mathbf{x} \varphi$ according to 3.3 or 3.5 or the use of Skolem sets yield an equivalent linear formula φ' of the form $\bigvee_{j \in J} \varphi'_j$ with the following respective complexity bounds:*

$$at_k(\mathbf{x}, \varphi'_j) \ \leq \ at_k(\mathbf{x}, \varphi)$$

for $k = 1,2,3,4$ in all cases except theorem 3.5, if φ contains parameters; in this case,

$$at'(\mathbf{x}, \varphi'_j) \ \leq \ at'(\mathbf{x}, \varphi)$$

$$at(\varphi'_j) \leq \begin{cases} at(\varphi) & \text{for Skolem sets} \\ at(\varphi) & \text{for theorem 3.3} \\ at(\varphi) & \text{for theorem 3.5 } (\varphi \text{ w.o. param.}) \end{cases}$$

$at'(\varphi'_j) \leq at'(\varphi) + n \cdot at'(\mathbf{x},\varphi)$ *and* $at'_{lin}(\varphi'_j) \leq at'_{lin}(\varphi)$ *for theorem 3.5 (φ with parameters)*

$$\sharp J \leq \begin{cases} \alpha^n & \text{for Skolem sets} \\ \beta^n & \text{for theorem 3.3} \\ \gamma^n & \text{for theorem 3.5 } (\varphi \text{ without parameters}) \\ \delta^n & \text{for theorem 3.5 } (\varphi \text{ with parameters}) \end{cases}$$

where

$$\alpha = at(\mathbf{x},\varphi))^2 + 2 \cdot at(\mathbf{x},\varphi), \quad \beta = \beta_{12} + \frac{1}{2}\beta_{34} + 2\beta_{34},$$

with $\beta_{12} = at_1(\mathbf{x},\varphi) + at_2(\mathbf{x},\varphi)$, $\beta_{34} = at_3(\mathbf{x},\varphi) + at_4(\mathbf{x},\varphi)$,

$$\gamma = at(\mathbf{x},\varphi) + 1, \quad \delta = 2 \cdot at'(\mathbf{x},\varphi) + 1.$$

Corresponding results hold for blocks of universal quantifiers.

Proof. Induction on n, taking into account that an existential quantifier can be interchanged with a disjunction. ■

So for arbitrary positive quantifier-free linear formulas φ the method of theorem 3.5 and corollary 3.6 provide elimination sets of size linear in $at(\mathbf{x},\varphi)$, whereas the corresponding Skolem sets, as well as the elimination sets described in theorem 3.3 and corollary 3.4, have size quadratic in $at(\mathbf{x},\varphi)$.

3.4. Taking the Boolean Structure into Account

So far, elimination sets were totally determined by the set of atomic formulas and the quantifier to be eliminated. Next, we consider linear formulas φ of special boolean structure that admit even smaller elimination sets. Our first example is just a slight variant of [15], lemma 2.3, case 1:

THEOREM 3.8. *Let φ be a quantifier-free linear formula (not necessarily positive), let x be a linear variable, and let $A(x,\varphi) = \{a_i x = b_i : i \in I\} = A_1(x,\varphi)$. Then $S = \{\frac{b_i}{a_i} : i \in I\} \cup \{\infty\}$ is a Skolem set for $x, A(x,\varphi)$, and hence an elimination set for $\exists x \varphi$ and for $\forall x \varphi$. Morover, if φ is positive, then $S = \{\frac{b_i}{a_i} : i \in I\}$ is an elimination set for $\exists x \varphi$.*

Proof. Easy. ■

DEFINITION 3.2. *Let φ be a positive quantifier-free linear formula and $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-tuple of linear variables. Then we say φ is an* **generalized equation, generalized weak inequality, generalized strict inequality** *wrt $\mathbf{x}$, if all atomic subformulas of φ that contain some x_i are identical to a*

single equation, weak inequality, strict inequality, respectively. Typical examples of such formulas are the formulas arising from the formal substitution of $\pm\infty$ into some atomic linear formula.

Next, we consider quantifier-free linear formulas of the form

$$\varphi_1 \implies \varphi_2, \tag{1}$$

where φ_1 is a positive quantifier-free linear formula with $A_3(\mathbf{x},\varphi) = A_4(\mathbf{x},\varphi) = \emptyset$ and φ_2 is a conjunction of generalized equations and generalized weak inequalities wrt $\mathbf{x}$.

THEOREM 3.9. *Let φ be a quantifier-free linear formula of the form (1) wrt the linear variable x, and assume that $A_1(\mathbf{x},\varphi_1) = \{a_i x = b_i : i \in I_1\}$ and $A_2(\mathbf{x},\varphi_1) = \{a_i x \leq b_i : i \in I_2\}$. Then*

$$S = \{\frac{b_i}{a_i} : i \in I_1 \cup I_2\} \cup \{\infty, -\infty\}$$

is an elimination set for $\forall x\varphi$.

Proof. For fixed values of all linear variables and parameters in φ except x in some ordered field F, φ_2 defines a closed interval M_2 in F and φ_1 defines a finite disjoint union M_1 of closed intervals in F. So $\forall x\varphi$ asserts that $M_1 \subseteq M_2$. This is equivalent to the assertion that all the endpoints of the closed intervals constituting M_1 are contained in M_2. ∎

By induction on n this theorem can be generalized to the elimination of a block of universal quantifiers. (This uses essentially the assumption that φ_2 is a conjunction of **generalized** equations and weak inequalities in order to handle the formal substitution of $\pm\infty$.)

THEOREM 3.10. *Let φ be a quantifier-free linear formula of the form (1) wrt the n-tuple $(x_1,\ldots,x_n)$ of linear variables. Then quantifier elimination for the linear formula $\forall\mathbf{x}\varphi$ according to the preceeding theorem yields an equivalent linear formula φ' of the form $\bigwedge_{j\in J}\varphi'_j$ with the following respective complexity bounds:*

$$at_k(\mathbf{x},\varphi'_j) \leq at_k(\mathbf{x},\varphi) \quad \text{for} \quad k = 1,2.$$

$$at(\varphi'_j) \leq at(\varphi) + 2n \cdot at(\mathbf{x},\varphi_1)$$

$$\sharp J \leq \alpha^n, \quad \text{where } \alpha = at(\mathbf{x},\varphi_1) + 2$$

An even simpler case has also applications:

THEOREM 3.11. *Let x be a linear variable, t a term with $x \notin X(t)$, $0 \neq q$ a rational number, and let φ be a quantifier-free formula (not necessarily positive nor linear), Then $S = \{t\}$ is an elimination set for the formulas*

$$\forall x\{q \cdot x = t \implies \varphi(x)\},$$

$$\exists x\{q \cdot x = t \wedge \varphi(x)\},$$

since both formulas are equivalent to $\varphi(x//t \cdot q^{-1})$.

Proof. trivial. ∎

The theorem can be used to eliminate a block of universal quantifiers or a block of existential quantifiers in linear formulas of the form

$$\forall x_1 \ldots \forall x_r(q_1 x_1 = t_1 \wedge \ldots \wedge q_r x_r = t_r \implies \varphi(x_1,\ldots,x_r))$$

or

$$\exists x_1 \ldots \exists x_r(q_1 x_1 = t_1 \wedge \ldots \wedge q_r x_r = t_r \wedge \varphi(x_1,\ldots,x_r))$$

where $q_1,\ldots q_r$ are non-zero rational numbers.

Finally, we consider quantifier-free linear formulas φ of the form

$$(**) \quad \varphi_1 \implies \varphi_2,$$

where φ_1 is a positive quantifier-free linear formula with $A_k(\mathbf{x},\varphi) = \emptyset$ for $k = 1,2$, and φ_2 is a conjunction of generalized strict inequalities wrt $\mathbf{x}$.

Using the same idea as in the proof of theorem 3.9, one finds that the following set

$$S = \{\frac{b_i}{a_i} \pm \epsilon : i \in I\}$$

is an elimination set for $\forall x\varphi$, where $A(x,\varphi_1) = \{a_i x < b_i : i \in I\}$. By way of contrast, the elimination set provided by the general method described in corollary 3.6 is of the form

$$S' = \{\frac{b_i}{a_i} : i \in I\} \cup \{\infty, \frac{b_i}{a_i} - \epsilon : i \in I'\},$$

where $A(x,\varphi_2) = \{a_i x < b_i : i \in I'\}$. So, the present method is superior, if $at(x,\varphi_2) \geq at(x,\varphi_1)$.

3.5. Avoiding Inverses that involve Parameters

So far, we have made extensive use of inverses a^{-1} in elimination sets. By our convention that $0^{-1} = 0$, there is no need to introduce case distinctions $a = 0, a \neq 0$, if a containes parameters. This has two effects: On the one hand, it helps to keep the number of atomic formulas smaller during quantifier elimination; on the other hand, the nesting of inverses prohibits simplification of formulas by detection of trivially true or false atomic subformulas or of trivial equivalences between atomic subformulas. It turns out that in practice the second, adverse effect may outweigh the advantages of the first one. The reason for this fact is apparent at least when the quantifier elimination is performed according to theorem 3.5 and corollary 3.6: Here, the modified substitution of the improper terms occuring in the elimination sets require a corresponding case distinction anyway.

So we indicate now how to avoid the occurence of inverses involving parameters altogether by treating them as improper terms. For this purpose, we replace the inverse a^{-1} formally by a syntactically different inverse $inv(a)$. Then the resulting terms in the elimination sets provided be theorem 3.5 and corollary 3.6 are of the form $d \cdot inv(c)$ and $d \cdot inv(c) + \epsilon$, where c, d are linear L_{OF}-terms, $X(c) = \emptyset$ and ϵ is a positive infinitesimal. The modified substitution of these improper

terms for $x \in X$ in formulas of the form $a \cdot x \ \rho \ b$ with $\rho \in \{=, \leq, <, \neq\}$ is then defined as follows:

If c contains no parameters, modified substitution is defined as before; otherwise:

$$a(d \cdot inv(c)) \ \rho \ b \ :=$$
$$(c = 0 \wedge 0 \ \rho \ b) \vee (c \neq 0 \wedge ad \ \rho \ cb),$$

if $\rho \in \{=, \neq\}$;

$$a(d \cdot inv(c)) \ \rho \ b \ :=$$
$$(c = 0 \wedge 0 \ \rho \ b) \vee (0 < c \wedge ad \ \rho \ cb) \vee$$
$$(c < 0 \wedge -ad \ \rho \ -cb),$$

if $\rho \in \{\leq, <\}$;

$$a(d \cdot inv(c)) \pm \epsilon) \ \rho \ b \ :=$$
$$(c = 0 \wedge a(0 \pm \epsilon) \ \rho \ b) \vee (c \neq 0 \wedge a(d \pm \epsilon') \ \rho \ cb),$$

if $\rho \in \{=, \neq\}$;

$$a(d \cdot inv(c)) \pm \epsilon) \ \rho \ b \ :=$$
$$(c = 0 \wedge a(0 \pm \epsilon) \ \rho \ b) \ \vee (0 < c \wedge a(d \pm \epsilon') \ \rho \ cb) \vee$$
$$(c < 0 \wedge -a(d \pm \epsilon') \ \rho \ -cb),$$

if $\rho \in \{\leq, <\}$;

In both cases, the occurence of the improper terms $0 \pm \epsilon$ and $d \pm \epsilon'$ is purely formal and explained as above in the modified substitution, where ϵ and ϵ' denote positive infinitesimals.

The elimination of inverses achieved in this way is of great importance for the simplification of the formulas arising from quantifier elimination by terms.

Among the various simplification strategies that will be employed in the examples are the following:

1. Replacing atomic formulas that are obviously true or false by the elementary properties of equality and order (as e.g. $a = a$, $a \leq a$, $a < a$, $a \neq a$) by their respective truth values and logical simplification of the resulting formula according to the rules $(T \vee \varphi \iff T)$, $(T \wedge \varphi \iff \varphi)$, $(F \vee \varphi \iff \varphi)$, $(F \wedge \varphi \iff F)$. On a more sophisticated level, the system may recognize sums of squares with positive rational coefficients in formulas of the form $a \ \rho \ 0$ and simplify accordingly.

2. Elimination of many multiple occurences of the same atomic formula or of closely related atomic formulas (as e.g. $a = 0$, $a \neq 0$, $a \leq 0$, $a < 0$, $-a = 0$, $-a \neq 0$, $-a \leq 0$, $-a < 0$) in a linear formula φ. This is achieved by a version of the **semantic tableau method** that uses some very elementary properties of ordered fields: We replace φ by $(a = 0 \wedge \varphi_1) \vee (a < 0 \wedge \varphi_2) \vee (-a < 0 \wedge \varphi_3)$, where φ_i results from φ by first replacing each of the atomic formulas listed above by the truth values T of F they get from the respective hypothesis $a = 0$, $a < 0$, $-a < 0$, and then applying the logical simplification described above. In this way, an arbitrary number of occurences of related formulas is replaced by exactly three such occurences at the cost of multiplying the number of occurences of all other atomic formulas by three. This procedure can be iterated as long as atomic formulas with many occurences are present. In case φ contains no order-inequalities, it suffices to adjoin in a similar way the case distinction $a = 0 \ \vee \ a \neq 0$ instead of $a = 0 \ \vee \ a < 0 \ \vee \ a > 0$.

3.6. Extensions by Linearly Defined Functions

In some applications (such as the recursive definition of sequence with period 9 in [2]) it is useful to enrich the language by terms involving function symbols, which can be in principle defined in a simple manner by linear formulas. In many cases, however, the elimination of such a function symbol by its definition leads to an enormous increase in the length of the formula. Typical examples are the absolute value function, the sign function, the max and the min function.

Consider e.g. the absolute value function $|.|$. An occurence of the function symbol $|.|$ in an "extended" linear formula $\varphi(|t|)$ can be eliminated by replacing $\varphi(|t|)$ by

$$(0 \leq t \wedge \varphi(t)) \vee (t \leq 0 \wedge \varphi(-t))$$

or equivalently by

$$(t < 0 \vee \varphi(t)) \wedge (0 < t \vee \varphi(-t))$$

In either case, an elimination of k nested occurences of the absolute value function symbol in an "extended" linear formula will increase the number $at(\varphi)$ of atomic subformulas of φ to more than $at(\varphi)2^k$ in the resulting linear formula φ'. For the elimination of a quantifier in an extended linear formula by this method one has then to substitute all terms of a suitable elimination set S for the resulting formula into the latter formula.

A good deal of the complexity arising from this method can be avoided by substituting the terms of the elimination set S obtained in this way directly into the (much shorter) original "extended" formula. It is not difficult to verify that S does indeed form a semantically correct elimination set for the original formula.

4. APPLICATIONS

The following examples illustrate the scope of the method in general and of a REDUCE-implementation of the method in particular. The implementation is based on the Diplomarbeit [1]. The following computations use a thoroughly revised, extended and optimized version of these programs that is due to Thomas Sturm at the University of Passau. For the 3-dimensional planar transportation problem he was generously supported by Dr. H. Melenk, ZIB, Berlin. All the computations in Passau were done on an IBM RS6000/520 with 16 megabyte main memory.

4.1. An Arithmetic Component in an Expert System

The first application comes from Ph. Hanschke [7]. For the production of the work plan for certain mechanical parts (milling of metal parts) skeleton work plans incorporating geometrical and technological properties have to be available in a knowledge base. From the query consistency tests are generated automatically which are handled by an arithmetical component of the knowledge base. The formulas describe in parametrized form geometrical properties composed in a systematic fashion from geometrical primitives relevant for the construction of the mechanical parts. The formulas are all linear but involve free parameters in the coefficients. The formulas are in general quantified with both quantifiers and include negation and disjunction. As it turns out, already simple properties involve a large number of free variables whereas the number of quantified variables is small.

In the following example we describe the primitive property "Einschnitt ohne Hinterschneidung" (cut without undercutting) as a quantified formula. The quantifier-free result contains 68 atomic subformulas if no inverses containing variables (TRECIPs) are admitted. Using TRECIPs the result contains only 32 atomic subformulas. This is not surprising in view of the fact that TRECIPs implicitly code case distinctions.

The input formula is

$$\exists x \exists y (0 < x \wedge y < 0 \wedge x \cdot r - x \cdot tt + tt = q \cdot x - s \cdot x + s$$
$$\wedge x \cdot b - x \cdot d + d = a \cdot y - c \cdot y + c)$$

Quantifier elimination without infinitesimals (`off eps`) and with infinitesimals (`on eps`) yields:

method	# *atomic formulas*
`off eps`	32
`on eps`	37

Elimination of `TRECIPs` applied to these results yields:

input formula results from	# *atomic formulas*
`off eps`	68
`on eps`	68

After elimination of all TRECIP's, the results are in fact identical for both types of elimination sets.

An automatic iterative tableau-method with respect to the term $a - c$ yields the following simplified formula containing only 56 atomic subformulas:

$((s - tt > 0 \wedge q - r - s + tt < 0 \vee s - tt < 0 \wedge q - r - s + tt > 0) \wedge (b \cdot s - b \cdot tt + c \cdot q - c \cdot r - c \cdot s + c \cdot tt - d \cdot q + d \cdot r < 0 \wedge q - r - s + tt < 0 \vee b \cdot s - b \cdot tt + c \cdot q - c \cdot r - c \cdot s + c \cdot tt - d \cdot q + d \cdot r > 0 \wedge q - r - s + tt > 0 \vee c - d > 0 \wedge q - r - s + tt = 0) \wedge (s - tt = 0 \wedge q - r - s + tt = 0 \vee q - r - s + tt \neq 0) \wedge (a \cdot c - a \cdot d - c^2 + c \cdot d = 0 \wedge q - r - s + tt = 0 \vee a \cdot b \cdot s - a \cdot b \cdot tt + a \cdot c \cdot q - a \cdot c \cdot r - a \cdot c \cdot s + a \cdot c \cdot tt - a \cdot d \cdot q + a \cdot d \cdot r - b \cdot c \cdot s + b \cdot c \cdot tt - c^2 \cdot q + c^2 \cdot r + c^2 \cdot s - c^2 \cdot tt + c \cdot d \cdot q - c \cdot d \cdot r = 0 \wedge q - r - s + tt \neq 0) \vee (b - d = 0 \wedge c - d > 0 \vee b - d < 0) \wedge s - tt = 0 \wedge q - r - s + tt = 0) \wedge a - c > 0 \vee (b - d = 0 \wedge c - d < 0 \vee b - d > 0) \wedge q - r - s + tt = 0 \wedge s - tt = 0 \wedge a - c < 0 \vee ((s - tt > 0 \wedge q - r - s + tt < 0 \vee s - tt < 0 \wedge q - r - s + tt > 0) \wedge (s - tt = 0 \wedge q - r - s + tt = 0 \vee q - r - s + tt \neq 0) \wedge (b \cdot s - b \cdot tt + c \cdot q - c \cdot r - c \cdot s + c \cdot tt - d \cdot q + d \cdot r = 0 \wedge q - r - s + tt \neq 0 \vee c - d = 0 \wedge q - r - s + tt = 0) \vee (c - d < 0 \wedge b - d < 0 \vee c - d > 0 \wedge b - d > 0) \wedge (b \cdot s - b \cdot tt + c \cdot q - c \cdot r - c \cdot s + c \cdot tt - d \cdot q + d \cdot r = 0 \wedge b - d \neq 0 \vee s - tt = 0 \wedge b - d = 0) \wedge (c - d = 0 \wedge b - d = 0 \vee b - d \neq 0) \vee s - tt = 0 \wedge q - r - s + tt = 0 \wedge c - d = 0 \wedge b - d = 0) \wedge a - c = 0$

CPU-Time $\leq$ 6 sec.

4.2. The Collins/Johnson-Problem

The input formula is

$$\exists r (0 < r \wedge r < 1 \wedge 0 < a_2 \wedge a_1 < 0)$$

where

$$a_1 = (2 - 3r)(a^2 + b^2) + 4ar - 2a - r$$
$$a_2 = (1 - 3r)(a^2 + b^2) + 2ar$$

Quantifier elimination yields:

method	# *atomic formulas*
`off eps`	22
`on eps`	28

Elimination of `TRECIPs` applied to these results yields:

input formula results from	# *atomic formulas*
`off eps`	142
`on eps`	66

The shortest output without `TRECIPs` contains 66 atomic subformulas and reads:

$(a^2 - 2 \cdot a + b^2 + 1 \leq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee a^2 - 2 \cdot a + b^2 + 1 \geq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 > 0 \vee 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 = 0) \wedge (a^2 - a + b^2 < 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee a^2 - a + b^2 > 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 > 0) \wedge ((a^2 - a + b^2 \leq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 = 0 \vee 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 \neq 0) \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 = 0 \wedge a^2 - a + b^2 < 0) \wedge ((3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 \geq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee 3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 \leq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 > 0 \vee a^2 + b^2 \geq 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 = 0) \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \vee (3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 > 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee 3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 < 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 > 0 \vee a^2 + b^2 > 0 \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 = 0) \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 \leq 0) \vee (a^2 - a + b^2 \leq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 < 0 \vee a^2 - a + b^2 \geq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \vee 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 = 0) \wedge (a^2 + b^2 < 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 < 0 \vee a^2 + b^2 > 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0) \wedge ((3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 \geq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 < 0 \vee 3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 \leq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \vee a^2 - a + b^2 \leq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 = 0) \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 < 0 \vee (3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 > 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 < 0 \vee 3 \cdot a^4 - 6 \cdot a^3 + 6 \cdot a^2 \cdot b^2 + 3 \cdot a^2 - 6 \cdot a \cdot b^2 + 3 \cdot b^4 - b^2 < 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \vee a^2 - a + b^2 < 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 = 0) \wedge 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 \geq 0) \wedge ((a^2 + b^2 \geq 0 \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 = 0 \vee 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 \neq 0) \wedge 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \vee 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 = 0 \wedge a^2 + b^2 > 0) \vee (3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 <$

$0 \wedge a^2 - 2 \cdot a + b^2 + 1 \geq 0 \vee 3 \cdot a^2 - 4 \cdot a + 3 \cdot b^2 + 1 \geq 0 \wedge a^2 - 2 \cdot a + b^2 + 1 > 0) \wedge (3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 > 0 \wedge a^2 - a + b^2 \leq 0 \vee 3 \cdot a^2 - 2 \cdot a + 3 \cdot b^2 \leq 0 \wedge a^2 - a + b^2 < 0)$

The automatic iterative tableau-method was not able to reduce the size of this formula. An automatic simplifier that recognizes explicit sums of squares (in this case $a^2 + b^2$) was able to cut down this formula to a formula with only 63 atomic subformulas. It missed, however, certain implicit sums of squares occuring in the formula such as $a^2 - 2ab + b^2 + 1$.

CPU-Time = 1 sec.

For comparison: The output formula obtained by Collins' original CAD-method contains 649 atomic subformulas; the vastly improved version in [8] only 4 atomic subformulas. The respective timings are about 102 and 8 seconds (on a SUN Sparc station SLC).

4.3. The Davenport/Heintz-Example

The input formula for real quantifier elimination is

$$\exists c \forall b \forall a((a = d \wedge b = c) \vee (a = c \wedge b = 1)) \rightarrow a \cdot a = b)$$

It was presented in [4] as an example, where the original CAD-method of Collins performed badly; the improved method in [3] drastically reduced the computing time to 228 seconds on a SUN3/50.

We proceed as follows: First

In the given input formula

$$\exists c \forall b \forall a((a = d \wedge b = c) \vee (a = c \wedge b = 1)) \rightarrow a \cdot a = b)$$

the universal quantifiers can be interchanged. The resulting formula

$$\exists c \forall a \forall b((a = d \wedge b = c) \vee (a = c \wedge b = 1)) \rightarrow a \cdot a = b)$$

is linear in b und equivalent to the positive formula

$$\exists c \forall a \forall b((a \neq d \vee b \neq c) \wedge (a \neq c \vee b \neq 1) \vee a \cdot a = b)$$

The following table shows the effect of eliminating the quantifier $\forall b$ in this formula using different elimination sets.

Elimination Set	Time (ms)	*# At form*
Skolem	90	30
Improved	80	30
With ∞	80	20
With ∞ and ε	80	20

The next table shows the results of the same elimination procedures additionally calling a primitive simplifying algorithm for propositional logic during elimination.

Elimination Set	Time (ms)	*# At form*
Skolem	80	18
Improved	80	18
With ∞	70	8
With ∞ and ε	60	8

Elimination of the quantifier $\forall b$ using the last option yields the formula

$\varphi_0 := \exists c \forall a($
$((c - 1 \neq 0 \vee a - d \neq 0) \wedge a - c \neq 0 \vee a^2 - 1 = 0)$
$\wedge ((c - 1 \neq 0 \vee a - c \neq 0) \wedge a - d \neq 0 \vee a^2 - c = 0)$

In the this formula, the variable a has two non-linear occurences, viz. in the atomic formulas $a^2 - 1 = 0$ und $a^2 - c = 0$. We replace the first occurence equivalently by $a = 1 \vee a = -1$, the second by $a = \sqrt{c} \vee a = -\sqrt{c}$, where the squareroot symbol is purely formal. Elimination of the linear quantifier $\forall a$ can now be performed separately for the two parts of the conjunction. For the first part

$$\forall a((c \neq 1 \vee a \neq d) \wedge a \neq c \vee a = 1 \vee a = -1)$$

the resulting formula is

$$\varphi_1 := (c - 1 \neq 0 \wedge c - d \neq 0 \vee d + 1 = 0 \vee d - 1 = 0) \wedge (c + 1 = 0 \vee c - 1 = 0).$$

For the second part

$$\forall a((c \neq 1 \vee a \neq c) \wedge a \neq d \vee a = \sqrt{c} \vee a = -\sqrt{c})$$

the result is

$$\varphi_2 := (c - 1 \neq 0 \wedge c - d \neq 0 \vee \sqrt{c} + c = 0 \vee \sqrt{c} - c = 0) \wedge (\sqrt{c} + d = 0 \vee \sqrt{c} - d = 0)$$

In order to eliminate the existential quantifier $\exists c$ we have to eliminate the formal squareroot symbols. φ_2 is replaced by the equivalent formula

$$\varphi_2{}' := (c \neq 1 \wedge c \neq d \vee c = 0 \vee c = 1 \vee c = 0) \wedge c = d^2.$$

Elimination of the quantifier $\exists c$ in the formula $\exists c(\varphi_1 \wedge \varphi_2{}')$ yields

$(d - 1 = 0 \vee d + 1 = 0) \wedge d^2 - 1 = 0 \vee (d^2 - d \neq 0 \wedge d^2 - 1 \neq 0 \vee d - 1 = 0 \vee d + 1 = 0)$
$\wedge (d^2 - d \neq 0 \wedge d^2 - 1 \neq 0 \vee d^2 - 1 = 0 \vee d = 0) \wedge d^2 - 1 = 0$

containing 12 atomic subformulas.

Application of an automatic iterative tableau-simplifier (with respect to $d^2 - 1$) yields the formula

$$(d + 1 = 0 \vee d - 1 = 0) \wedge d^2 - 1 = 0$$

which is equivalent to the well-known solution $d = 1 \vee d = -1$.

CPU-time is $\leq$ 1 sec in this interactive quantifier elimination compared to 217 sec for the automatic quantifier elimination in [3].

4.4. Multidimensional Planar Transportation Problems

The following linear formulas $T_{i,m}$ describe the feasability conditions for parametrized planar transportation problems of dimension ≤ 3. The problem in each case is to find quantifier-free conditions on the parameters that are necessary and sufficient for the feasability of

the problem. For the communication of these problems we are indebted to Prof. Milan Vlach (comp. [14]).

In dimension 1, the formula is

$$T_{1,m} \;:=\; \exists x_1 \ldots \exists x_m (\sum_i x_i = a \;\wedge\; \bigwedge_i x_i \geq 0),$$

where the sum and the conjunction ranges from 1 to m.

Obviously, this formula is equivalent to $a \geq 0$.

In dimension 2, the formula is

$$T_{2,m} \;:=\; \exists \{x_{ij}\}_{1 \leq i,j \leq m} (\sum_i x_{ij} = a_i \;\wedge\; \sum_j x_{ij} = b_j \;\wedge\; \bigwedge_{i,j} x_{ij} \geq 0),$$

where the sums and the conjunction range from 1 to m.

Again it is known that $T_{2,m}$ is equivalent to the obvious necessary condition:

$$\bigwedge_i a_i \geq 0 \;\wedge\; \bigwedge_j b_j \geq 0 \;\wedge\; \sum_i a_i = \sum_j b_j$$

In dimension 3 there are two possible feasability conditions:

$$T_{3,m} \;:=\; \exists \{x_{ijk}\}_{1 \leq i,j,k \leq m} (\sum_{j,k} x_{ijk} = a_i \wedge \sum_{i,k} x_{ijk} = b_j \;\wedge\; \sum_{i,j} x_{ijk} = c_k \;\wedge\; \bigwedge_{i,j,k} x_{ijk} \geq 0),$$

and

$$T_{4,m} \;:=\; \exists \{x_{ijk}\}_{1 \leq i,j,k \leq m} (\sum_i x_{ijk} = a_{jk} \;\wedge\; \sum_j x_{ijk} = b_{ik} \;\wedge\; \sum_k x_{ijk} = c_{ij} \;\wedge\; \bigwedge_{i,j,k} x_{ijk} \geq 0),$$

where the sums and the conjunction range from 1 to m.

The first formula is known to be equivalent to

$$\bigwedge_i a_i \geq 0 \;\wedge\; \bigwedge_j b_j \geq 0 \;\wedge\; \bigwedge_k c_k \geq 0 \;\wedge\; \sum_i a_i = \sum_j b_j = \sum_k c_k$$

The second formula obviously implies

$$\bigwedge_{j,k} a_{jk} \geq 0 \;\wedge\; \bigwedge_{i,k} b_{ik} \geq 0 \;\wedge\; \bigwedge_{i,j} c_{ij} \geq 0 \;\wedge\; \sum_{j,k} a_{jk} = \sum_{i,k} b_{ik} = \sum_{i,j} c_{ij},$$

but it is known that for $m \geq 3$, $T_{4,m}$ is not equivalent to this condition (see [12, 14]).

A computer program for finding a quantifier-free equivalent to $T_{4,3}$ by a completely different specialized method was developed in [11] but did not produce any result after 50 hours computing time on an IBM 360/50.

As first input formula for linear quantifier elimination we take $T_{2,3}$, i.e. the two-dimensional transport problem for $m = 3$.

$$\begin{aligned} T_{2,3} \;:=\; & \exists x_{11} \exists x_{12} \exists x_{13} \exists x_{21} \exists x_{22} \exists x_{23} \exists x_{31} \exists x_{32} \exists x_{33} (\\ & x_{11} + x_{12} + x_{13} = a_1 \wedge x_{21} + x_{22} + x_{23} = a_2 \wedge \\ & x_{31} + x_{32} + x_{33} = a_3 \wedge x_{11} + x_{21} + x_{31} = b_1 \wedge \\ & x_{12} + x_{22} + x_{32} = b_2 \wedge x_{13} + x_{23} + x_{33} = b_3 \wedge \\ & 0 \leq x_{11} \wedge 0 \leq x_{12} \wedge 0 \leq x_{13} \wedge 0 \leq x_{21} \wedge 0 \leq x_{22} \wedge 0 \leq \\ & x_{23} \wedge 0 \leq x_{31} \wedge 0 \leq x_{32} \wedge 0 \leq x_{33}) \end{aligned}$$

Linear quantifier elimination using infinitesimals yields an output formula containing 486 atomic subformulas. Application of an automatic iterative tableau method reduces the number of atomic subformulas to 406. The equation $a_1 + a_2 + a_3 - b_1 - b_2 - b_3 = 0$ that appears as a conjunctive part in the result admits the substitution of b_3 for the term $a_1 + a_2 + a_3 - b_1 - b_2$ in the remaining formula. Since the formula contains moreover $a_1 \geq 0 \wedge a_2 \geq 0 \wedge a_3 \geq 0 \wedge b_1 \geq 0 \wedge b_2 \geq 0 \wedge b_3 \geq 0$ as a conjuctive part, the remaining formula ψ is equivalent to $\psi[\mathrm{true}/a_1 \geq 0, \mathrm{true}/a_2 \geq 0, \mathrm{true}/a_3 \geq 0, \mathrm{true}/b_1 \geq 0, \mathrm{true}/b_2 \geq 0, \mathrm{true}/b_3 \geq 0]$ Automatic boolean simplification of the latter formula results in the following final output formula containing 196 atomic subformulas.

$a_1 \geq 0 \wedge a_2 \geq 0 \wedge a_3 \geq 0 \wedge b_1 \geq 0 \wedge b_2 \geq 0 \wedge b_3 \geq 0 \wedge ((a_2 - b_2 \leq 0 \wedge b_1 - a_3 \geq 0 \vee a_2 - b_2 \geq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_2 - b_1 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_1 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \vee a_1 - b_1 - b_2 + a_3 \leq 0 \; landa_1 - b_2 \geq 0 \vee a_2 - b_1 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \wedge a_1 - b_2 \geq 0 \vee a_1 - b_1 - b_2 + a_3 \leq 0 \wedge a_1 - b_1 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \vee a_1 - b_2 \leq 0 \wedge b_1 - a_3 \geq 0 \vee a_2 - b_2 \geq 0 \wedge a_1 - b_1 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \vee a_2 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 - a_3 \geq 0 \vee b_2 - a_3 \geq 0 \wedge a_2 - b_1 \leq 0 \vee b_2 - a_3 \geq 0 \wedge a_2 - b_2 + a_3 \geq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_2 - b_1 \geq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_1 - b_1 - b_2 + a_3 \geq 0 \wedge b_2 - a_3 \geq 0 \vee a_1 - b_1 - b_2 + a_3 \leq 0 \wedge b_2 - a_3 \geq 0 \wedge a_1 - b_2 + a_3 \geq 0 \vee a_2 - b_1 \geq 0 \wedge a_1 - b_2 + a_3 \geq 0 \wedge b_2 - a_3 \geq 0 \vee a_1 - b_1 - b_2 + a_3 \leq 0 \wedge a_1 - b_1 \geq 0 \vee b_2 - a_3 \geq 0 \wedge a_1 - b_1 \leq 0 \vee a_2 - b_2 + a_3 \geq 0 \wedge b_2 - a_3 \geq 0 \wedge a_1 - b_1 \geq 0 \vee a_2 - b_1 - b_2 + a_3 \geq 0 \wedge b_2 - a_3 \geq 0 \vee b_2 - a_3 \leq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_2 - b_1 + a_3 \geq 0 \wedge a_2 - b_1 \leq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee b_2 - a_3 \leq 0 \wedge a_1 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 + b_2 - a_3 \geq 0 \vee b_1 - a_3 \leq 0 \wedge a_1 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 + b_2 - a_3 \geq 0 \vee b_2 - a_3 \leq 0 \wedge a_1 - b_1 - b_2 + a_3 \leq 0 \vee a_1 - b_2 \leq 0 \wedge a_1 - b_2 + a_3 \geq 0 \wedge a_1 - b_1 - b_2 + a_3 \leq 0 \vee b_2 - a_3 \leq 0 \wedge a_2 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 + b_2 - a_3 \geq 0 \vee a_2 - b_2 \leq 0 \wedge a_2 - b_2 + a_3 \geq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee b_1 - a_3 \leq 0 \wedge a_2 - b_1 - b_2 + a_3 \leq 0 \vee a_1 - b_1 + a_3 \geq 0 \wedge a_1 - b_1 \leq 0 \wedge a_1 - b_1 - b_2 + a_3 \leq 0 \vee b_1 - a_3 \leq 0 \wedge a_1 - b_1 - b_2 + a_3 \leq 0 \vee b_1 - a_3 \leq 0 \wedge a_2 - b_1 - b_2 + a_3 \geq 0 \wedge b_1 + b_2 - a_3 \geq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_2 - b_1 \leq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_2 - b_2 \leq 0 \vee a_2 - b_2 \geq 0 \wedge a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_2 - b_1 - b_2 \leq 0 \vee a_2 - b_1 \geq 0 \wedge a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_2 - b_1 - b_2 \leq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_1 - b_1 - b_2 \leq 0 \wedge a_1 - b_2 \geq 0 \vee a_2 - b_1 \geq 0 \wedge a_1 - b_2 \geq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_1 - b_1 - b_2 \leq 0 \wedge a_1 - b_1 \geq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_1 - b_1 \leq 0 \vee a_1 + a_2 - b_1 - b_2 \geq 0 \wedge a_1 - b_2 \leq 0 \vee a_2 - b_2 \geq 0 \wedge a_1 - b_1 \geq 0 \vee a_1 + a_2 - b_1 - b_2 \leq 0 \wedge a_1 - b_2 \geq 0 \vee a_2 - b_1 + a_3 \geq 0 \wedge a_2 - b_1 \leq 0 \wedge a_1 - b_2 \geq 0 \vee a_1 + a_2 - b_1 - b_2 \leq 0 \wedge a_2 - b_2 \leq 0 \wedge a_1 + a_2 - b_2 \geq 0 \vee b_1 - a_3 \leq$

$0\wedge a_1+a_2-b_2 \geq 0\wedge a_2-b_2 \leq 0\vee a_1+a_2-b_1-b_2 \leq 0\wedge a_2-b_2 \geq 0 \vee a_1+a_2-b_1-b_2 \leq 0 \wedge a_1-b_2 \leq 0 \wedge a_1+a_2-b_2 \geq 0 \vee a_2-b_1-b_2+a_3 \geq 0 \wedge a_2-b_1-b_2 \leq 0 \wedge a_2-b_2 \geq 0\vee a_1-b_1-b_2+a_3 \geq 0\wedge a_1-b_1-b_2 \leq 0\wedge a_1-b_2 \geq 0\vee b_1-a_3 \leq 0 \wedge a_1-b_2 \geq 0 \vee a_1-b_1+a_3 \geq 0 \wedge a_1-b_1 \leq 0 \wedge a_2-b_2 \geq 0 \vee b_1-a_3 \leq 0 \wedge a_1+a_2-b_2 \geq 0 \wedge a_1-b_2 \leq 0 \vee b_1-a_3 \leq 0\wedge a_2-b_2 \geq 0\vee a_1+a_2-b_1-b_2 \leq 0\wedge a_1+a_2-b_1 \geq 0\wedge a_2-b_1 \leq 0 \vee a_1+a_2-b_1-b_2 \leq 0 \wedge a_1-b_1 \geq 0 \vee a_1+a_2-b_1-b_2 \leq 0\wedge a_1+a_2-b_1 \geq 0\wedge a_1-b_1 \leq 0\vee a_1+a_2-b_1-b_2 \leq 0\wedge a_2-b_1 \geq 0\vee b_2-a_3 \leq 0\wedge a_1+a_2-b_1 \geq 0\wedge a_2-b_1 \leq 0\vee a_2-b_2+a_3 \geq 0\wedge a_2-b_2 \leq 0\wedge a_1-b_1 \geq 0\vee a_2-b_1-b_2+a_3 \geq 0\wedge a_2-b_1-b_2 \leq 0\wedge a_2-b_1 \geq 0\vee a_1-b_2 \leq 0\wedge a_2-b_1 \leq 0\vee a_2-b_1-b_2+a_3 \geq 0\wedge a_2-b_1 \leq 0\vee a_2-b_2 \leq 0\wedge a_1-b_1 \leq 0\vee a_2-b_1-b_2+a_3 \geq 0 \wedge a_2-b_2 \leq 0 \vee a_1-b_1-b_2+a_3 \geq 0 \wedge a_1-b_1-b_2 \leq 0 \wedge a_1-b_1 \geq 0 \vee b_2-a_3 \leq 0 \wedge a_1+a_2-b_1 \geq 0 \wedge a_1-b_1 \leq 0 \vee a_1-b_2+a_3 \geq 0 \wedge a_1-b_2 \leq 0 \wedge a_2-b_1 \geq 0 \vee b_2-a_3 \leq 0\wedge a_1-b_1 \geq 0\vee b_2-a_3 \leq 0\wedge a_2-b_1 \geq 0\vee a_1-b_1-b_2+a_3 \geq 0\wedge a_1-b_1 \leq 0\vee a_1-b_1-b_2+a_3 \geq 0\wedge a_1-b_2 \leq 0\vee a_2-b_1+a_3 \leq 0\vee a_1-b_1+a_3 \leq 0\vee a_2-b_2+a_3 \leq 0\vee a_1-b_2+a_3 \leq 0\vee a_1-b_1-b_2 \geq 0\vee a_2-b_1-b_2 \geq 0\vee a_1+a_2-b_1 \leq 0\vee a_1+a_2-b_2 \leq 0 \vee b_1+b_2-a_3 \leq 0) \wedge a_1+a_2-b_1-b_2+a_3-b_3 = 0)$

CPU-time = 13 sec.

Finally, we describe the result of our method for the 3-dimensional planar transportion problem with $m = 3$, $T_{4,3}$, where no quantifier-free equivalent has been known.

The input formula $T_{4,3}$ contains 27 existential quantifiers $\exists x_{ijk}$, 27 parameters a_{jk}, b_{ik}, c_{ij} and 54 atomic subformulas.

Crucial for the success of our method is the extensive use of theorem 3.11. We apply an automatic procedure (developed by Th. Sturm) that searches for the innermost existential quantifier $\exists x_{ijk}$ such that the corresponding variable x_{ijk} occurs in the given formula conjunctively in an equation of the form $qx_{ijk} = t$ with $0 \neq q \in \mathbf{Q}$ and $x \notin X(t)$. If the search is successful, the procedure moves the corresponding quantifier to the innermost position and eliminates this quantifier using theorem 3.11 (i.e. essentially "Gauss' elimination"). This simple method works for 19 of the 27 quantifiers (CPU-time = 6 sec). The resulting formula ψ has only $54 - 19 = 35$ atomic subformulas, 8 existential quantifiers, and contains no strict inequalities. Moreover, ψ is of the form $\psi_0 \wedge \psi_1$, where ψ_0 is a conjunction of 8 atomic subformulas that contain none of the quantified variables x_{ijk} and ψ_1 is a prenex existential formula with 8 quantifiers and 27 atomic subformulas. So for the sake of the remaining quantifier elimination ψ_0 can be discarded. Elimination of two more quantifiers from ψ_1, using the elimination set of theorem 3.5 (CPU-time = 12 sec.) yields a disjunction of 48 prenex existential formulas ρ_j, each with 6 quantifiers and 25 atomic subformulas. So the elimination of the remaining quantifiers can be done separately for each of these disjunctive parts ρ_j. This is crucial for the success of the method, since otherwise the space requirement for the elimination would be exorbitant.

In view of the homogeneity of the problem in the x_{ijk}, it is no significant restriction to substitute the value 1 for one of the remaining quantified variables x_{ijk}. Denote the formulas obtained by this substitution by ρ'_j.

For two of the 48 formulas ρ'_j the elimination of the remaining 5 existential quantifiers was carried out by H. Melenk at the ZIB, Berlin, on an HP-workstation. It required less than 6000 seconds each and resulted in quantifier-free output formulas containing 249318 and 238146 atomic subformulas, respectively.

Since the structure of all the 48 formulas ρ'_j is very similar, a corresponding quantifier elimination for the remaining 46 formulas should not raise any problems.

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