On the Role of Implication in Formal Logic¹

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April 24, 1998

¹This research was supported in part by grants number EQ1648, EQ41840 and CE110 of the program Fonds pour la Formation de Chercheurs et l'aide à la Recherche (F.C.A.R.) of the Québec Ministry of Education and in part by grant OGP0023391 from the Natural Sciences and Engineering Research Council of Canada.

Abstract

Evidence is given that implication (and its special case, negation) carry the logical strength of a system of formal logic. This is done by proving normalization and cut elimination for a system based on combinatory logic or λ -calculus with logical constants for and, or, all, and exists, but with none for either implication or negation. The proof is strictly finitary, showing that this system is very weak. The results can be extended to a "classical" version of the system. They can also be extended to a system with a restricted set of rules for implication: the result is a system of intuitionistic higher-order BCK logic with unrestricted comprehension and without restriction on the rules for disjunction elimination and existential elimination. The result does not extend to the classical version of the BCK logic.

1991 AMS (MOS) Classification: 03B40, 03F05, 03B20

Key words: Implication, negation, combinatory logic, lambda calculus, comprehension principle, normalization, cut-elimination, BCK logic

The main aim of this paper is to provide evidence that implication and implication alone carries the logical strength of a system of formal logic. By this I mean that if implication and its rules are left out of a system of formal logic, the system is very weak. This conclusion requires that negation be treated as a special case of implication by means of the definition of $\neg A$ as $A \supset \bot$. (The intuitionistic property of \bot , namely that any formula follows from it, adds no logical strength to the system, as we shall see below.) Furthermore, before we are done, we will see that there are some restricted rules for implication that can be assumed without adding logical strength to the system.

As an indication of how the argument will go, consider a standard natural deduction formulation of first-order minimal logic, and drop the rules for implication. The remaining rules are as follows:

$$\wedge \mathbf{I} \qquad \frac{A_1}{A_1 \wedge A_2}, \qquad \wedge \mathbf{E} \qquad \frac{A_1 \wedge A_2}{A_i},$$

$$\vee \mathbf{I} \qquad \frac{A_i}{A_1 \vee A_2}, \qquad \qquad \vee \mathbf{E} \qquad \underbrace{\begin{bmatrix} A_1 \end{bmatrix} \quad \begin{bmatrix} A_2 \end{bmatrix}}_{A_1 \vee A_2} \\ \qquad \qquad \underbrace{A_1 \vee A_2 \quad C \quad C}_{C}, \\ \forall \mathbf{I} \qquad \frac{A(a)}{(\forall x)A(x)}, \qquad \qquad \forall \mathbf{E} \qquad \underbrace{\frac{(\forall x)A(x)}{A(t)}}_{A(t)}, \\ \exists \mathbf{I} \qquad \frac{A(t)}{(\exists x)A(x)}, \qquad \qquad \exists \mathbf{E} \qquad \underbrace{\begin{bmatrix} A(a) \end{bmatrix}}_{C},$$

where in $\forall I$ and $\exists E$ a is a free variable which does not occur free in any undischarged assumption and where in $\land E$ and $\lor I$, i=1 or 2.

Now suppose that we transform deductions by modifying each inference by $\vee E$ or $\exists E$ by placing a copy of the deduction of the major (left) premise above each assumption discharged by the rule: i.e., change

$$\begin{array}{ccc} & & 1 & & 2 \\ & & [A_1] & & [A_2] \\ D_1 & & D_2 & & D_3 \\ \hline A_1 \lor A_2 & & C & & C \\ \hline & & & C & & VE-1-2 \end{array}$$

to

$$\begin{array}{cccc} & D_1 & D_1 \\ & \frac{A_1 \vee A_2}{A_1} * 1 & \frac{A_1 \vee A_2}{A_2} * 2 \\ D_1 & D_2 & D_3 \\ & \frac{A_1 \vee A_2}{C} & \frac{C}{C} \vee \mathbf{E} * -1 - 2 \end{array}$$

and

$$\begin{array}{cc}
 & 1 \\
 [A(a)] \\
D_1 & D_2(a) \\
\underline{(\exists x)A(x)} & C \\
\hline
C & \exists E - 1
\end{array}$$

to

$$\frac{D_1}{(\exists x)A(x)} *1$$

$$\frac{D_1}{A(a)} *1$$

$$\frac{D_2(a)}{(\exists x)A(x)} \frac{C}{C} \exists E * -1$$

These changes could be made systemmatically throughout any deduction, say by proceeding from the top down (so that each transformation is carried out on an inference by $\forall E$ or $\exists E$ for which there are no untransformed inferences by either rule above any premise).

Now consider the standard (non-permutative) proof reduction steps. The steps for \land -reductions and \forall -reductions are unchanged, and are as follows:

$$\wedge$$
-reductions

$$\begin{array}{c|c}
D_1 & D_2 & \text{reduces to} \\
A_1 & A_2 \\
\hline
A_1 \wedge A_2 \\
\hline
A_i \\
D_3
\end{array} \wedge \mathbf{E}$$

$$\begin{array}{c}
D_i \\
A_i \\
D_3, \\
\end{array}$$

where i = 1 or 2, and

$$\forall$$
-reductions $D_1(a)$ reduces to $D_1(t)$ $A(a)$ $A(t)$ D_2 .

The steps for \vee -reductions and \exists -reductions are transformed respectively into the following:

where i = 1 or 2, and

$$\exists^*\text{-reductions} \qquad \qquad \text{reduces to} \qquad \qquad D_0 \qquad \qquad D_0 \qquad \qquad A(t) \qquad \qquad A(t) \qquad \qquad A(t) \qquad \qquad D_1(t) \qquad \qquad D_1(t) \qquad \qquad D_1(t) \qquad \qquad C \qquad \qquad D_2. \qquad \qquad D_2. \qquad D_$$

Note that Each of these reduction steps shortens the deduction. This means that any sequence of these reduction steps terminates, and this is proved without any reference to the complexity of the formulas involved. The

result will still hold if instead of first-order quantifiers we have second- or higher-order quantifiers. Furthermore, although the normalization is not complete (in the sense of Prawitz [16], it is sufficient to prove the consistency of the system.

At first this result seems to contradict Gödel's Second Theorem, since the proof is completely finitary but implies the consistency of a system with higher-order quantifiers. However, the absence of implication and negation means that we cannot represent this proof within the system. In fact, the system is so weak that we cannot even represent the famous Aristotelian premise "All men are mortal." Furthermore, this proof fails if implication (or negation) is included in the system. If implication is present, the proof requires a transformation at each implication cut formula that takes

$$\frac{1}{[A]}$$

$$D_{1}$$

$$\frac{B}{A \supset B} \supset I - 1$$

$$\frac{D_{2}}{A}$$

$$B$$

$$D_{3}$$

to

$$\frac{D_2}{A \times 1}$$

$$\frac{A}{A} * 1$$

$$\frac{D_1}{A \supset B} \supset I - 1$$

$$\frac{D_2}{A}$$

$$A \supset E$$

$$D_3,$$

and these transformations must all be carried out before the reduction process starts. But since new implication cut formulas can be created as part of the reduction process, there is no way all of them can be transformed at that stage. The problem is that the assumption discharged in the creation of the cut formula does not occur in the same rule as the part of the deduction to be duplicated. Hence, this proof cannot be carried out this way if implication (or its special case, negation) occurs with its usual rules.

However, implication (and negation) can be permitted in the system if the introduction rule is restricted so that the assumption discharged by the rule cannot occur more than once (after the transformation that precedes the reduction).

What Gödel's Second Theorem really tells us about this proof is that the system is quite weak, no matter what the order of the quantifiers. This, in turn, suggests the main theme of this paper, that the strength of a system of logic is determined by its rules for implication (and negation). The purpose of this paper is to explore this importance of implication in connection with systems of logic based on combinatory logic or λ -calculus in the tradition of H. B. Curry.

There is previous evidence in Curry's work for the importance of implication in this regard. It is well known that after Kleene and Rosser [11] proved inconsistent the original systems of Curry and Church, Curry was the only one (except for F. B. Fitch) who remained interested in using combinatory logic or λ -calculus as a basis for logic and mathematics. This story is told in [19]. Actually, the paradox of Kleene and Rosser did not apply to Curry's original system in [3] but to its extension in [4]; however, this extension was so much a part of Curry's original objectives that the part of his original system dealing with logical connectives and quantifiers was not very interesting without it. Curry's first assumption was that the cause of the contradiction lay in his postulates for the universal quantifier. Later, however, he derived in [5] a contradiction from the postulates for implication alone.

In [8, Theorem 16C3, p. 441] we claimed to have a proof that if implication (and its special case, negation) is left out of the system, no further restrictions are needed to avoid the contradiction. If the proof had been valid, this result would also have applied to other systems, such as second order logic, higher order logic (type theory), logic with comprehension terms for set theory, etc. But Curry found a gap in that proof in 1975. (The proof was, in fact, my personal responsibility, and originally appeared in [17, Theorem 5C3, p. 130f]. The error is that Stage 2 case (β) does not go through as claimed in [8, p. 202].) This paper arose from an attempt to give a new proof of the same result.

In Section 1, a system of logic based on combinatory logic or λ -calculus with conjunction, disjunction, and the universal and existential quantifiers is defined, first as an L-system and then as a natural deduction system, and reduction rules are given for the latter formulation. The system is formalized as minimal logic, but the quantifiers are essentially of infinite order. In Section 2, the above sketch of a proof of normalization is carried out in detail. This is done, following a suggestion of G. E. Mints (in private correspondence) by defining in(D), the *index* of a deduction, which is the length (number of formulas) in the tree-form of the deduction obtained from D by carrying out the transformation suggested above. It is then shown that each reduction step reduces the index of the deduction, and so normalization follows. It is then shown that this implies cut-elimination for the L-system. In Section 3, these results are extended to a "classical" version of the system: for the L-system this means allowing more than one formula on the right-hand side of the sequent, but without implication a new way of defining the classical version of the natural deduction formulation is needed. It turns out that this classical system is essentially a version (without implication or negation) of the logic of constant domains. Finally, in Section 4, the system with implication restricted as suggested above is considered. This system turns out to be a variation of intuitionistic higher-order BCK logic with unrestricted comprehension. BCK logic is a logic in which the postulates for implication correspond under the formulas-as-types notion [10] to the types of a system of combinators in which no combinator duplicates an argument or to a system of λ -calculus in which λx . M is well-formed only when x occurs free at most once in M. This particular BCK logic is unlike other formulations of BCK logic such as that of [20] in that there are no restrictions on the rules for other connectives and quantifiers. This result does not hold for the corresponding classical system. The result is stronger than the similar result of White [20] because in White's system the rules restricting the number of occurrences of discharged assumptions for the rule of implication introduction that characterize BCK logic also apply to disjunction elimination and existential elimination, whereas in the system considered here, the restructions apply only to the rule for implication introduction (and its special case of negation introduction).

Because this paper is a part of the program of H. B. Curry, it is written in his language. This means that I am following Curry in using ' Λ ' for ' \wedge ', 'V' for ' \vee ', ' Π ' for ' \forall ', and ' Σ ' for ' \exists ' in the names of rules. Furthermore, I

follow Curry in using 'i' and 'e' instead of 'I' and 'E' in the names of natural deduction rules, and I use an asterisk as in '* Λ ' to indicate the L-rule for conjunction on the left. Except where otherwise specified, the other basic definitions and conventions are those of [9] and [8]. In particular, as in [9], conversion will be denoted by ' $=_*$ ' and reduction by ' \triangleright '. The major change is that, for the reasons given in [18, p. 31, footnote 1], I will not use the symbol ' \vdash ' in some places where it is used in [8].

A preliminary version of this paper was presented under the title "Normalization for Combinatorially Complete Systems Without Implication" at the special symposium on proof theory held at the joint annual meeting of the American Mathematical Society and the Association for Symbolic Logic in Denver, 5–9 January 1983. I would like to thank G. E. Mints, M. W. Bunder, Garrel Pottinger, and Roger Hindley for their helpful comments and suggestions.

1 The System for Logic without Implication

Let us begin with the system called \mathcal{F}_{33} in [8, §16C2].

Definition 1 The system \mathcal{F}_{33}^L (or, when more precision is needed, \mathcal{F}_{33}^{LA}) is formed from the terms of a system of combinatory logic or λ -calculus with the following non-redex constants: Λ (conjunction), V (disjunction), Π (universal quantifier), and Σ (existential quantifier). The provability relation is given by an L-system with the following axioms and rules, where M and M' are sequences of terms:

Axiom Scheme: $X \Vdash X$, for each term X.

Rules:

$$* \land \quad \frac{M, X, Y \Vdash Z}{M, \land XY \Vdash Z,} \qquad \qquad \land * \quad \frac{M \Vdash X}{M \Vdash \land XY,} \stackrel{M \Vdash Y}{}$$

$$* \lor \quad \frac{M, X \Vdash N \qquad M, Y \Vdash Z}{M, \lor XY \Vdash Z,} \qquad \qquad \lor * \quad \frac{M \Vdash X_i}{M \Vdash \lor X_1 X_2,}$$

$$* \sqcap \quad \frac{M, XY \Vdash Z}{M, \sqcap X \Vdash Z,} \qquad \qquad \sqcap * \quad \frac{M \Vdash Xx}{M \Vdash \sqcap X,}$$

$$* \Sigma \quad \frac{M, Xx \Vdash Z}{M, \Sigma X \vdash Z,} \qquad \qquad \Sigma * \quad \frac{M \Vdash XY}{M \Vdash \Sigma X,}$$

$$\text{Cut} \quad \frac{M \Vdash X \qquad M, X \Vdash Z}{M \Vdash Z,}$$

where in Π^* and ${}^*\Sigma$, x is a variable which does not occur free in X, M, or Z, where in $\vee *$, i = 1 or 2, where in rules ${}^*Exp^*$, there is the condition that $X \rhd Y$, and where in *C , M' is a permutation of M.

Remark The main departure from the conventions of [8] is that I am not writing the range of quantification inside the symbol ' \Vdash ' and I am writing ' \triangleright ' for reduction.) By [8, Theorem 12C7, p. 193], the rules

*Eq
$$\frac{M, Y \Vdash N}{M, X \Vdash N}$$
 Eq* $\frac{M \Vdash Y, L}{M \Vdash X, L}$

where the convention here is that $X =_* Y$, are admissible in the system, and, indeed, in any other system of this kind; I shall use this fact throughout the paper without further mention. Of course, in the singular system, rules C^* , W^* , and K^* are not postulated, but Exp^* is, and so Eq^* is admissible.

Remark This is not quite the system \mathcal{F}_{33} of [8, §16C2], which has, in addition, the non-redex constants Q and and P and the rules *Q* of [8, §12C2] and *P of §4 below. The system presented here will be modified in §4 so that Q can be defined in such a way as to make rules *Q* of [8, §12C2] valid, and a natural deduction rule corresponding to *P will be one of the rules postulated in that modification.

The natural deduction system, which is of more interest to us here, is defined as follows:

Definition 2 The system \mathcal{F}_{33}^T (or, to be more precise, \mathcal{F}_{33}^{TA}), is defined from the same terms as \mathcal{F}_{33}^T of Definition 1. Its proof system is a natural deduction system with the following rules:

Eq
$$\frac{X}{Y}$$
, Condition: $Y =_* X$.

Ae $\frac{\Delta XY}{X}$, $\frac{\Delta XY}{Y}$, Ai $\frac{X}{\Delta XY}$,

Ve $[X]$ $[Y]$ Vi $\frac{X}{\nabla XY}$, $\frac{Y}{\nabla XY}$,

 $\frac{\nabla XY}{Z}$, $\frac{Z}{Z}$

The $\frac{\Pi X}{XZ}$, The $\frac{\Pi X}{Z}$ The $\frac{XX}{Z}$ $\frac{X}{Z}$ The $\frac{XX}{Z}$ The $\frac{X}{Z}$ The \frac

Here, in Π i and Σ e, x is a variable which does not occur free in X, Z, or any undischarged assumption.

We are interested in the following proof-reduction steps, where in each case the deduction on the left reduces to the one on the right:

 Λ -reductions For i = 1 or i = 2,

$$\frac{D_1}{X_1} \frac{D_2}{X_2} \frac{D_i}{X_1} \frac{X_i}{Y_i} \text{Eq}$$

$$\frac{\frac{\Lambda X_1 X_2}{\Lambda Y_1 Y_2}}{Y_i} \Lambda e$$

$$\frac{D_i}{X_i} \frac{X_i}{Y_i} \text{Eq}$$

$$D_3.$$

V-reductions For i = 1 or i = 2,

$$\begin{array}{c|ccccc} D_1 & 1 & 2 & D_1 \\ \frac{X_i}{VX_1X_2} & \text{Vi} & [Y_1] & [Y_2] & \frac{X_i}{Y_i} & \text{Eq} \\ \frac{VY_1Y_2}{Z} & Eq & Z & Z & Ve - 1 - 2 & D_{i+1} \\ D_4 & & D_4. & & D_4. \end{array}$$

Π -reductions

$$D_1(x)$$
 $D_1(Z)$
$$\frac{Xx}{\Pi X} \prod_{Eq} \prod_{Eq} \frac{XZ}{YZ} Eq$$

$$D_2$$
 D_2

Σ -reductions

$$\begin{array}{c|c}
D_1 & 1 \\
\underline{XU} & [Yx] \\
\underline{\overline{\Sigma X}} & \Sigma i & D_2(x) \\
\underline{\overline{\Sigma Y}} & Eq & Z \\
\underline{Z} & D_3
\end{array}$$

$$\begin{array}{c|c}
D_1 \\
\underline{XU} \\
YU \\
D_2(U) \\
Z \\
D_3.$$

Note that we do not have the permutative reduction steps of [16]; I know of no way to extend the normalization proof to cover these rules. This means that the standard proof in [16] that normalization implies cut elimination cannot be applied to this system without modification. Nevertheless, if we modify the definition of "branch", then it is true that in a normal deduction all of the e-rules precede all of the i-rules in a given branch. Here a branch is the first part of a thread down to the first minor premise for an inference by Ve or Σ e if there is one. (In §4 below it will mean the first part of a thread down to the first minor premise for an inference by Ve, Σ e, or Pe.) Because of the above property of normal deductions, it is easy to show by the usual methods than an "atomic formula" (in this case, a term in normal form which does not convert to one of the forms ΛXY , ∇XY , ΠX , or ΣX) cannot be proved without an undischarged assumption

Remark It follows that if \bot is taken to be an abbreviation for Π I (which is $(\forall x)x$ in the usual notation), then there is no proof of \bot ; for if there were, then there would be the following proof of x for any variable x, which is ruled out.

 $\frac{\prod \mathbf{I}}{\mathbf{I}x} \prod \mathbf{Eq}$

Note that this is a property of intuitionistic logic not shared by minimal logic. This is why the name of the system does not refer to minimal logic, but to what Curry in [6, Chapter 5] calls *absolute* logic. A different definition of \bot would give us a form of minimal logic.

2 The Proof of Normalization

We now come to the definition of in(D) for a deduction D as explained in the Introduction, where until the end of the proof of Theorem 1, "deduction" means deduction of \mathcal{F}_{33}^{TA} . The definition requires first a function of a formula, an specification of occurrences of that formula as an undischarged assumption (e.g., by the numbers normally used to indicate where they are discharged), and a deduction, which tells us how many times that assumption is duplicated in the transformation described in the introduction.

Definition 3 (Index of an assumption) If D is a deduction, X a term, and O is a specification of occurrences of X as an undischarged assumption, then the *index of* X *with respect to* O *and* D, in(X, O, D), is defined by

induction on the length of D as follows:

- (a) if $D \equiv X$, then in(X, O, D) = 1 if this one occurrence of X is specified by O and in(X, O, D) = 0 otherwise;
- (b) if $D \equiv Z \not\equiv X$, then in(X, O, D) = 0;
- (c) if the last inference in D is by any of the rules Eq, Λ e, Vi, Π e, Π i, or Σ i (i.e., any of the rules with one premise), if D' is the result of deleting the last inference from D, and if O' specifies the occurrences of X in D' specified by O in D, then in(X, O, D) = in(X, O', D');
- (d) if D is the deduction

$$\begin{array}{c} D_1 & D_2 \\ \underline{Y_1} & \underline{Y_2} \\ \overline{\Lambda Y_1 Y_2}, \Lambda \mathrm{i} \end{array}$$

then $in(X, O, D) = in(X, O_1, D_1) + in(X, O_2, D_2)$, where O_1 and O_2 specify the occurrences of X in D_1 and D_2 respectively specified by O in D;

(e) if D is the deduction

(2)
$$\begin{array}{cccc} & & & 1 & & 2 \\ & & & [Y_1] & & [Y_2] \\ & & D_1 & D_2 & D_3 \\ & & & VY_1Y_2 & Z & Z \\ \hline & & Z, & & Ve-1-2 \end{array}$$

then $in(X, O, D) = [in(Y_1, O'_1, D_2) + in(Y_2, O'_2, D_3) + 1] \cdot in(X, O_1, D_1) + in(X, O_2, D_2) + in(X, O_3, D_3)$, where, for i = 1, 2, 3, O_i specifies the occurrences of X as an undischarged assumption in D_i specified by O (which, by the assumption about O, do not include any of the assumptions discharged by the rule Ve) and, for $j = 1, 2, O'_j$ specifies the occurrences of Y_j in D_{j+1} discharged by the inference by the rule Ve; and

(f) if D is the deduction

(3)
$$\begin{array}{c} 1 \\ [Yx] \\ D_1 \quad D_2(x) \\ \underline{\Sigma Y} \quad \underline{Z} \\ Z, \end{array}$$

then $in(X, O, D) = [in(Yx, O', D_2(x)) + 1] \cdot in(X, O_1, D_1) + in(X, O_2, D_2(x))$, where O' specifies the occurrences of Yx in $D_2(x)$ discharged by the inference by Σ e and where, for $i = 1, 2, O_i$ specifies the occurrences of X in D_i specified by O in D.

Note that $in(X, O, D) \neq 0$ if and only if X occurs as an undischarged assumption in D.

Definition 4 (Index of a deduction) The *index* of a deduction D, in(D), is defined by induction on the length of D as follows:

- (a) if $D \equiv X$, then in(D) = 1;
- (b) if the last inference in D is by Rule Eq, and if D' is the result of deleting the last inference from D, then in(D) = in(D');
- (c) if the last inference in D is by any of the rules with one premise except Eq, and if the result of deleting the last inference from D is D', then

$$in(D) = in(D') + 1;$$

(d) if D is the deduction (1), then

$$in(D) = in(D_1) + in(D_2) + 1;$$

(e) if D is the deduction (2), then

$$in(D) = [in(Y_1, O_1, D_2) + in(Y_2, O_2, D_3) + 1] \cdot in(D_1) + in(D_2) + in(D_3) + 1,$$

where, for i = 1, 2, O_i specifies the occurrences of Y_i in D_{i+1} discharged by the inference by Ve; and

(f) if D is the deduction (3), then

$$in(D) = [in(Yx, O, D_2(x)) + 1] \cdot in(D_1) + in(D_2(x)) + 1,$$

where O specifies the occurrences of Yx in D_2 discharged by the inference by Σe .

Remark Note that in (e) and (f) of Definition 4, if there are no inferences by Ve or Σ e in D_2 , D_3 , or $D_2(x)$, then the number by which $in(D_1)$ is multiplied in the formula for in(D) is one more than the total number of occurrences of the assumptions discharged by the inferences in question. Note also that in(D) does not count the inferences by Rule Eq in D.

Theorem 1 If D' is obtained from D by a reduction step, then in(D') < in(D).

The proof requires three lemmas.

Lemma 1 (a) in([Z/x]X, O', [Z/x]D) = in(X, O, D) if O' specifies the occurrences of [Z/x]X in [Z/x]D corresponding to the occurrences of X specified by O in D; and (b) in([Z/x]D) = in(D).

Proof By an easy induction on the length of D.

Lemma 2 Let D_1 and D_2 be the deductions

$$\begin{array}{ccc} D_1 & & X \\ X & & D_2 \\ & Z. \end{array}$$

Then if D is

$$D_1 \\ X \\ D_2 \\ Z,$$

we have $in(D) = in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2)$, where O specifies the indicated occurrences of X in D_2 (i.e., the occurrences over which D_1 is placed to form D).

Proof By a straightforward induction on the length of D_2 .

As an immediate corollary of Lemma 2, we have the following result:

Lemma 3 If D and D' are the deductions

$$\begin{array}{ccc} D_1 & & & D_1' \\ X & & and & & X \\ D_2 & & & D_2, \end{array}$$

and if $in(D_1) < in(D'_1)$, then in(D) < in(D').

Theorem 1 now follows by a straightforward use of these lemmas for each kind of reduction step.

Corollary 1.1 Every deduction can be reduced to a normal deduction (a deduction which is irreducible).

Proof An easy induction on the index of the deduction.

This normalization result makes it possible to prove cut elimination for the (singular) L-system introduced in §2.

Theorem 2 The cut elimination theorem holds for \mathcal{F}_{33}^L .

The proof consists of two lemmas:

Lemma 4 *If* (4)

 $M \Vdash X$

is provable in the \mathcal{F}_{33}^L , then

$$(5) M \vdash X$$

holds in \mathcal{F}_{33}^T .

Proof A straightforward induction on the length of the proof of (4). If (4) is the conclusion of *Exp or Exp*, then (5) follows by Rule Eq. The other cases are similar to those in [6, Theorems 5D6 and 7C1 (necessity)]. ■

Lemma 5 If (5) holds in \mathcal{F}_{33}^T , then there is a cut-free proof of (4) in \mathcal{F}_{33}^L .

Proof By Corollary 1.1, if (5) holds there is a normal deduction of it. Let this normal deduction be D. The proof is an induction on (in(D), ln(D)), where ln(D) is the length of D (the number of nodes in the tree diagram of D) and where the pairs are ordered by the usual lexicographic order, so that (a, b) < (c, d) if a < c or else a = c and b < d. Note that a proper part of a deduction has an index no higher than that of the entire deduction. For the basis of the induction, note that the result is trivial if (5) is is a deduction consisting of a single formula, since (4) an instance of the axiom scheme. For the induction step, we have the following cases:

Case 1. The last inference is an i-inference. Let D' be the result of deleting the last inference (D' may consist of two separate deductions). By

applying the corresponding rule on the right to the induction hypothesis (whose index(es) is (are) lower than in(D)), we obtain a cut-free proof of (4).

Case 2. The last inference is by Eq. Similar to Case 1, but now in(D') = in(D) and ln(D') < ln(D), and the corresponding rule on the right is Eq*.

Case 3. The last inference is an e-inference. Then the only inferences which occur in the left branch of D are e-inferences and inferences by Eq. Let the top formula of this left branch be Y (note that it is not discharged in D) and let the first inference be by rule R. If R is Λ e, Π e, or Eq, then deleting the inference results in a deduction of lower index or equal index and lower length than D, and so we can apply the corresponding rule on the left to the induction hypothesis to obtain a cut-free proof of (4). The only cases left are those in which R is Ve and Σ e.

If R is Ve, then Y is VY_1Y_2 and D is

$$\begin{array}{ccc} & 1 & 2 \\ & [Y_1] & [Y_2] \\ & D_1 & D_2 \\ \hline & & Z & Z \\ & & Z$$

Now the two deductions

have lower indexes than does D. Furthermore, although they may not be in normal form (because our normalization procedure does not eliminate all maximum segments but only cut formulas), they can certainly be normalized by Corollary 1.1, and the resulting normal deductions will have still lower

indexes. Hence, if M' is all of M except Y, then by the induction hypothesis there are cut-free proofs of

$$M', Y_1 \Vdash X,$$
 $M', Y_2 \Vdash X.$

By V there is a cut-free proof of (4).

The case for Σ e is similar.

Note that Lemmas 4 and 5 also imply the equivalence of \mathcal{F}_{33}^T and \mathcal{F}_{33}^L .

An examination of the proof of Lemma 5 shows that we can, in fact, prove the following stronger result:

Corollary 2.1 If (5) holds in \mathcal{F}_{33}^T , then there is a cut-free proof of (4) in \mathcal{F}_{33}^L in which rule *W is not used.

Corollary 2.2 Rule *W is redundant in \mathcal{F}_{33}^L .

Proof Redefine \mathcal{F}_{33}^L without this rule. Then Lemma 4 is proved as before, and Lemma 5 holds by Corollary 2.1. Hence, Theorem 2 holds for this modified \mathcal{F}_{33}^L . It is then possible to derive rule *W as follows:

$$\frac{M,X,X \Vdash Z}{M, \forall XX \Vdash Z} * \forall \frac{X \Vdash X}{X \Vdash \forall XX} \forall * \\ \frac{M,X \Vdash Z}{}.$$
 Cut

3 The Classical Version

Finding a classical version of \mathcal{F}_{33}^L is easy; take the system with more than one formula on the right of a sequent.

Definition 5 The system \mathcal{F}_{33}^{LC} is defined by taking the same terms as in Definition 1 for \mathcal{F}_{33}^{LA} . The proof system is defined by the following axioms and rules, where M, N, and L are sequences of terms:

Axiom Scheme: $X \Vdash X$, for each term X.

Rules:

*C
$$\frac{M \Vdash N}{M' \vdash N},$$

$$C^* \qquad \frac{M \Vdash N}{M \Vdash N'},$$

$$W^* \qquad \frac{M \vdash X, X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, L}{M \vdash X, L},$$

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$$W^* \qquad \frac{M \vdash X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, Y, L}{M \vdash X, Y, L},$$

$$W^* \qquad \frac{M \vdash X, Y, L}{M \vdash X, Y, L},$$

$$W^* \qquad \frac{M \vdash X, Y, L}{M \vdash X, L},$$

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$$W^* \qquad \frac{M \vdash X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, L}{M \vdash X, L},$$

$$W^* \qquad \frac{M \vdash X, L}{M \vdash X,$$

where, in rules Π^* and ${}^*\Sigma$, x is a variable which does not occur free in M, X, N, or L, in rules ${}^*Exp^*$ there is the condition that $X \triangleright Y$, and in rules ${}^*C^*$, M' and N' are permutations of M and N respectively.

Remark This statement of the rules follows a convention from [8] in that for a system that is singular (i.e., with only one term on the right of a sequent), N is to consist of one formula and L is to be void. This makes it possible to state the rules, except for rule V^* , for singular and multiple systems together.

However, for the natural deduction system, it is not so easy, since all of the usual natural deduction rules which lead to classical logic when added to intuitionistic logic involve implication or negation. (A rule with implication alone is Pk of [6].) One way to find such a rule with neither implication nor negation is to try to prove that if

$$(6) M \Vdash Y_1, Y_2, \dots, Y_n$$

holds in \mathcal{F}_{33}^{LC} , then

$$(7) M \vdash \mathsf{V}Y_1(\mathsf{V}Y_2(\dots(\mathsf{V}Y_{n-1}Y_n)\dots))$$

holds in \mathcal{F}_{33}^{TA} . If we are trying an induction on the length of the proof of (6), then the only case that fails is the case in which (6) is the conclusion of Π^* . This reflects the fact that in an L-system for first-order intuitionistic logic, the rules which must be singular on the right are Π^* and the rules for implication and negation. To complete this case of the proof here, it is sufficient to add the rule

$$\frac{\mathsf{V}(Xx)Y}{\mathsf{V}(\mathsf{\Pi}X)Y,}$$

where x does not occur free in X, Y, or any undischarged assumption.

Now in first-order predicate calculus, adding this rule to intuitionistic logic leads to the logic of constant domains, and as Lopez-Escobar shows in [13], there is no cut-free, complete, and sound L-system for this logic. But Lopez-Escobar is working in a mixed system in which the rules for implication and negation must be singular on the right but Π^* need not be. Since we are dealing with a system without implication or negation that is not mixed (all rules on the right are multiple), we do not have this problem.

Remark One might have thought that cut elimination could be proved for the mixed system of Lopez-Escobar by the methods of [6, Theorem 5D3, pp. 213–215]. But this is not the case. Curry's proof works only for systems with the following property: if the rule for a connective or quantifier on the left fails to be invertible, then the rule for the same connective or quantifier on the right must be singular. Since $^*\Pi$ is not invertible, Curry's proof does not apply here.

Definition 6 The system \mathcal{F}_{33}^{TC} is defined by adding rule ΠV to \mathcal{F}_{33}^{TA} .

For this new rule we take the following reduction steps, where the deduction on the left reduces to the one on the right:

ΠV_1 -reductions

$$\frac{U}{VUV} Vi \qquad \qquad \frac{U}{Xx} Eq \\
\frac{V(Xx)Y}{V(\Pi X)Y} \Pi V \qquad \qquad \frac{\Pi X}{V(\Pi X)Y} Vi \\
D_2 \qquad \qquad D_2.$$

ΠV_2 -reductions

$$\frac{V}{VUV} Vi \qquad \qquad \frac{V}{Y} Eq \\
\frac{V(Xx)Y}{V(\Pi X)Y} \Pi V \qquad \qquad \frac{V}{Y} Eq \\
D_{2}.$$

With these reduction rules, we can show that in a normalized deduction no inference by ΠV will follow an i-inference. Hence, although ΠV may look like an introduction rule, it behaves like an elimination rule. It differs from other e-rules in that because of the eigenvariable it cannot be the top inference in a branch of a deduction.

To extend the definition of index to this system, include ΠV in clause (c) of Definition 3 and add the following clause to Definition 4:

(c) if the lst inference of D is by ΠV , and if D' is the result of deleting the last inference from D, then in(D) = in(D') + 2.

(This is "in(D') + 2" rather than "in(D') + 1" for technical reasons; see the proof of Theorem 1 below.)

The proofs of Lemmas 1, 2, and 3 for this system are straightforward. We can prove Theorem 1 as follows:

Proof of Theorem 1 ΠV_1 -reductions. Let D reduce to D' by a ΠV_1 -reduction where by Lemma 3 we may ignore D_2 . Then

$$in(D) = in(D_1) + 3 > in(D_1) + 2 = in(D').$$

 ΠV_2 -reductions. Let D reduce to D' by a ΠV_2 -reduction where by Lemma 3 we may ignore D_2 . Then

$$in(D) = in(D_1) + 3 > in(D_1) + 1 = in(D').$$

Corollary 1.1 now follows as before.

To extend Theorem 2 to the classical system, note first that rules ${}^*\Lambda^*$, ${}^*V^*$, Π^* , and ${}^*\Sigma$ are invertible but ${}^*\Pi$ and Σ^* are not; see [6, Theorem 7B4, p. 329]. To restate Lemma 4 for the multiple system, let N^{V} be $\mathsf{V}Y_1(\mathsf{V}Y_2(\ldots(\mathsf{V}Y_{n-1}Y_n)\ldots))$ where N is $Y_1,Y_2,\ldots,Y_{n-1},Y_n$. Then the revised form of the lemma is the following:

Lemma 3.1' *If*

$$(6) M \Vdash Y_1, Y_2, \dots, Y_n$$

holds in the \mathcal{F}_{33}^{LC} , then

$$(7) M \vdash \mathsf{V}Y_1(\mathsf{V}Y_2(\dots(\mathsf{V}Y_{n-1}Y_n)\dots))$$

can be deduced in \mathcal{F}_{33}^{TC} .

The proof is an easy induction on the length of the proof of (6).

In order to prove Lemma 5 for this system, we need the definitions of (immediate) ancestor, quasi-parametric ancestor, and semiparametric ancestor from [8, p. 191]. Note that if B is a parametric or quasi-parametric ancestor of A, then A and B are identical, and if B is a semiparametric ancestor of A then $A \rhd B$. (Note also that Curry uses the phrases "principal constituent", "subaltern", and "parametric constituent" where most of us today follow Prawitz [16] in using "principal formula", "side formula", and "parameter".)

Theorem 3 Let D be a normal deduction in \mathcal{F}_{33}^{TC} of

$$(8) M \vdash X.$$

Let there be for any K and N a cut-free proof in \mathcal{F}_{33}^{LC} of

$$(9) K, X \Vdash N.$$

Then there is a cut-free proof in \mathcal{F}^{LC}_{33} of

$$(10) M, K \Vdash N.$$

Proof We may assume without loss of generality that no variable which occurs free in D [in the proof of (9)] is an eigenvariable in the proof of (9) [in D]. (If necessary, we can change the eigenvariables until this is true.)

The proof is by induction on (in(D), ln(D)) as in Lemma 5. The basis is immediate, since if D consists only of X, then (10) is just (9). For the induction step there are three cases:

Case 1. The last inference in D is by an i-rule. If the rule is Λ i, then X is $\Lambda X_1 X_2$, and D is

$$D_1$$
 D_2 X_1 X_2 Λ i

Now for i = 1, 2, we have that $in(D_i) < in(D)$. Also, by (9) and the invertibility of $*\Lambda$, there is a cut-free proof of

$$K, X_1, X_2 \Vdash N$$
.

By this, D_1 , and the induction hypothesis, there is a cut-free proof of

$$M, K, X_2 \Vdash N$$
.

By this, D_2 , and the induction hypothesis, there is a cut-free proof of

$$M, M, K \Vdash N$$
,

and (10) follows by *C and *W.

If the rule is Vi or Σi the proof is similar.

If the rule is Π i, then X is ΠX_1 and D is

$$D_1(x)$$

$$\frac{X_1x}{\Pi X_1} \Pi i$$

Here $in(D_1(x)) < in(D)$. Now let A_1, A_2, \ldots, A_n be a cut-free proof of (9) in which each A_i for i < n is used exactly once as the premise of an inference. Let A_k be

$$K_k, U_k \Vdash N_k$$

where U_k consists of the semiparametric ancestors of X in A_k (if Y is in any U_k , then $X \triangleright Y$, and so Y is ΠY_1 where $X_1 \triangleright Y_1$). For each k let A'_k be

$$M, K_k \Vdash N_k$$
.

Then A'_n is (10), and it is sufficient to prove for each k by a secondary induction on k that there is a cut-free proof of A'_k . There are the following cases in this secondary induction:

- (α) U_k is void. Then A'_k follows from A_k by *K, and there is a cut-free proof of A_k because it is part of the proof of (9).
- (β) U_k is not void and A_k is an axiom. Then K_k is void and U_k and N_k each consist of one term, ΠY_1 , where $X_1 \rhd Y_1$. If x is a variable which is not free in any of the terms of the context, then $Y_1x \Vdash Y_1x$ is an axiom. By *Exp we get $X_1x \Vdash Y_1x$. Then by $D_1(x)$ and the main induction hypothesis, there is a cut-free proof of $M \Vdash Y_1x$, and A'_k now follows by Π^* .
- (γ) U_k is not void and A_k is derived from A_i, A_j by a rule R for which all of the terms in U_k are parametric. By the induction hypothesis on k there are cut-free proofs of A'_i, A'_j , and A'_k now follows by R.
- (δ) U_k is not void and A_k is derived from A_i by a structural rule (*C, *K, or *W) or *Exp whose principal formula is in U_k . By the induction hypothesis on k, there is a cut-free proof of A'_i , and A'_k is identical to A'_i .
- (ϵ) U_k is not void and A_k is derived by one of the rules Λ^* , V^* , Π^* , or Σ^* whose principal formula is in U_k . The only such rule possible is Π , and hence there is only one premise, say A_i . The principal formula is ΠY_1 , where $X_1 \triangleright Y_1$, and A_i is

$$K_k, U_i, Y_1Z \Vdash N_k$$
.

By the argument of (γ) above, there is a cut-free proof of

$$M, K_k, Y_1Z \Vdash N_k$$
.

By *Exp there is a cut-free proof of

$$M, K_k, X_1Z \Vdash N_k$$
.

Now substituting Z for x in $D_1(x)$ gives us a deduction $D_1(Z)$ of X_1Z where $in(D_1(Z)) < in(D)$. Hence, by the main induction hypothesis, there is a cut-free proof of

$$M, M, K_k \Vdash N_k$$

and A'_k follows by *W and *C.

Case 2. D ends in an inference by Eq. This case is easy using Eq*.

Case 3. D ends in an e-inference (which may be by ΠV). Then since D is normalized, the left branch consists entirely of such inferences or inferences by Eq. Consider the first inference of the branch (which cannot be by ΠV because of the eigenvariable). If it is by Λe or Eq, then deleting it we get a deduction with a lower index or an equal index but lower length, and we can apply the induction hypothesis and the corresponding rule on the left.

If the first inference is by Ve, then M is M_1 , VY_1Y_2 and D is

Now the deductions

may not be normalized, but each has index less than in(D), and normalizing them reduces the indexes still more. Hence, by the induction hypothesis, there is a cut-free proof of

$$M_1, Y_i, K \Vdash N, \qquad \qquad i = 1, 2,$$

and then (10) can be obtained by *V.

If the first inference is by Σ e the proof is similar.

If the first inference is by Π e, then M is $M_1, \Pi Y$ and D is

$$\frac{\Pi Y}{YZ}$$
 Пе D_1 X .

There are the following subcases:

- (i) No eigenvariable of an inference by ΠV in the main branch of D occurs free in Z. Then D_1 is a valid deduction and $in(D_1) < in(D)$. Hence, we can proceed as in the case for Λe .
- (ii) There is an eigenvariable of an inference by ΠV in the main branch of D which occurs free in Z. Then D_1 is not a valid deduction, but D can be written as follows:

$$\frac{\prod Y}{YZ} \, \Pi \mathbf{e}$$

$$\frac{D_2(x)}{\mathsf{V}(Ux)W}$$

$$\frac{\mathsf{V}(Ux)W}{\mathsf{V}(\Pi U)W} \, \Pi \mathsf{V}$$

$$\frac{D_3}{X},$$

where there are no inferences by ΠV (and hence no eigenvariables) in the main branch of D_3 . Now the deductions

(11)
$$\frac{\Pi Y}{YZ} \Pi e$$
$$D_2(x)$$
$$\mathsf{V}(Ux)W$$

and

(12)
$$V(\Pi U)W$$
$$D_3$$
$$X$$

are valid normal deductions with indexes less than in(D). By (12), (9), the induction hypothesis, and perhaps *K, there is a cut-free proof of

$$M_1, V(\Pi U)W, K \Vdash N.$$

Because *V is invertible, this implies that there are cut-free proofs of

$$(13) M_1, \Pi U, K \Vdash N$$

and

$$(14) M_1, W, K \Vdash N.$$

Let $A_1, A_2, \ldots A_n$ be a cut-free proof of (13) in which each A_i for i < n is used exactly once as the premise for an inference. Let A_k be

$$M'_k, U_k \Vdash N_k,$$

where, as before, U_k consists of the semiparametric ancestors of ΠU that occur in A_k . Let A'_k be

$$M'_k, M_1, \Pi Y, K \Vdash N_k, N.$$

Then (10) follows from A'_n by *W* and *C*. As before, we prove by a secondary induction on k that there is a cut-free proof of A'_k , and as before we have the following subcases:

- (α) U_k is void. Then A'_k follows from A_k by *K*.
- (β) U_k is not void and A_k is an axiom. Then M'_k is void and U_k and N_k both consist of the same term, which is $\Pi U'$ where $U \rhd U'$. We have the following cut-free proof:

$$\frac{U'x \Vdash U'x}{M_1, U'x, K \Vdash U'x, N} *K* \frac{(14)}{M_1, W, K \Vdash U'x, N} K* \frac{M_1, V(U'x)W, K \Vdash U'x, N}{M_1, V(Ux)W, K \Vdash U'x, N} *Exp$$

Hence, by (11) and the main induction hypothesis, there is a cut-free proof of

$$M_1, \Pi Y, K \Vdash U'x, N.$$

Applying Π^* , we get a cut-free proof of A'_k .

- (γ) U_k is not void and A_k follows from A_i , A_j by a rule for which all the terms in U_k are parametric. Then A'_k follows from A'_i , A'_j by the same rule.
- (δ) U_k is not void and A_k follows from A_i by a structural rule or *Exp. Then A'_k is identical to A'_i .
- (ϵ) U_k is not void and A_k follows from A_i by * Π and the principal formula is in U_k . Then the principal formula is $\Pi U'$ where U > U'. Since U_k is $U_i, \Pi U', A_i$ is

$$M'_k, U_i, U'V \Vdash N_k$$
.

By the argument of (γ) above, there is a cut-free proof of A''_k :

$$M'_k, M_1, \Pi Y, U'V, K \Vdash N_k, N.$$

We now have the following cut-free proof:

$$\frac{A_k''}{M_k', M_1, \Pi Y, UV, K \Vdash N_k, N} * \operatorname{Exp} \frac{(14)}{M_k', M_1, \Pi Y, W, K \Vdash N_k, N} * K * M_k', M_1, \Pi Y, \mathsf{V}(UV)W, K \Vdash N_k, N.} * \mathsf{V}$$

If we substitute V for x in (11), we get

$$\frac{\prod Y}{Y([V/x]Z)} \, \text{Tle}$$

$$\frac{D_2(V)}{V(UV)W},$$

which has an index less than in(D). Thus, by the main induction hypothesis, there is a cut-free proof of

$$M'_k, M_1, \Pi Y, \Pi Y, K \Vdash N_k, N,$$

and A'_k follows from this by *W.

Remarks

- 1. Note that rule *W is used in the proof in an essential way. It does not appear that we can prove Corollary 2.2 for the classical system.
- 2. Since the theorem is, in a sense, a special case of the cut-elimination theorem, it is perhaps not surprising that parts of the proof (for Π i in Case 1 and for Π e in Case 3) resemble part of the proof (Stage 1) of proofs of the elimination theorem in [6] and [8].
- 3. This proof cannot be applied to the logic of constant domains because subcases (β) and (η) of Case 3 fail if there is a rule in the system whose right-hand side must be singular.

If in this system we let (9) be the axiom $X \Vdash X$, then (10) becomes $M \Vdash X$; hence Lemma 5 holds, and we have the following result:

Corollary 3.1 Theorem 2 holds for \mathcal{F}_{33}^{LC} .

Note that as in the case of the intuitionistic system, the proof also implies the equivalence of \mathcal{F}_{33}^{TC} and \mathcal{F}_{33}^{LC} .

Remark It is possible to deduce the cut-elimination theorem directly from Theorem 3 without using Lemma 5. Thus, suppose we have cut-free proofs of

$$(15) M, X \Vdash N$$

and

$$(16) K \Vdash X, L.$$

If L is void, then by (16) and Lemma 3.1', we have $K \vdash X$. The deduction of this can be normalized, and then by (15) and Theorem 3 there is a cut-free proof of

$$M, K \Vdash N$$
.

If L is not void, then by (16) and Lemma 3.1',

$$K \vdash VXL^{V}$$
.

By using (15), the axiom $L^{V} \Vdash L^{V}$, and *V, we can get a cut-free proof of

$$M, VXL^{\mathsf{V}} \Vdash N, L^{\mathsf{V}}.$$

Then, by Theorem 3, there is a cut-free proof of

$$M, K \Vdash N, L^{\mathsf{V}}$$
,

and by the invertibility of V^* there is a cut-free proof of

$$M, K \Vdash N, L$$
.

4 Introducing Implication with Restrictions

By the contradiction Curry found in [5], the theory we have developed so far cannot be applied if implication is added to the system without any restriction. In order to see if there are any restrictions under which the rules for implication can be added, it is worth looking for the place at which the argument of the paper so far breaks down when implication is present in the system.

On the other hand, as we saw in the introduction to the paper, if the discharged assumption of the inference by Pi occurs at most once (after all the duplicating; i.e., occurs at most once and is not duplicated later in the deduction, which means that it has no descendent in the deduction which is a major premise for an inference by Ve or Σ e), then a P-reduction step will shorten the proof. This suggests that we adopt the following restriction:

Restriction on Rule Pi The discharged assumption is to occur at most once and have no descendant which is a major premise for an inference by Ve or Σ e (down to the inference by Pi).

Remark Since we now have implication, we also have negation. $\neg X$ is defined to be $\mathsf{P}X(\mathsf{\Pi}\mathsf{I})$. However, the rule for negation introduction is a special case of the rule for implication introduction, so that in this logic implication and negation do not satisfy all of their usual properties, even in intuitionistic logic.

Definition 7 The system \mathcal{F}_{33}^{TJ} is obtained from \mathcal{F}_{33}^{TA} by adding the rules Pi and Pe, where rule Pi is subject to the Restriction on Rule Pi.

To see that this works, extend Definitions 3 and 4 by treating Pi with the other one-premise rules in case (c) of each definition and treat Pe like Λ i in

case (d) of each definition. Lemmas 1–3 can be proved as before. We also want the following lemmas:

Lemma 6 Let D be a deduction, X a term, and O a specification of occurrences of X as an undischarged assumption in D. If the specified occurrence(s) of X satisfy the Restriction on Rule Pi, then $in(X, O, D) \leq 1$.

The proof is a straightforward induction on the length of D using Definition 3. Note that if O specifies no occurrences of X in D, then in(X, O, D) = 0. Hence, it is sufficient to suppose that O specifies one occurrence of X in D and to prove that in(X, O, D) = 1.

Lemma 7 Suppose D and D' are deductions such that D reduces to D', and suppose X is an undischarged assumption in both deductions. If X satisfies the Restriction on Rule Pi in D, then it satisfies it in D'.

The proof is a straightforward induction on the number of reduction steps from D to D', with cases according to the reduction step.

Now to complete the proof of Theorem 1, it remains to consider the case for P-reductions. Let D be the first and D' the second deduction in a P-reduction and disregard D_3 (as we can because of Lemma 3). It is easy to see that if O specifies the occurrences of X in D_1 discharged by the inference by Pe,

$$in(D) = in(D_1) + in(D_2) + 2,$$

 $in(D') = in(X, O, D_1) \cdot [in(D_2) - 1] + in(D_2).$

By Lemma 6 and the restriction on Rule Pi, $in(X, O, D_1) \leq 1$. It follows that in(D') < in(D).

This proves the following result:

Theorem 4 Theorem 1 holds for \mathcal{F}_{33}^{TJ} .

In [8, Corollary 16C3.1, p. 442] a corresponding result was claimed for the L-formulation, where the rules for P are

$$*\mathsf{P} \quad \frac{M \Vdash X \qquad M,Y \Vdash Z}{M,\mathsf{P}XY \Vdash Z,} \qquad \qquad \mathsf{P}^* \qquad \quad \frac{M,X \Vdash Y}{M \Vdash \mathsf{P}XY,}$$

and where the restriction on P^* is that the X in the premise not have any ancestor which is the principal formula of an operational rule on the left. This clearly does not correspond to the restriction on the rule Pi stated above.

Instead, let us note that Bunder and da Costa give in [2] a natural deduction system with just the above restriction on rule Pi as a formulation of BCK logic. Furthermore, White [20] gives a natural deduction formulation for a variant of higher-order BCK logic with comprehension that differs from the system given here in that his rules for disjunction and existential elimination satisfy the same restriction on discharged assumptions that we have here for Pi, so that his system is weaker than this one. Elsewhere, for example in [15], [12], and [14], BCK logic is identified with an L-system without rules *W*. This suggests that we might find an L-formulation equivalent to our natural deduction formulation by leaving out these rules. But there is a technical problem with this: consider the following deduction in \mathcal{F}_{33}^{TA} of the distributive rule:

$$\frac{\frac{\Lambda X(\mathsf{V}YZ)}{X} \wedge \mathsf{e} \quad \frac{1}{[Y]} \wedge \mathsf{i}}{\frac{\mathsf{V}YZ}{\mathsf{V}Z} \wedge \mathsf{e} \quad \frac{\mathsf{I}}{\mathsf{V}(\mathsf{N}XY)} \wedge \mathsf{i}} \qquad \frac{\frac{\Lambda X(\mathsf{V}YZ)}{X} \wedge \mathsf{e} \quad \frac{2}{[Z]}}{\frac{\mathsf{X}}{\mathsf{V}(\mathsf{N}XZ)} \wedge \mathsf{i}} \wedge \mathsf{i}} \\ \frac{\frac{\mathsf{V}YZ}{\mathsf{V}Z} \wedge \mathsf{e} \quad \frac{\mathsf{V}(\mathsf{N}XY)}{\mathsf{V}(\mathsf{N}XZ)} \wedge \mathsf{i}}{\mathsf{V}(\mathsf{N}XY)} \wedge \mathsf{i}} \qquad \frac{\mathsf{N}X(\mathsf{V}YZ)}{\mathsf{V}(\mathsf{N}XZ)} \wedge \mathsf{i}}{\mathsf{V}(\mathsf{N}XY)} \wedge \mathsf{i}$$

The restriction on Pi prevents us from discharging the assumption $\Lambda X(VYZ)$ because there are three occurrences of it and one of them has a descendant which is a major premise for an inference by Ve. But now consider the equivalent proof in \mathcal{F}_{33}^{LA} :

$$\frac{\frac{X \Vdash X}{X,Y \Vdash X} * \mathsf{K} \quad \frac{Y \Vdash Y}{X,Y \Vdash Y} * \mathsf{K}}{\frac{X,Y \Vdash \Lambda XY}{X,Y \Vdash \mathsf{V}(\Lambda XY)(\Lambda XZ)} \mathsf{V} *} \qquad \qquad \frac{\frac{X \Vdash X}{X,Z \Vdash X} * \mathsf{K} \quad \frac{Z \Vdash Z}{X,Z \Vdash Z} * \mathsf{K}}{\frac{X,Z \Vdash \Lambda XZ}{X,Z \Vdash \mathsf{V}(\Lambda XY)(\Lambda XZ)}} * \mathsf{V} *}{\frac{X,VYZ \Vdash \mathsf{V}(\Lambda XY)(\Lambda XZ)}{\Lambda X(\mathsf{V}YZ) \Vdash \mathsf{V}(\Lambda XY)(\Lambda XZ)}} * \mathsf{\Lambda}}$$

There is nothing to prevent the use of P* here.

On the other hand, the occurrence of $\Lambda X(VYZ)$ on the left of the conclusion does have an ancestor which is a principal formula for an inference by *V. This suggests that we form the L-system by deleting rules *W* and adding rules *P* with the restriction that no ancestor of X on the left of the

premise is the principal formula for an inference by *V or $^*\Sigma$. However, this is still not quite right; there is nothing corresponding to the condition in the Restriction on Rule Pi that the discharged assumption occur at most once

The solution turns out to be to assume rules *W* but take as an restriction on P* that the left side formula (X in *P) not have an ancestor which is a principal formula for an inference by *W. Also, to make all this work properly, it is necessary to modify some of the rules.

Definition 8 The system \mathcal{F}_{33}^{LJ} will be singular. (See below for the reason we cannot extend the result to the multiple [classical] version of the calculus.) Its rules are *C, *K, *W, and *Exp* of Definition 1, and

where, in Πi^* and $^*\Sigma$, x does not occur free in M, X, or Z; where in $^*\Lambda$ and V^* , i=1 or i=2; and where there is a restriction on P^* to be stated below.

To state the restriction on P^* , define X to be *side-connected* to Y if X is a parametric constituent for the left premise of $^*\mathsf{P}$ (i.e., is in M_1) and Y is the side formula of the right premise, or else X is a parametric constituent for the left premise of Cut (i.e., is in M_1) and Y is the side formula of the right premise (i.e., the X in the right premise). Then define X to be *chained* if (i) X has an ancestor which is a principal formula for an inference by $^*\mathsf{W}$, $^*\mathsf{V}$, or $^*\Sigma$; or (ii) X is side-connected to Y and Y is chained.

Restriction on Rule P* The side formula on the left is not chained.

(The "side formula on the left" of Rule P^* is the occurrence of X on the left of the premise.)

Remark Note that in this system there are no provable sequents with void right-hand sides. This formulation is what Curry in [6] calls an F-formulation, where his F is Π I. What in another system would be a sequent with a void right-hand-side corresponds here to a sequent whose right-hand-side is Π I.

To prove Theorem 2, we need the following modifications of Lemmas 4 and 5.

$$(2) M \Vdash X$$

is provable in \mathcal{F}_{33}^{LJ} , then there is a deduction of

$$(3) M \vdash X$$

in \mathcal{F}_{33}^{TJ} such that for each Y in M, if Y satisfies the Restriction on Rule P* in the proof of (4), then it satisfies the Restriction on Rule Pi in the deduction of (5).

Proof A straightforward induction on the proof of (4). In each case of the induction step, it is necessary to check that satisfying the restriction on Rule Pi is preserved by the induction step. In this process, note that *W corresponds to grouping occurrences of an assumption together for the purpose of discharging them together; the Restriction on Rule Pi precludes this for that rule, but it can be done for Rules Ve and Σ e. One case should illustrate the way to complete the proof from the proof of Lemma 4: the case for Λ *. Here $X \equiv \Lambda X_1 X_2$, $M \equiv M_1, M_2$, and the premises are

$$M_1 \Vdash X_1,$$
 $M_2 \Vdash X_2.$

By the induction hypothesis, there are deductions

$$D_1$$
 D_2 X_1 , X_2 ,

where, for i = 1, 2, the undischarged assumptions of D_i are in M_i and where, for each Y in M_i , if Y satisfies the Restriction on Rule P^* then Y satisfies

the restriction on Rule Pi in D_i . Then

$$D_1$$
 D_2 X_1 X_2 Λ i

is a deduction of (5) in which each Y satisfies the Restriction on Rule Pi if it satisfies the restriction on Rule P* in (4). (Note that Y satisfies the Restriction on Rule P* in (4) if and only if it does in the premise in which it occurs.)

Lemma 3.2" If there is a deduction of (5) in \mathcal{F}_{33}^{TJ} , then there is a cut-free proof of (4) in \mathcal{F}_{33}^{LJ} in which, for each Y in M, if Y satisfies the Restriction on Rule Pi in the deduction of (5), then it is not chained in the proof of (4).

Proof A straightforward modification of the proof of Lemma 5, in which a case for Pi is added in Case 1 and a case for Pe is added in Case 3, and where, in each case, we have to check the condition on not being chained. One subcase each from Cases 1 and 3 should make clear how the rest of the proof goes.

Under Case 1, let us consider the case for Λ i. Here $X \equiv \Lambda X_1 X_2$ and D is

$$D_1$$
 D_2 X_1 X_2 Λ i

For i = 1, 2, let M_i be a sequence consisting of the undischarged assumptions in D_i . By the hypothesis of induction, there is a cut-free proof of

$$M_1 \Vdash X_1, \qquad \qquad M_2 \Vdash X_2$$

in which, for each Y in one of the M_i which satisfies the Restriction on Rule Pi in D_i satisfies the Restriction on Rule *P. By Λ^* , we get a cut-free proof of

$$M_1, M_2 \Vdash \Lambda X_1 X_2$$

in which each Y in one of the M_i which satisfies the Restriction on Rule Pi satisfies the Restriction on Rule P*. Now if M cannot be obtained from M_1, M_2 by adding assumptions (for which we can use Rule *K), it can only

be because there is an assumption Y which is in both M_1 and M_2 ; this assumption is an undischarged assumption in both D_1 and D_2 , and is to be considered as one undischarged assumption until it is discharged. In this case, we use *W; the terms which occur as principal formulas of these inferences by *W, and which do not satisfy the Restriction on Rule P* in (4), do not satisfy the Restriction on Rule Pi in (5).

Under Case 3, let us consider the case for Pe. Here D is

$$\frac{PYZ}{Z} \qquad \frac{D_1}{Y} \text{Pe}$$

$$\frac{D_2}{X} \qquad Y$$

For i = 1, 2, let M_i be the undischarged assumptions of D_i other than the indicated occurrence of Z in D_2 . By the hypothesis of induction, there is a cut-free proof of

$$(17) M_1 \Vdash Y, M_2, Z \Vdash X$$

such that, for each Y in M_i , if Y satisfies the Restriction on Rule in Pi in D_i , then Y satisfies the Restriction on Rule P* in whichever of (17) it occurs on the left, and the same is true for Z in the second premise (and in D_2). Now by *P we get

$$(18) M_1, M_2, PYZ \Vdash X.$$

If M cannot be obtained from M_1, M_2 , PYZ by adding terms (using *K), then either there is an overlap between M_1 and M_2 or there is an overlap between M_i and PYZ. In either case, we can use *W to get (4) from (18). Now suppose W in M satisfies in D the Restriction on Rule Pi. Then W is in M_i or is PYZ, and it is not one of the formulas in an overlap as described above (since it can occur at most once). Hence, it does not have an ancestor in the steps from (18) to (4) which is the principal formula for an inference by *W. If it is in M_i , then it satisfies in D_i the Restriction on Rule Pi, and hence it satisfies in the proof of (17) the Restriction on Rule P*. If W is in M_2 , then it satisfies in the Restriction on Rule P*. If W is in M_1 then it satisfies the Restriction on Rule Pi in D_1 . If it satisfies

the Restriction on Rule Pi in D, then the indicated occurrence of Z in D_2 must satisfy this restriction in D_2 , and in this case W will satisfy the Restriction on Rule P* in the cut-free proof of (4). Similarly, if PYZ satisfies the Restriction on Rule Pi in D, then Z satisfies this restriction in D_2 , and therefore PYZ satisfies the Restriction on Rule P* in the cut-free proof of (18) and hence (since there is no overlap) in the cut-free proof of (4).

This proves

Theorem 5 Theorem 3 holds for \mathcal{F}_{33}^{LJ} .

This intuitionistic variant of BCK logic, \mathcal{F}_{33}^{TJ} , is strong enough to prove some interesting results. For example, here are proofs of the implication formulas (PI), (PK), (PB), and (PC):

(PI):

$$\frac{1}{\frac{[X]}{PXX}} Pi - 1$$

(PK):

$$\frac{\frac{[X]}{PYX} Pi - v}{PX(PYX).} Pi - 1$$

(PB):

$$\frac{\frac{3}{[\mathsf{P}XY]} \frac{2}{[X]}}{\frac{Y}{\mathsf{P}}} \mathsf{Pe}$$

$$\frac{\frac{Z}{\mathsf{P}XZ} \mathsf{Pi} - 1}{\frac{\mathsf{P}(\mathsf{P}XY)(\mathsf{P}XZ)}{\mathsf{P}(\mathsf{P}XZ)} \mathsf{Pi} - 2} \mathsf{Pi} - 3$$

$$\frac{\mathsf{P}(\mathsf{P}YZ)(\mathsf{P}(\mathsf{P}XY)(\mathsf{P}XZ))}{\mathsf{P}(\mathsf{P}XZ)(\mathsf{P}(\mathsf{P}XY)(\mathsf{P}XZ))} \mathsf{Pi} - 3$$

(PC):

$$\frac{ \frac{3}{[PX(PYZ)]} \frac{1}{[X]} \operatorname{Pe}}{\frac{PYZ}{PXZ} \frac{[Y]}{Pi-1} \operatorname{Pe}} \operatorname{Pe} \\ \frac{\frac{Z}{PXZ} \operatorname{Pi} - 1}{\frac{PY(PXZ)}{P(PX(PYZ))(PY(PXZ))}} \operatorname{Pi} - 3$$

If we define Q by

$$Q \equiv \lambda xy \cdot P(\lambda z \cdot P(zx)(zy)),$$

then, using the obvious abbreviations, we have

$$\mathsf{Q}XY =_* (\forall z)(zX\supset zY),$$

and we can interapret the system \mathcal{Q} of [8, §12C]. Finally, if we let $X \sim Y$ be an abbreviation for $\Lambda(\mathsf{P}XY)(\mathsf{P}YX)$, then we can prove a comprehension scheme

$$(\exists z)(\forall y)(zy \sim X),$$

where X is any term in which z does not occur free (but y may occur free):

$$\frac{\frac{1}{[(\lambda y \cdot X)y]} \operatorname{Eq}}{X} \operatorname{Eq} \frac{\frac{[X]}{(\lambda y \cdot X)y} \operatorname{Eq}}{PX((\lambda y \cdot X)y)X} \operatorname{Pi} - 1 \frac{\frac{[X]}{PX((\lambda y \cdot X)y} \operatorname{Pi} - 2}{PX((\lambda y \cdot X)y)} \operatorname{Ai} \frac{\frac{(\lambda y \cdot X)y \sim X}{(\lambda y \cdot (\lambda y \cdot X)y \sim X)y} \operatorname{Eq}}{\Pi(\lambda y \cdot (\lambda y \cdot X)y \sim X)} \operatorname{Ii} \frac{\frac{(\lambda z \cdot \Pi(\lambda y \cdot zy \sim X))(\lambda y \cdot X)}{(\lambda z \cdot \Pi(\lambda y \cdot zy \sim X))} \operatorname{Eq}}{\Sigma(\lambda z \cdot \Pi(\lambda y \cdot zy \sim X)).}$$

If we think in terms of formulas-as-types [9, §§14D, 15D] (also known as the Curry-Howard isomorphism), then the implication fragment of this system is equivalent to a system of typed combinatory logic in which the basic combinators are B, I, C, and K, or, equivalently, to a typed λ -calculus in which λx . X is only defined when x occurs at most once in X. This system has long been known to be consistent; see [7, Theorem 10C3, p. 364].

Since I can be defined in terms of B, C, and K (as, e.g., CKB), this shows something of the origin of the name of BCK logic.

Note that if we define \neg to be λx . $Px\bot$, where \bot is ΠI , then we get properties of intuitionistic negation.

It is well known that if we add

(PW)
$$P(PX(PXY))(PXY)$$

as a new axiom scheme, then the system becomes inconsistent; see [8, §12B3, pp. 180-1]. Bunder shows in [1] that we can obtain a contradiction by adding the scheme

(PS)
$$P(PX(PYZ))(P(PXY)(PXZ))$$

or, surprisingly, either of the schemes

$$(Pc)$$
 $P(P(PXY)X)X$

or

$$(\neg V)$$
 $VX(\neg X)$.

To see this, let Y be any arbitrary term, and define

$$X \equiv Y(\lambda x \cdot PxY),$$

where Y is a fixed-point operator. Then

$$X =_{*} \mathsf{P} X Y.$$

Hence, we can prove $PXY \vdash Y$:

$$\frac{PXY}{Y} \frac{\frac{PXY}{X}}{Pe} Eq$$

This means that if we can prove that PXY follows from a scheme, any arbitrary term can be proved from that scheme and the system is inconsistent.

The proof that PXY follows from (PS) is

$$\frac{P(PX(PXY))(P(PXX)(PXY))}{P(PXX)(PXY)} \frac{\frac{[X]}{PXY} Eq}{PX(PXY)} Pi - 1 \qquad (PI)$$

$$\frac{P(PXX)(PXY)}{PXY} Pe$$

$$\frac{PXX}{PXY} Pe$$

The proof that it follows from (Pc) is

$$\frac{\sum_{\substack{[X]\\ \overline{PXY}}}^{2} \operatorname{Eq}}{\frac{PXY}{PXY}} \operatorname{Pi} - 2 \qquad \frac{\operatorname{P(P(PXY)X)X}}{Y} \frac{\sum_{\substack{[PXY]\\ \overline{P(PXY)X}}}^{1} \operatorname{Eq}}{Y} \operatorname{Pi} - 1$$

$$\frac{P(P(PXY)X)X}{X} \operatorname{Pe} = \frac{P(P(PXY)X)X}{Y} \operatorname{Pe} = \frac{P(PXY)X}{Y} \operatorname{Pe} = \frac{$$

The proof that it follows from $(\neg V)$ is

$$\frac{ \begin{bmatrix} \frac{2}{|\neg X|} & \text{Eq} & 3 \\ \frac{|\neg X|}{| PX \perp} & \text{Eq} & \frac{3}{|X|} & \text{Pe} \\ & & \frac{1}{|X|} & \text{Eq} & \\ \frac{|X|}{|Y|} & \text{Te} & \frac{|X|}{|Y|} & \text{Eq} & \\ \frac{|X|}{|Y|} & \frac{|X|}{|Y|} & \text{Eq} & \frac{|X|}{|Y|} & \text{Pi} - 3 \\ \frac{|X|}{|Y|} & \frac{|X|}{|Y|} & \text{Pi} - 1 - 2 \end{bmatrix}$$

This proves

Theorem 6 Theorem 1 cannot be extended to the classical version of this variant of BCK logic.

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