

The continuity of cupping to $0'$

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Abstract

It is shown that, if a, b are recursively enumerable degrees such that $0 < a < 0'$ and $a \cup b = 0'$, then there exists a recursively enumerable degree c such that $c < a$ and $c \cup b = 0'$.

1 Introduction

By analogy with the notion of major subset in the context of the lattice of r.e. sets, the r.e. degree c is called a *major subdegree* of the r.e. degree a if $c < a$ and for every r.e. degree b

$$a \cup b = 0' \Rightarrow c \cup b = 0'.$$

This paper represents modest progress towards answering the question: Does every r.e. degree which is neither 0 nor $0'$ have a major subdegree? This question was first posed by the second author in 1967, although it does not seem to have appeared in print. In the 70's and 80's efforts were made to answer the question but bore little fruit.

In this paper we prove:

*The second author was supported by NSERC (Canada) Grant A3040, and the third author by National Science Foundation Grant DMS 88-07389. The third author presented the results in this paper and those in the companion paper [1] by Harrington and Soare, at the *Workshop on Set Theory and the Continuum*, October 16–20, 1989, held at the Mathematical Sciences Research Institute in Berkeley, California, during the Special Year in Mathematical Logic, from September 1, 1989 through August 24, 1990.

Theorem 1.1 *If a, b are recursively enumerable degrees such that $0 < a < 0'$ and $a \cup b = 0'$, then there exists a recursively enumerable degree c such that $c < a$ and $c \cup b = 0'$.*

This theorem can be seen as non-uniform version of the assertion that every r.e. degree in the interval $(0, 0')$ has a major subdegree. After this work was done Seetapun [3] showed that every non-zero low_2 r.e. degree has a major subdegree.

Earlier Harrington and Soare [2] proved the dual of Theorem 1.1 which says that, if a, b are a minimal pair of r.e. degrees, then there exists an r.e. degree $c > a$ such that c and b are a minimal pair. Subsequently, Seetapun [4] showed that for every low_2 r.e. degree a there is an r.e. degree $c > a$ such that there is no minimal pair a, b with $b < c$. An easy corollary is the dual of his theorem mentioned in the last paragraph, i.e., for every low_2 r.e. degree a there is an r.e. degree $c > a$ such that every r.e. degree which forms a minimal pair with a also forms a minimal pair with c .

The results discussed above can also be seen as reflecting continuity properties of the operations \cap and \cup on the r.e. degrees. This aspect is pursued at length by Harrington and Soare who point out in Corollary 2.5 of [2] the following consequence of their theorem and ours: Let $F(x, y)$ be an open formula in the language $\{<, \cup, \cap, 0, 0'\}$ and a, b be distinct r.e. degrees such that $F(a, b)$ holds in the upper semilattice \mathcal{L} of r.e. degrees. Then there exist r.e. degrees a_0, a_1, b_0, b_1 such that

$$a_0 < a < a_1 \wedge b_0 < b < b_1$$

and

$$\mathcal{L} \models \forall x \forall y ((a_0 < x < a_1 \wedge b_0 < y < b_1) \rightarrow F(x, y)).$$

An r.e. degree a is called *cuppable* if there is an r.e. degree $< 0'$ such that $a \cup b = 0'$, in which case b is said to *cup* a . Dually, An r.e. degree a is called *cappable* if there is an r.e. degree > 0 such that a and b form a minimal pair. Thus Theorem 1.1 can also be seen as a contribution to the theory of cupping. The most remarkable discovery about cupping is that for an r.e. degree a , there is some low r.e. degree which cups a if and only if a is non-cappable, see [1] or [5, page 296].

2 Formulation of the problem

Let recursively enumerable sets A, B and partial recursive functionals Φ and Θ^n ($n < \omega$) be given such that A is not recursive. In §3, 4 we shall describe the construction of recursively enumerable sets P, C and partial recursive functionals $?, \Delta^n$ ($n < \omega$) which satisfy the requirements:

$$\begin{aligned} \mathcal{R} : & && C \text{ is recursive in } A \\ \mathcal{S} : & && P = \Phi(A, B) \Rightarrow A = ?(C, B) \\ \mathcal{T}^n : & && [P = \Phi(A, B) \wedge A = \Theta^n(C)] \Rightarrow K =^* \Delta^n(A) \end{aligned}$$

for all $n < \omega$, where K denotes $\{e : e \in W_e\}$. Here $=^*$ indicates that the two sides of the equation agree except on a finite subset of ω .

Suppose that A, B represent recursively enumerable degrees a, b such that $0 < a < 0'$ and $a \cup b = 0'$, and that $\langle \Theta_n : n < \omega \rangle$ is an effective enumeration of all partial recursive functionals. Since our construction is uniform in the sense that indexes for $P, C, ?,$ and $\langle \Delta^n : n < \omega \rangle$ can be found effectively from indexes for $A, B, \Phi,$ and $\langle \Theta^n : n < \omega \rangle$, by the recursion theorem Φ can be chosen such that $P = \Phi(A, B)$. Let $c = \text{deg}(C)$. From \mathcal{R} , we have $c \leq a$, and, from \mathcal{S} , $a \leq c \cup b$. Hence $0' \leq a \cup b \leq c \cup b$. Since $0 < a < 0'$, A is nonrecursive and $K \neq^* \Delta^n(A)$. So, from \mathcal{T}^n , $A \neq \Theta^n(C)$ which yields $a \not\leq c$. From these remarks it is clear that the plan is sufficient to prove Theorem 1.1.

3 The basic module and the idea of the construction

We describe a strategy for satisfying the requirements \mathcal{R}, \mathcal{S} , and a single requirement

$$\mathcal{T} : [P = \Phi(A, B) \wedge \Theta(C) = A] \Rightarrow K =^* \Delta(A).$$

Recall that $A, B, K, \Phi,$ and Θ are given and that suitable $C, P, ?,$ and Δ are to be enumerated. As usual the construction will consist of *stages* numbered $0, 1, 2, \dots$. However, before we can describe what happens in the stages we must describe various elements which play a part in the overall scheme.

First we will establish some notation. In describing the construction we will use $A, B, C, K, P, \Phi, \Theta$ to denote the current approximations to these objects. Thus at any point in the construction A denotes the finite set of numbers which have already been enumerated in A , and Φ denotes the functional determined by the finite set of instructions which have already been enumerated in Φ . A_s and Φ_s denote the approximations to A and Φ respectively which obtain immediately before stage s . Similarly, for B, C, K, P , and Θ . The convention of using a subscript s to indicate the value immediately before stage s will be extended to all objects occurring in the construction.

At any point of the construction let $a(0), a(1), \dots$ be an enumeration of $\overline{A} = \omega - A$ in order of magnitude.

Use-functions. As usual, the functionals Φ and Θ have associated use-functions φ and θ . We regard use functions as “local” to the construction we are describing. In keeping with this viewpoint, we do not think of A and B as being arguments of φ . However, when we feel that there is an advantage in reminding the reader that A and B are the arguments of Φ in \mathcal{S} , then we will write $\varphi^{A,B}$ instead of φ . The essential features which we wish φ to have are as follows. At any point in the construction $\varphi(x) \downarrow$ implies $\Phi(A, B; x) \downarrow$ and indeed that

$$\Phi(A \upharpoonright (\varphi(x) + 1), B \upharpoonright (\varphi(x) + 1); x) = 0.$$

Conversely, $\Phi(A, B; x) \downarrow$ and $x \notin P$ imply $\varphi(x) \downarrow$. Further, once $\varphi(x)$ is defined it remains defined with the same value unless some number $\leq \varphi(x)$ enters A or B . It is convenient to choose the use-function φ and the enumeration of instructions for the corresponding functional Φ so that:

- i) if $\varphi(x)$ becomes defined, then $x \notin P$, and $\varphi(y)$ is already defined for all $y < x$ such that $y \notin P$,
- ii) φ is increasing on its domain,
- iii) if $\varphi(x) \downarrow$, then it becomes undefined whenever a number $\leq \varphi(x)$ is enumerated in A or B ,
- iv) the successive values taken by $\varphi(x)$ are non-decreasing and $\geq x$.

Similarly, for Θ we shall have the use-function θ with properties corresponding to those listed for φ above. Now, whereas in the case of Φ we were concerned with whether $P = \Phi(A, B)$, in the case of Θ we are concerned with the equation $A = \Theta(C)$. Thus in the conditions for θ the role played above by the sets A and B will be played by C , and the role played by P will be played by A .

The functionals $?$ and Δ with which we hope to satisfy the requirements will be defined implicitly by defining in the course of the construction suitable use-functions γ and δ . Of course, in considering γ the relevant equation is $A = ?(C, B)$. Thus $\gamma(x) \downarrow$ implies $?(C, B; x) \downarrow$, etc. γ will satisfy the same conventions as the use-functions for Φ and Θ with one exception explained below. Of the properties (i) through (iv) specified for φ , δ will only be required to satisfy the property corresponding to (iii).

Once x has been enumerated in A , $\gamma(x)$ will never become defined, although of course it may have a value prior to x being enumerated in A which may persist for some time after the arrival of x in A . Similarly, once k has been enumerated in K , $\delta(k)$ will never become defined.

All use-functions will have the property that a value attained in stage s is $\leq s$.

The only numbers enumerated in C will be values of γ , and x will be enumerated in C only if some $y \leq x$ is enumerated in A in the same stage. This ensures that the requirement \mathcal{R} is satisfied. For the rest we may suppose that it turns out that $P = \Phi(A, B)$ since otherwise the other requirements are satisfied by default.

With each member $a(m)$ of \bar{A} we shall associate a number $p(m) \in \bar{P}$. For $p(m)$ we use the least member of

$$\omega - (P \cup \{p(0), \dots, p(m-1)\}).$$

Once $\varphi(p(m))$ becomes defined, in a subsequent stage we let $\gamma(a(m))$ be the number of the current stage. This ensures that

$$\varphi(p(m)) < \gamma(a(m)). \tag{1}$$

This inequality plays a key role in everything which follows. If $a(m)$ is enumerated in A , then in the same stage $\gamma(a(x))$ is enumerated in C for some $x \leq m$. If at some stage a number $\leq \varphi^{A,B}(p(m))$ is enumerated in A , causing $\varphi^{A,B}(p(m))$ but not $\gamma^{C,B}(a(m))$ to become undefined, then *we*

wish to enumerate $\gamma(a(x))$ in C for some $x \leq m$ in order to allow (1) to be re-established next time $\varphi^{A,B}(p(m))$ becomes defined. We will always follow our inclination in this regard except in one case which is described below.

Once $a(m)$, $p(m)$, and $\varphi(p(m))$ have reached their final values, we will not redefine $\gamma(a(m))$ unless some number $\leq \gamma(a(m))$ is enumerated in C . In this regard, γ may violate (iii) above because if x satisfying $\varphi(p(m)) < x \leq \gamma(a(m))$ is enumerated in B , then $\gamma(a(m))$ remains defined with the same value. This is the exception to the conventions governing the use function γ to which we referred above.

Suppose that x is not yet in K , $\varphi(p(x)) \downarrow$, and $\Theta(C)$ and A agree on all arguments $\leq \varphi(p(x))$. Then we set $\delta(x) = \theta(\varphi(p(x)))$. If k is enumerated in K after we have defined $\delta(k)$, then $\Delta(A)$ and K disagree at k until some number $\leq \delta(k)$ enters A . We react to this critical situation as follows. Let m be the greatest y if any such that $\gamma(a(y)) \leq \delta(k)$. We enumerate $p(m)$ in P . Assuming $\Phi(A, B) = P$, our action forces the enumeration of some $x \leq \varphi(p(m))$ in either A or B . If x is enumerated in A , then $\delta(k)$ becomes undefined since $\varphi(p(m)) < \gamma(a(m)) \leq \delta^A(k)$, and so the injury to Δ can be repaired. If x is enumerated in B , then $\varphi^{A,B}(p(m))$ and $\gamma^{C,B}(a(m))$ become undefined but $\delta^A(k)$ is unaffected. So we repeat the procedure, i.e., letting m be the greatest y if any such that $\gamma(a(y)) \leq \delta(k)$ we enumerate $p(m)$ in P , etc. Note that this m is less than the previous one. Eventually either the injury to Δ is repaired or γ has no values $\leq \delta(k)$. In the latter case the new values of γ are all $\geq \delta(k) = \theta_t(\varphi_t(p_t(k)))$, where t is the stage at which $\delta(k)$ was last defined. Since only values of γ can enter C , no number $\leq \theta_t(\varphi_t(p_t(k)))$ subsequently enters C . Thus as soon as γ has no values $\leq \delta(k)$ we know that $\Theta(C)$ will never change subsequently on arguments $\leq \varphi_t(p_t(k))$. If the scenario just sketched is repeated infinitely often, then the values of $\varphi_t(p_t(k))$ are unbounded and so $\Theta(C)$ becomes fixed on longer and longer initial segments. But A is non-recursive and so there are, in fact, only a finite number of such injuries to Δ which go unrepaired.

There is one point yet to be resolved. Assuming that A is not recursive we will repair all but a finite number of injuries to Δ . However, if, for fixed m , we enumerate $p(m)$ in P infinitely often in the course of repairing Δ , each time eliciting a response $\leq \varphi(p(m))$ in A , then the construction may fail. The problem is that, when a number $\leq \varphi(p(m))$ is enumerated in A , then we want to enumerate $\gamma(a(m))$ in C in order to be able to maintain the fundamental inequality (1). However, if $\gamma(a(m))$ is enumerated in C infinitely often, then

$\gamma(a(m))$ does not exist in the limit. This would be a fatal injury to the construction since it would mean that $\Theta(C, B)$ is not total. To avoid this trap we modify the plan laid out above as follows. Suppose that in response to the enumeration of k in K we enumerate $p(m)$ in P for some $m \leq k$ (making $p(m)$ temporarily undefined), and that in turn $x = a(j) \leq \varphi(p(m))$ is enumerated in A as a response to the enumeration of $p(m)$ in P . If $m < j$, and either $m = 0$, or $m > 0$ and $\gamma(a(m-1)) < x$, then we *restrain* $\gamma(a(m))$ (from C). At the same time we enumerate $\gamma(a(m+1))$ in C . The restraint remains in effect either forever or until some number $\leq a_m$ is enumerated in A . Moreover, until the restraint is lifted, we do not bother to redefine $p(m)$, we suspend the enumeration of pairs in the use-function δ and we do not bother to react to the enumeration of numbers in K . If and when the restraint on $\gamma(a(m))$ is lifted, we make up for these lapses. The only activity we maintain during the restraint of $\gamma(a(m))$ is the enumeration of $\gamma(a(y))$ in C in response to the enumeration of $a(y)$ in A .

Why does this device work? Since

$$m < j \Rightarrow a(m) < a(j) \Rightarrow \gamma(a(m+1)) \leq \gamma(a(j))$$

and $\gamma(a(m+1))$ is enumerated in C , our plan for satisfying \mathcal{S} is not impeded. Turning attention to \mathcal{T} notice that since $m \leq k$,

$$x = a(j) \leq \varphi(p(m)) \leq \varphi(p(k)),$$

and

$$\theta(a(j)) \leq \theta(\varphi(p(m))) \leq \theta(\varphi(p(k))) = \delta(k),$$

the computation $\Theta(C; x) = 0 = A_t(x)$ will be preserved from the stage t in which $\delta(k)$ last became defined. Thus the enumeration of x in A establishes a disagreement between A and $\Theta(C)$ which will persist forever unless some number $\leq \gamma(a(m))$ has to be enumerated in C , i.e., until the restraint on $\gamma(a(m))$ is lifted. Finally, using induction on m , we can argue that $p(m)$ is enumerated in P only finitely often as follows. It is clear that if, for some i , $\gamma(a(i))$ is permanently restrained from C then the construction is successful and eventually there is no activity which affects the partial function p . So without loss of generality suppose that no restraint is permanent. Further, for induction suppose that the following parameters have reached their final values before stage s :

$$a(i), \gamma(a(i)), p(i), \varphi(p(i)) \quad (i < m)$$

that $a(m)$ has reached its final value, that no number $\leq \gamma(a(m-1))$ is enumerated in A at any stage $\geq s$, nor any number $< m$ in K . It is clear that, if $p(m)$ is enumerated in P in stage s , then the number x enumerated in A in response exceeds $\gamma(a(m-1))$. This causes the permanent restraint of $\gamma(a(m))$ from C . So $p(m)$ is not enumerated in P at any stage $\geq s$. Further, once all members of A which are $\leq \varphi(p(m))$ have been enumerated, then $\gamma(a(m))$ will not change again. So the induction is complete.

In the next section we simultaneously play the strategy we have described for \mathcal{T} against all the requirements $\mathcal{T}^0, \mathcal{T}^1, \dots$. The needs of the different requirements do not conflict strongly. Nevertheless care must be exercised. To satisfy \mathcal{T} we employed δ and the partial function p . For \mathcal{T}^i we use δ^i and a partial function p^i . We give the convergence of $\gamma(a(i))$ priority over \mathcal{T}^m for each $i < m$. Hence $p^m(i)$ is only defined for $i \geq m$. The intended inequality corresponding to (1) is:

$$\varphi(p^0(m)) < \varphi(p^1(m)) < \dots < \varphi(p^m(m)).$$

However, at times some of these quantities may be undefined.

Generally speaking, we pursue satisfaction of the requirements \mathcal{R}^i independently. However, if a value $\gamma(a(m))$ is restrained from C on behalf of \mathcal{T}^i , where necessarily $i \leq m$, then that restraint will be respected by the substrategies charged with satisfying \mathcal{T}^j for all $j > i$. Thus, if in the course of trying to correct a value of δ^j , $j > i$, we would like to enumerate $p^j(k)$ in P where $j \leq k \leq m$, then we delay the action until the restraint on $\gamma(a(m))$ is lifted.

4 The construction

The notations and conventions established in the last section are adapted to the full problem in the obvious way. Instead of Θ, θ, Δ , and δ we now have $\Theta^n, \theta^n, \Delta^n$, and δ^n , where n runs through ω . While building ? via γ as in the sketch above we will attempt to satisfy the requirements \mathcal{T}^n in the same way that \mathcal{T} was satisfied in the previous section. Instead of the auxiliary function p we will have, for each n , an auxiliary function p^n devoted to the satisfaction of \mathcal{T}^n . Before giving instructions for the stages in the construction we make precise some notions which will be needed.

Restraining values of γ from C . During the construction a particular value of γ may be restrained from being enumerated in C on behalf of \mathcal{T}^n . If $\gamma(a(m))$ is restrained from C on behalf of \mathcal{T}^n , then we say that $\gamma(a(m))$ is *n-restrained*.

The input enumerations. A recursive enumeration of A without repetitions will be generated in the course of the construction. During the construction the next member of A will be enumerated on demand. Similarly for B and K . In the same way we will enumerate the next instruction in the functional Φ (or the functional Θ^n) on demand. However, we note that the enumerations of A and B will be tailored to the needs of the construction. This will be seen in the instructions for an R-stage below.

The auxiliary partial functions $p^y(x)$. At any point in the construction $p^y(x)$ will be defined for a finite set of pairs x, y with $y \leq x$. If $p^y(x) \downarrow$, then $p^z(x) \downarrow$ and $p^z(x) < p^y(x)$ for each $z < y$. Further, if $p^y(x)$ and $p^v(u)$ are both defined and $x < u$, then $p^y(x) < p^v(u)$. Also, $p^y(x) \downarrow$ implies $p^y(x) \notin P$. During the construction, if $p^y(x) \downarrow$, then $p^y(x)$ becomes undefined as soon as any of the following events occurs: some number $\leq p^y(x)$ is enumerated in P , $\varphi(p^y(x))$ becomes undefined, $\gamma(a(z))$ becomes undefined for some $z < x$. The value of the partial function p^y immediately before stage s will be denoted p_s^y . At any point in the construction a member of \overline{P} is said to be *used* if it is in $\bigcup_{y \in \omega} \text{Rng}(p^y)$.

Special behaviour of the use-function γ . According to (iii) of the convention for use-functions mentioned above $\gamma(a(m))$ should become undefined if a number $\leq \gamma(a(m))$ is enumerated in either B or C . This requirement is relaxed to the following extent. Suppose that in stage s some $x \leq \gamma(a(m))$ is enumerated in B . Then $\gamma(a(m))$ becomes undefined only if either there exists $y \leq m$ such that $\varphi_s(p_s^y(m)) \downarrow$ and $x \leq \varphi_s(p_s^y(m))$, or $m > 0$ and $x \leq \gamma(a(m-1))$.

(One of the basic ideas of the construction is to maintain the inequality $\varphi(p^y(m)) < \gamma(a(m))$ when both sides are defined. Thus we have no need to redefine $\gamma(a(m))$ because of the enumeration in B of a number $\leq \gamma(a(m))$ unless one of the values $\varphi(p^y(m))$ is disturbed. On the other hand it is convenient to reset $\gamma(a(m))$ if $m > 0$ and a number $\leq \gamma(a(m-1))$ is enumerated in B .)

Action prompted by enumeration of a number in A . Let $x = a(j)$ be

enumerated in A in stage s . There are two cases:

Case 1. There exist y, z such that

$$y < j \wedge p_s^z(y) \downarrow \wedge \varphi_s(p_s^z(y)) \downarrow \wedge \gamma_s(a_s(y)) \downarrow \\ \wedge x \leq \varphi_s(p_s^z(y)) < \gamma_s(a_s(y)).$$

Let n denote the least possible value of z . With $z = n$ let m denote the least possible value of y . Enumerate $\gamma_s(a_s(m))$ in C unless it becomes n -restrained in stage s . If $\gamma_s(a_s(m))$ does become n -restrained in stage s , then enumerate $\gamma_s(a_s(m+1))$ in C if it is defined.

Case 2. Otherwise. If $\gamma_s(a_s(j))$ is defined, then enumerate it in C .

The Repair List. If k is enumerated in K while $\delta^n(k) \downarrow$, then the pair (k, n) is added to the Repair List (called the *R-list* for short). (k, n) remains on the R-list until $\delta^n(k)$ becomes undefined through some number $\leq \delta^n(k)$ being enumerated in A . When (k, n) is on the R-list, $\mu^n(k)$ denotes the greatest m such that $\gamma(a(m)) \leq \delta^n(k)$. If $\mu^n(k) \uparrow$ or $\mu^n(k) < n$, then we say that (k, n) is *complete*. If $\mu^n(k) \geq n$ and $\gamma(a(\mu^n(k)))$ is i -restrained for some $i \leq n$, then we say that (k, n) is *stopped by i* . If (k, n) is neither complete nor stopped by any $i \leq n$, then we say that (k, n) *requires attention*.

Labels for the stages. Each stage of the construction will have one of the labels:

$$R, A, B, K, \gamma, \delta^n, \Phi, \Theta^n$$

where m runs through ω . The label R signifies that the stage is dedicated to correcting one of the functions δ^n . Every stage in which a pair on the R-list requires attention will be an R -stage. The remaining stages will be used to complete little by little the other tasks which are part of the construction. We shall arrange (see Lemma 5.2 below) that there are infinitely many stages with each possible label.

Action taken in stage s . In the rest of the section we describe the actions which take place in stage s according to its particular label:

R-stage. Let (k, n) be the first pair on the R-list which requires attention and let m denote $\mu^n(k)$. Enumerate $p^n(m)$ in P . Now we search for a number $x \leq \varphi(p^n(m))$ such that either $x \in A$ but has not yet been enumerated in A , or $x \in B$ but has not yet been enumerated in B . We may, of course,

assume that such x is found. Otherwise, $\Phi(A, B) \neq P$ which means that all the requirements are satisfied. Enumerate x in A if it is found first in A , and in B otherwise. In case the number $x = a(j)$ is enumerated in A (rather than B), $m < j$, $m \leq k$, and

$$[n > 0 \Rightarrow \varphi(p^{n-1}(m)) < x] \wedge [m > 0 \Rightarrow \gamma(a(m-1)) < x],$$

then $\gamma(a(m))$ is declared to be n -restrained. $\gamma(a(m))$ remains n -restrained either forever or until $\gamma(a(m))$ becomes undefined. Note that this imposition of n -restraint on $\gamma(a(m))$ saves the latter from enumeration in C which is the prescribed reaction to the enumeration of x in A in other circumstances.

A-stage. Enumerate in A its next member.

B-stage. Enumerate in B its next member.

K-stage. Enumerate the next member, k say, in K . For each n such that $\delta^n(k) \downarrow$ we add (k, n) to the R-list.

γ -stage. Let m be least such that $\gamma(a(m)) \uparrow$. There are three cases.

Case 1. For each $n \leq m$, $p^n(m) \downarrow$ and $\varphi(p^n(m)) \downarrow$. Define $\gamma(a(m)) = s$.

Case 2. $\varphi(q) \downarrow$, where q is the least unused member of \overline{P} , and there exists $n \leq m$ such that $p^n(m) \uparrow$. Set $p^n(m) = q$ for the least such n .

Case 3. Otherwise. Pass to the next stage.

δ^n -stage. Let m be least in \overline{K} such that $\delta^n(m) \uparrow$ and $\varphi(p^n(m)) \downarrow$. If $\theta^n(\varphi(p^n(m))) \downarrow$, and no value of γ is n -restrained, and $A(x) = \Theta^n(C; x)$ for all $x \leq \varphi(p^n(m))$, then set $\delta^n(m) = \theta^n(\varphi(p^n(m)))$. Otherwise pass to the next stage.

Φ -stage. Enumerate in Φ its next instruction.

Θ^n -stage. Enumerate in Θ^n its next instruction.

5 The verification

Since we have used $A, B, K, \gamma, \delta^n, \Phi, \Theta^n, p^n$, and μ^n to denote the current approximations to these quantities at an instant during the construction, we will use the subscript ω to denote limiting values where it seems helpful. Throughout the verification we assume that at the end of the construction $0 < \text{deg}(A) < \theta'$ and $P = \Phi(A, B)$.

By inspection of the construction, if a number x is enumerated in C in stage s , then a number $\leq x$ is enumerated in A in the same stage. Therefore C is recursive in A .

Lemma 5.1 *Immediately before a stage the following hold:*

- i) $p^y(x) \downarrow \Rightarrow [y \leq x \wedge p^y(x) \notin P \wedge \varphi(p^y(x)) \downarrow]$
- ii) $\gamma(a(x)) \downarrow \Rightarrow \gamma(a(x)) \notin C$
- iii) $[p^y(x) \downarrow \wedge \gamma(a(x)) \downarrow] \Rightarrow \varphi(p^y(x)) < \gamma(a(x))$
- iv) $[x < y \wedge p^z(x), p^u(y) \downarrow] \Rightarrow p^z(x) < p^u(y)$
- v) $[p^z(x) \downarrow \wedge y < z] \Rightarrow [p^y(x) \downarrow \wedge p^y(x) < p^z(x)]$
- vi) $Dom(\gamma)$ is an initial segment of \bar{A}
- vii) if $\gamma(a(x)) \downarrow$ and $p^y(x) \uparrow$ for some $y \leq x$, then $\gamma(a(x))$ is y -restrained, where y is least such that $p^y(x) \uparrow$.

Proof. We proceed by induction on stages. The stages with labels R, A, B, γ are those we need to examine.

Consider a stage s which is either a B-stage or an R-stage and in which x is enumerated in B . Let y be the least number if any such that $x \leq \gamma(a(y))$. If there is no such y , then the conclusion of the lemma is unaffected. So suppose y exists. Then, for all $z > y$ and $u \leq z$, $p^u(z)$ and $\gamma(a(z))$ become undefined in stage s if they were defined at the beginning of the stage. On the other hand for all $z < y$ and $u \leq z$, $p^u(z)$ and $\gamma(a(z))$ are unaffected. Immediately before stage s ,

$$\varphi(p^0(y)) \leq \dots \leq \varphi(p^n(y)) \leq \gamma(a(y))$$

where $n = \max \{u : p_s^u(y) \downarrow\}$. If n is not defined or $\varphi(p^n(y)) < x$, then all of $p^0(y), \dots, p^n(y)$, $\gamma(a(y))$ remain defined. However, if n is defined and $j \leq n$ is least such that $\varphi(p^j(y)) \geq x$, then $p^0(y), \dots, p^{j-1}(y)$ remain defined in stage s , while $p^j(y), \dots, p^n(y)$, and $\gamma(a(y))$ become undefined. Taking all these cases into account we see that the conclusions of the lemma are still true at the end of stage s .

Now consider a stage s which is either an A -stage or an R -stage and in which $x = a(j)$ is enumerated in A . Let (m, n) be the least pair (y, u) if any in the lexicographical order of pairs such that $\varphi_s(p_s^u(y)) \downarrow$ and $x \leq \varphi(p^u(y))$. There are three cases:

- Case 1. $\gamma_s(x) \uparrow$ and $(m, n) \uparrow$. The conclusion of the lemma is not affected.
Case 2. $\gamma_s(x) \downarrow$, and $(m, n) \uparrow$ or $m \geq j$. Then, in stage s , $\gamma_s(x)$ is enumerated in C and $p^u(y)$ becomes undefined for all $y > j$ and $u \leq y$ such that $y > j$ and $p_s^u(y) \downarrow$. By inspection the conclusion of the lemma remains true at the end of stage s .
Case 3. $(m, n) \downarrow$ and $m < j$. If $\gamma(a(m))$ becomes n -restrained in stage s , then $\gamma(a(m+1))$ is enumerated in C during stage s if it is defined. If $\gamma(a(m))$ does not become n -restrained in stage s , $\gamma(a(m))$ is enumerated in C if it is defined. In any case, $p^u(y)$ becomes undefined for all $(y, u) \geq (m, n)$. By inspection the conclusion of the lemma remains true at the end of stage s .

Finally, it is easy to see that γ -stages preserve the conditions stated in the various clauses of the lemma.

Lemma 5.2 *A pair (k, n) on the R -list receives attention at at most a finite number of stages. Hence for each possible label there are infinitely many stages with that label.*

Proof. When a pair (k, n) on the R -list receives attention, then a number x such that

$$x \leq \varphi(p^n(\mu^n(k))) \leq \gamma(a(\mu^n(k))) \leq \delta^n(k)$$

is enumerated in either A or B . In the former case (k, n) leaves the R -list, while in the latter the number of values of γ which are $\leq \delta^n(k)$ is diminished. Since any new values of γ are $> \delta^n(k)$, (k, n) can receive attention only a finite number of times.

Lemma 5.3 *For all x and $y \leq x$, $p^y(x)$ is eventually constant or eventually never defined, $\gamma(a(x))$ is eventually constant, and, for all $i \leq x$, $\gamma(a(x))$ is eventually permanently i -restrained or eventually never i -restrained.*

Proof. We proceed by induction on x . Consider m and n such that the conclusion holds for all $x < m$ and such that $p^y(m)$ is eventually (defined

and) constant for all $y < n \leq m$. Let u be chosen such that by the end of stage u the following quantities have achieved their final status:

$$\begin{aligned} a(x) & & (x \leq m) \\ p^y(x), \varphi(p^y(x)) & & (x < m, \text{ or } x = m \text{ and } y < n) \\ \gamma(a(x)) & & (x < m). \end{aligned}$$

Using Lemma 5.2 we shall also suppose that u is large enough so that no pair (k, n) with $k < m$ receives attention at a stage $\geq u$.

Claim 1. *If $p^n(m)$ is eventually never defined, then $\gamma(a(m))$ is eventually constant.*

Proof of Claim 1. Suppose that $p^n(m)$ is eventually never defined. Consider $v > u$, if any, such that $p_v^n(m) \uparrow$ and $\gamma_v(a(m)) \uparrow$. Let q be the least member of \overline{P} which is unused immediately before stage v . There exists w , the first γ -stage $\geq v$ in which $\varphi(q)$ is defined. Between stages v and w , neither γ nor any of the p_z 's acquires any new values. So q remains unused. In stage w , $p^n(m)$ is set equal to q . We infer that v is bounded, i.e., that $\gamma(a(m))$ is eventually always defined. It follows that $\gamma(a(m))$ is eventually constant since it cannot change value without first becoming undefined.

Claim 2. *If there exists q such that $p^n(m) = q$ at infinitely many stages, then eventually $p^n(m)$ is constant.*

Proof of Claim 2. Suppose that $p^n(m)$ takes the value q arbitrarily late in the construction. Choose $s > u$ such that $p^n(m) = q$ and $\varphi(q) \downarrow$ immediately before stage s , and such that $\varphi(q)$ is never subsequently undefined. It is clear that $p^n(m) = q$ for the rest of the construction.

Claim 3. *$p^n(m)$ takes only a finite number of different values.*

Proof of Claim 3. Suppose that $p^n(m) = q$ in a stage $v > u$. If there is a stage $w > v$ in which $p^n(m)$ is defined and $\neq q$, then there must be an R-stage s such that $v \leq s < w$ in which $p^n(m) = q$ is enumerated in P . (Otherwise, whenever $p^n(m)$ becomes undefined at a stage $> v$, it will always be given the same value when it is reset.) Let (k, n) be the pair which receives attention in stage s . It follows that $\mu^n(k) = m$ immediately before stage s , and hence that $\varphi(p^n(m)) < \gamma(a(m)) \leq \delta^n(k)$ also. Let t be the stage in which $\delta^n(k)$ is given the value which obtains immediately before stage s . Then in stage t , $\delta^n(k)$ is set equal to $\theta^n(\varphi(p^n(k)))$. Notice that between stages t and s , the quantities $p^n(m)$, $\varphi(p^n(m))$, and $\gamma(a(m))$ are unchanged. Otherwise, $\delta^n(k) < \gamma(a(m))$ immediately before stage s which would mean that $\mu^n(k) \neq m$, contradiction. It follows that A_s and A_t agree on arguments $\leq \varphi_s(p_s^n(m))$. There are now

two cases according as the number $x \leq \varphi(p^n(m))$ enumerated in stage s is enumerated in A or B :

Case 1. $x = a(j)$ is enumerated in A . By choice of u , $m < j$, $m \leq k$, and immediately before stage s

$$\left[n > 0 \wedge \varphi(p^{n-1}(m)) < x \right] \vee \left[n = 0 \wedge m > 0 \wedge \gamma(a(m-1)) < x \right].$$

Therefore $\gamma(a(m))$ becomes n -restrained in stage s . This situation will persist for the remainder of the construction and so $p^n(m)$ will never be defined after stage s .

Case 2. x is enumerated in B . At stages $\geq t$ and $< s$, no number $\leq \delta_{t+1}^n(k) = \delta_s^n(k)$ is enumerated in C . Otherwise some number $< \delta^n(k)$ is enumerated in A at a stage $\geq t$ and $< s$, which contradicts the choice of t . In stage s , $\gamma(y)$ becomes undefined for all $y \geq a(m)$. It follows that no number $< \delta_{t+1}^n(k) = \theta_t^n(\varphi_t(p_t^n(k)))$ is enumerated in C at a stage $\geq t$. Therefore

$$\theta_\omega^n(y) = \theta_t^n(y) \wedge \Theta_\omega^n(C_\omega; y) = \Theta_t^n(C_t; y) = A_t(y) \quad (y \leq \varphi_t(p_t^n(k))). \quad (2)$$

Since $m \leq k$ by choice of u , we have

$$p_s^n(m) = p_t^n(m) \leq p_t^n(k) \leq \varphi_t(p_t^n(k)). \quad (3)$$

From (2) and (3), $\Theta_\omega(C_\omega)$ and A_t agree on arguments $\leq p_s^n(m)$.

Suppose that there are infinitely many stages $> u$, like the stage s just discussed, in which $p^n(m)$ is enumerated in P . As s increases to ∞ so do $p_s^n(m)$ and t , where t is related to s as above. This shows that for each particular pair (s, t) , A_t and A_ω agree on arguments $\leq p_s^n(m)$. Since the pairs (s, t) are effectively enumerable, we infer that A_ω is recursive, a contradiction. This completes the proof of Claim 3 since it shows there can be only a finite number of such stages s .

We can now complete the proof of the lemma. Fix m such that the conclusions of the lemma hold for all $x < m$. Let $n \leq m$ be the least y if any such that $p^y(m)$ is not eventually constant. The three claims show that $p^n(m)$ is eventually never defined while $\gamma(a(m))$ is eventually constant. The desired conclusions now follow easily.

This leaves only the case in which $p^y(m)$ is eventually (defined and) constant for all $y \leq m$. We have to prove that $\gamma(a(m))$ will also be eventually defined and constant. Let u be chosen as at the beginning of the proof of the

lemma taking $n = m + 1$. Consider the first γ -stage $v > u$. Then $\gamma(a(m))$ is certainly defined at the end of stage v . By induction on stages, $\gamma(a(m))$ is never subsequently undefined and keeps its value from stage v for the rest of the construction. The desired conclusions are again almost immediate.

This completes the proof of the lemma.

Corollary 5.4 *A_ω is recursive in B_ω and C_ω .*

Proof. Suppose that we already know which numbers $< x$ belong to A_ω . Let there be m numbers $< x$ which are in $\overline{A_\omega}$. To discover whether $x \in A_\omega$ we look for s such that $a_s(m) = x$, $\gamma_s(x) \downarrow$, and no number $\leq \gamma_s(x)$ is enumerated in either B or C at a stage $\geq s$. From the previous lemma either we find such s or x is enumerated in A . Suppose that s is found. By induction on stages $\geq s$, for all $y \leq m$, $a(y)$ and $\gamma(a(y))$ retain their values from immediately before stage s for the rest of the construction. For, if one of these quantities takes on a new value or becomes undefined, it must be the case that one of the γ -values is enumerated in C or some $y \leq \gamma_s(x)$ is enumerated in B . This is impossible by choice of s . Thus, if s is found, then $x \notin A_\omega$. It is clear from this that A_ω is recursive in B_ω, C_ω .

Lemma 5.5 *For each n at most a finite number of values of γ are permanently n -restrained.*

Proof. Let $\gamma_s(a_s(m))$ become permanently n -restrained in stage s . Then $a_s(m) = a_\omega(m)$, $\gamma_s(a_s(m)) = \gamma_\omega(a_s(m))$, stage s is an R-stage, and during stage s , $p^n(m)$ is enumerated in P and some number $\leq \varphi(p^n(m))$ is enumerated in A . Let (k, n) be the pair which receives attention in stage s . Then $\delta_s^n(k) \downarrow$, $m = \mu_s^n(k)$, and

$$\varphi_s(p_s^n(m)) \leq \gamma_s(a_s(m)) \leq \delta_s^n(k).$$

Thus (k, n) leaves the R-list at stage s and of course never returns. On the other hand, at stages $\geq s$, δ^n acquires no new values. Hence, there are only finitely many k such that (k, n) is ever on the R-list. Once all the pairs (k, n) , which are ever going to leave, have left the R-list, no value of γ can become permanently n -restrained.

Lemma 5.6 *If some value of γ is permanently n -restrained, then $\Theta_\omega^n(C_\omega) \neq A_\omega$.*

Proof. As in the proof of the last lemma suppose that $\gamma_s(a_s(m))$ becomes permanently n -restrained in stage s . We shall use the same notation as before. Now we repeat part of the proof of the Lemma 5.3. Let t be the stage in which $\delta^n(k)$ is given the value it has immediately before stage s . Then in stage t , $\delta^n(k)$ is set equal to $\theta^n(\varphi(p^n(k)))$. Notice that between stages t and s , the quantities $p^n(m)$, $\varphi(p^n(m))$, and $\gamma(a(m))$ are unchanged. Otherwise, $\delta^n(k) < \gamma(a(m))$ immediately before stage s which would mean that $\mu^n(k) \neq m$, contradiction. It follows that A_s and A_t agree on arguments $\leq \varphi_s(p_s^n(m))$. Also, the conditions for the imposition of n -restraint require $m \leq k$.

Let $x \leq \varphi_s(p_s^n(m)) = \varphi_t(p_t^n(m))$ be the number enumerated in A in stage s . Since $\delta^n(k)$ does not become undefined between stage t and stage s , no number $\leq \delta_t^n(k)$ is enumerated in A between stage t and stage s , whence no number $\leq \delta_t^n(k)$ is enumerated in C between stage t and stage s . Since $\mu_s^n(k) = m$, we have

$$\delta_t^n(k) = \delta_s^n(k) \leq \gamma_s(a_s(m+1))$$

if $\gamma_s(a_s(m+1))$ is defined. It follows that no number $\leq \delta_t^n(k) = \theta_t^n(\varphi_t(p_t^n(k)))$ is enumerated in C at a stage $\geq t$. Therefore

$$\Theta_\omega^n(C_\omega; x) = \Theta_t^n(C_t; x) = 0$$

since $x \notin A_t$ and

$$x \leq \varphi_s(p_s^n(m)) = \varphi_t(p_t^n(m)) \leq \varphi_t(p_t^n(k)).$$

Since $x \in A_\omega$, A_ω and $\Theta_\omega^n(C_\omega)$ disagree at x . This completes the proof of the lemma.

Lemma 5.7 *For all n , $\Theta_\omega^n(C_\omega) \neq A_\omega$.*

Proof. Towards a contradiction suppose that $\Theta_\omega^n(C_\omega) = A_\omega$. From the previous lemma no value of γ is permanently n -restrained. Since some pair (k, n) leaves the R-list each time a value of γ becomes n -restrained, it follows that

there are infinitely many δ^n -stages in which no value of γ is n -restrained, i.e., stages in which the domain of δ^n is free to grow. From Lemma 5.3 it follows that for all but a finite number of $x \in \overline{K_\omega}$, $\delta^n(x)$ is eventually always defined and constant. By the same token, for each x , $\delta^n(x)$ becomes defined only finitely often. When $\delta^n(x)$ becomes defined, it is given the value $\theta^n(\varphi(p^n(x)))$. Since use-functions are required to majorize the identity function, the value assigned to $\delta^n(x)$ is $\geq p^n(x)$. Hence the new values assigned to $\delta^n(x)$ for various x as the construction proceeds tend to infinity.

Let m be the greatest number $\geq n$ such that for no $y \geq m$ is $\gamma(a(y))$ permanently i -restrained for any $i \leq n$. Fix s such that $a(m)$ and $\gamma(a(m))$ attain their final values before stage s , such that no number $\leq \gamma_\omega(a_\omega(m))$ is enumerated in A at any stage $\geq s$, and such that all values assigned to δ^n at stages $\geq s$ are $> \gamma(a(m))$. Let D denote the set of all x such that $\delta^n(x)$ becomes defined at some stage $t \geq s$ and no number $\leq \delta^n(x)$ is subsequently enumerated in A . Clearly, D is recursively enumerable in A_ω , and almost every member of $\overline{K_\omega}$ belongs to D . On the other hand, $D \subseteq \overline{K_\omega}$. Otherwise we should have $k \in D - K_\omega$ and a pair (k, n) eventually always on the R-list with $\delta^n(k) \geq \gamma(a(m))$. Since (k, n) receives attention only finitely often, at every sufficiently large stage (k, n) is stopped by i for some $i \leq n$. This means that $m \leq \mu^n(k)$, $\gamma(a(\mu^n(k))) \leq \delta^n(k)$, and $\gamma(a(\mu^n(k)))$ is i -restrained for some $i \leq n$. By choice of m we know that $\gamma(a(\mu^n(k)))$ is not permanently i -restrained, whence this value of $\gamma(a(\mu^n(k)))$ is eventually destroyed. However, new values of γ will all be $> \delta^n(k)$. Thus this state of affairs cannot continue until the end of the construction. This confirms the claim that $D \subseteq \overline{K_\omega}$. It follows that $K \leq_T A$. This contradiction completes the proof of the lemma.

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