On the exclusion of forest minors: a short proof of the path-width theorem

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Robertson and Seymour proved that excluding any fixed forest F as a minor imposes a bound on the path-width of a graph. We give a short proof of this, reobtaining the best possible bound of $|F|-2$.

1. Introduction

At the start of their Graph Minors series, Robertson and Seymour [1] proved, by a long and involved argument, that for every forest F there exists an integer n such that every graph without an F minor has path-width at most n. This bound was brought down to the best possible value by Bienstock, Robertson, Seymour and Thomas [2], who proved the following:

Theorem 1. [2] *For every forest F, every graph of path-width $\geqslant |F|-1$ has a minor isomorphic to F.*

The authors remark that this result is best possible in two ways. First, the value of $|F|-1$ is sharp, because the complete graph K_{n-1} has path-width $n-2$ but no (forest) minor on n vertices. Second, if F is *not* a forest then the exclusion of F as a minor does not bound the path-width of a graph: as noted without proof in [1], trees can have arbitrarily large path-width (but will never contain F as a minor if F contains a cycle).

The proof of Theorem 1 in [2], already much shorter than [1], relies on a non-trivial minimax theorem involving the concept of "blockages". (These were adapted from "tangles", a central concept in the Graph Minors series concerning tree-width.) Although interesting in its own right, it turns out that this minimax theorem is not needed for a proof of Theorem 1; the purpose of this note is to give a short direct proof.

2. Definitions

Graphs in this paper are finite, and they may have loops or multiple edges. Let G be a graph. If $X \subset V(G)$ or $X \subset G$, then $G[X]$ denotes the subgraph of G induced by the vertices in X. Following [2], we denote by $\mathrm{att}(X)$ (for "attachment") the set of those vertices in X that have a neighbour in $G - X$, and write $\alpha(X) := |\mathrm{att}(X)|$. A *minor* of G is a graph obtained from a subgraph of G by contracting edges.

A *path-decomposition* of G is a sequence $(W_1, \ldots, W_s)$ of subsets of $V(G)$ such that

(i) $W_1 \cup \ldots \cup W_s = V(G)$, and for every edge e of G there exists an $r \leqslant s$ such that both endvertices of e are in W_r;

(ii) $W_p \cap W_r \subset W_q$ whenever $1 \leqslant p \leqslant q \leqslant r \leqslant s$.

The *width* of a path-decomposition as above is the number

$$\max\{\, |W_r| - 1 : 1 \leqslant r \leqslant s \,\},$$

and the *path-width* of G is the smallest width of any path-decomposition of G.

For each positive integer n, we denote by $\mathcal{B}_n = \mathcal{B}_n(G)$ the unique minimal subset of the power set of G satisfying the following two conditions:

(i) $\emptyset \in \mathcal{B}_n$;

(ii) if $X \in \mathcal{B}_n$, $X \subset Y \subset V(G)$ and $\alpha(X) + |Y \backslash X| \leqslant n$, then $Y \in \mathcal{B}_n$.

Thus, a set $X \subset V(G)$ is in $\mathcal{B}_n$ if and only if there is a sequence

$$\emptyset = X_0 \subset \ldots \subset X_s = X$$

such that $\alpha(X_r) + |X_{r+1} \backslash X_r| \leqslant n$ for all $r < s$.

For example, if $(W_1, \ldots, W_s)$ is a path-decomposition of G of width $< n$, then all the sets $W_1 \cup \ldots \cup W_r$ for $r \leqslant s$, including $V(G)$ (for $r = s$), are in $\mathcal{B}_n$. This is easy to verify by induction on r from the axioms of a path-decomposition, but will not be used below. Its converse is also true, and will be needed later:

(2.1) *If $V(G) \in \mathcal{B}_n$ then G has path-width $< n$.*

Indeed, if $V(G) \in \mathcal{B}_n$ then there is a sequence $\emptyset = X_0 \subset \ldots \subset X_s = V(G)$ such that $\alpha(X_r) + |X_{r+1} \backslash X_r| \leqslant n$ for all $r < s$. With

$$W_r := \mathrm{att}(X_{r-1}) \cup (X_r \backslash X_{r-1})$$

for all $1 \leqslant r \leqslant s$, the sequence $(W_1, \ldots, W_s)$ is easily seen to be a path-decomposition of G of width $< n$.

3. Proof of the theorem

In our proof of Theorem 1, we essentially follow the proof of [2, (3.1)], albeit in a less elegant and rather more straighforward inductive set-up (for a bit of mathematical glasnost). Instead of [2]'s minimax theorem on blockages, we shall use the following lemma.

Lemma. *Let G be a graph, $Y \in \mathcal{B}_n(G)$, and $Z \subset Y$. Assume that there is a set $\{ P(z) : z \in \text{att}\,(Z) \}$ of disjoint paths in G such that each $P(z)$ starts in z, has no other vertex in Z, and ends in $\text{att}\,(Y)$. Then $Z \in \mathcal{B}_n(G)$.*

Proof. By definition of $\mathcal{B}_n$, there are sets $\emptyset = Y_0 \subset \ldots \subset Y_s = Y$ such that $\alpha(Y_r) + |Y_{r+1} \backslash Y_r| \leqslant n$ for all $r < s$. We claim that, with

$$Z_r := Y_r \cap Z,$$

we likewise have $\alpha(Z_r) + |Z_{r+1} \backslash Z_r| \leqslant n$ for all $r < s$, showing that $Z = Z_s \in \mathcal{B}_n$.

Fix r. Since

$$Z_{r+1} \backslash Z_r = Z_{r+1} \backslash Y_r \subset Y_{r+1} \backslash Y_r\,,$$

it suffices to show that $\alpha(Z_r) \leqslant \alpha(Y_r)$. We prove this by constructing a 1–1 map $z \mapsto y$ from $\text{att}\,(Z_r) \backslash \text{att}\,(Y_r)$ to $\text{att}\,(Y_r) \backslash \text{att}\,(Z_r)$.

Consider a vertex $z \in \text{att}\,(Z_r) \backslash \text{att}\,(Y_r)$. Then z has a neighbour in $Y_r \backslash Z_r = Y_r \backslash Z$, so $z \in \text{att}\,(Z)$. Now $P(z)$ is a path from $(Z_r \subset)\ Y_r$ to $\text{att}\,(Y)$, and $\text{att}\,(Y_r)$ separates these two sets in G. Therefore $P(z)$ has a vertex y in $\text{att}\,(Y_r)$; note that $y \neq z$ by the choice of z. As z is the only vertex of $P(z)$ in Z, we thus have $y \in \text{att}\,(Y_r) \setminus \text{att}\,(Z_r)$. By definition of the paths $P(z)$, the vertices y are distinct for different z, so $\alpha(Z_r) \leqslant \alpha(Y_r)$ as claimed. □

We are now ready to prove Theorem 1. Let us assume, without loss of generality, that F is a tree. Let G be a graph of path-width at least $n = |F| - 1$, and let $(v_1, \ldots, v_{n+1})$ be an enumeration of $V(F)$ such that $F\,[\,v_1, \ldots, v_i\,]$ is connected for all i. Then, for each $i \leqslant n$, exactly one vertex in $\{ v_1, \ldots, v_i \}$ is adjacent to v_{i+1}.

For every $i = 0, \ldots, n$, we shall define a set $\mathcal{C}^i = \{ C_0^i, \ldots, C_i^i \}$ of disjoint subgraphs of G, so that $C_j^k \subset C_j^\ell$ whenever $j \leqslant k \leqslant \ell$, and all C_j^i with $j > 0$ are connected. We shall write $X^i := V\big(\bigcup \mathcal{C}^i\big)$. For each i, the following four statements will hold:

(i) G contains a C_j^i–C_k^i edge whenever $1 \leqslant j < k \leqslant i$ and $v_j v_k \in E(F)$ (so $F\,[\,v_1, \ldots, v_i\,]$ is a minor of $G\,[\,C_1^i \cup \ldots \cup C_i^i\,]$);

(ii) $\alpha(X^i) = i$, and $|V(C_j^i) \cap \text{att}\,(X^i)| = 1$ for all $1 \leqslant j \leqslant i$;

(iii) $X^i \in \mathcal{B}_n$;

(iv) $\alpha(X) > i$ for all $X \in \mathcal{B}_n$ with $X^i \subsetneqq X$.

Let C_0^0 be an inclusion-maximal subgraph of G with $V(C_0^0) \in \mathcal{B}_n$ and $\alpha(C_0^0) = 0$ (possibly $C_0^0 = \emptyset$). Then (i)–(iv) hold for $i = 0$. Assume now that $\mathcal{C}^i$ has been defined so that (i)–(iv) holds, for some $i \leqslant n$. If $i = 0$, let x be any vertex of $G \backslash C_0^0$; note that $G \backslash C_0^0 \neq \emptyset$, since $V(C_0^0) \in \mathcal{B}_n$ but $V(G) \notin \mathcal{B}_n$ by (2.1). If $i > 0$, consider the unique $j \leqslant i$ such that $v_j v_{i+1} \in E(F)$, and let $x \in G - X^i$ be a neighbour of the unique vertex in $V(C_j^i) \cap \operatorname{att}(X^i)$. Let $X := X^i \cup \{x\}$.

If $i = n$, then (i) and the choice of x imply that F is a minor of $G[X]$ and we are done. So we assume that $i < n$. Then, by (ii), (iii) and the definition of $\mathcal{B}_n$, we have $X \in \mathcal{B}_n$. Thus, $\alpha(X) > i$ by (iv). As clearly $\operatorname{att}(X) \cap X^i \subset \operatorname{att}(X^i)$, this means that $\operatorname{att}(X) = \operatorname{att}(X^i) \cup \{x\}$ and $\alpha(X) = i + 1$. Let Y be inclusion-maximal in $\mathcal{B}_n$ with $X \subset Y$ and $\alpha(Y) = i + 1$. (This set Y will later be our X^{i+1}.)

By Menger's theorem, there exist a set $\mathcal{P}$ of disjoint $X - \operatorname{att}(Y)$ paths in $G[Y]$ and an $X - \operatorname{att}(Y)$ separator $S \subset Y$ in $G[Y]$ consisting of a choice of exactly one vertex from every path in $\mathcal{P}$. Let Z denote the union of S and all the vertex sets of components of $G - S$ meeting X. Then $X^i \subsetneq X \subset Z$. By the choice of S and definition of $\operatorname{att}(Y)$, we further have $Z \subset Y$. Since $\operatorname{att}(Z) = S$, this means that $Z \in \mathcal{B}_n$ by the Lemma. By (iv), then, $|\mathcal{P}| = |S| = \alpha(Z) > i$. By definition of $\operatorname{att}(X)$, each of the paths in $\mathcal{P}$ meets $\operatorname{att}(X)$. Thus $i < |\mathcal{P}| \leqslant \alpha(X) = i + 1$, so $|\mathcal{P}| = i + 1$ and the paths in $\mathcal{P}$ contain a perfect path matching between $\operatorname{att}(X)$ and $\operatorname{att}(Y)$.

We now define $\mathcal{C}^{i+1}$. Let $C_0^{i+1} := C_0^i \cup \big(Y \setminus (X \cup V(\bigcup \mathcal{P}))\big)$. For $1 \leqslant j \leqslant i$, let x_j be the unique vertex of C_j^i in $\operatorname{att}(X^i)$ and P_j the path in $\mathcal{P}$ containing it, and define C_j^{i+1} as the union of C_j^i with the final segment of P_j starting at x_j. Finally, let C_{i+1}^{i+1} be the final segment from x of the path in $\mathcal{P}$ containing x. Then $X^{i+1} = V(\bigcup \mathcal{C}^{i+1}) = Y$. Now $\mathcal{C}^{i+1} = \{C_0^{i+1}, \ldots, C_{i+1}^{i+1}\}$ satisfies, for $i + 1$, condition (i) by the choice of x and of C_{i+1}^{i+1}; conditions (ii) and (iii) since $X^{i+1} = Y$; condition (iv) by the choice of Y and $X^i \subset Y = X^{i+1}$ together with (iv) for i.

As remarked before, the assertion of the theorem follows from the definition of X in the case of $i = n$.

References

[1] N. Robertson and P.D. Seymour, Graph minors I: excluding a forest, *J. Combin. Theory B* **35** (1983), 39–61.

[2] D. Bienstock, N. Robertson, P.D. Seymour and R. Thomas, Quickly excluding a forest, *J. Combin. Theory B* **52** (1991), 274–283.

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