Emergent Statistical Wealth Distributions in Simple Monetary Exchange Models: A Critical Review

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Abstract: This paper reviews recent attempts at modlling inequality of wealth as an emergent phenomenon of interacting-agent processes. We point out that recent models of wealth condensation which draw their inspiration from molecular dynamics have, in fact, reinvented a process introduced by Angle (1986) in the sociological literature. We also emphasize some problematic aspects of simple wealth exchange models and contrast them with a monetary model based on economic principles of market mediated exchange. The paper also reports new results on the influence of market power on the wealth distribution in statistical equilibrium. As it turns out, inequality increases but market power alone is not sufficient for changing the exponential tails of single exchange models into Pareto tails.

1 Introduction

Since the days of Vilfredo Pareto, the frequency distribution of wealth among the members of a society has been the subject of intense empirical research. Besides the power law or Pareto distribution proposed by Pareto himself, frequently used stochastic models for wealth stratification are the Lognormal or Gamma distributions. Recent research, however, confirms that power-law behaviour with an exponent between 1 and 2 indeed seems to characterize the right tail of the distribution (Levy and Solomon, 1997; Milakovic, 2005). However, when applied to the entire shape of the empirical distribution, the power law would produce a rather mediocrese fit and would be outperformed by other candidate processes. Besides the elementary findings on the shape of the unconditional wealth distribution, one might even find additional stylized facts on the temporal development of wealth distribution in historical data. Angle, 1986, for example, reviews evidence of a gradual change in the shape of the distribution in response to industrial development. While rapid spurts of economic growth in the initial phase of transition from a mainly agricultural to an industrial economy might first lead to an increase of inequality, the subsequent build-up of human capital is believed to reduce the concentration of wealth again.

These and other findings should give rise to modelling efforts explaining the remarkebly similar wealth distribution of many developed countries as well as their long-run changes (if such changes can be confirmed statistically). Unfortunately, economic theory has been quite silent on this topic for a long time. Until recently, one had to go back the to literature of the fifties and sixties (e.g., Champernowne, 1953; Mandelbrot, 1961) to find stochastic models of wealth accumulation in modern societies. Recent advances in computer technology, however, open another avenue for analysis of the emergence of wealth distributions allowing this issue to be studied in a computational agent-based framework. Such a bottom-up approach could, in principle, be helpful in isolating the key mechanisms that apparently lead to a stratification of wealth in all but the simplest hunter and gatherer economies. As it appears, this path has been pursued recently by physicists rather than economists (cf. Bouchaud and Mézard, 2000; Drăgulescu and Yakovenko, 2000; Chakraborty and Chakrabarty, 2000; Silver, Slad and Takamoto, 2002, among others). Many of these models share a relatively simple structur of agent interactions. As had been entirely overloaded in the pertinent publications, these models have an important predecessor in the sociological literature. Investigating essentially the same structures already almost twenty years ago. Angle, 1986, might be considered as the first contribution to agent-based analysis of wealth formation. In the following, I will shortly review Angle's interesting work as the prototypical agent-based model of wealth dynamics, based on particle-like microscopic interactions of agents. However, the assumptions of these models would almost certainly be considered to be problematic by most economists (section 2). I will, therefore, move on to the isolated contribution by Silver et. al. (2002) which much better fits into standard economic reasoning, but nevertheless provides a similarly simple exchange model which could also explain 97% (if we believe the numbers on the range of validity of exponential and Pareto regime) of empirical data (section 3). Section 4 presents some additional results expanding on the seminal framework of Silver et. al. Conclusions are in section 5.

2 Angle's Surplus Theory of Social Stratification and the Inequality Process

In a long chain of papers covering more than 15 years, sociologist John Angle has elaborated on a class of stochastic processes which he first proposed in 1986 as a generating mechanism for the universal emergence of inequality in wealth distributions in human societies. His starting point is evidence he attributes to archeological excavations that inequality among the members of a community is typically first found with the introduction of agriculture and the prevalence of food abundance: While simpler hunter/gatherer societies appear to be rather egalitarian, production of a “surplus” beyond subsistence level immediately seems to lead to a “ranked society” or some kind of “chiefdom” (Angle, 1986, p. 298), inequal distribution of wealth is usually identified by archeologists via the observed distribution of grave goods).

So as soon as there is some excess capacity of food processes seem to be set into motion from which inequality emerges. Angle, surveying earlier narrative work in sociology, sees this as the result of redistribution by which some members of society succeed in grapping some of the surplus wealth of others. This dynamics he attemps to formalize in his “inequality process”. [1] The relevant empirical observations are summarized as follows:

> ”**Proposition 1:** *Where people are able to produce a surplus, some of the surplus would be fugitive and would leave the possession of the people who produce it.*
>
> **Proposition 2:** *Wealth confers on those who posssess it the ability to extract wealth from others. So netting out each person’s ability to do this in a general competition for surplus wealth, the rich tend to take surplus away from the poor.” (Angle, 1986, p. 298).*

According to Angle, the expropriation of the loosers happens via (1) theft, (2) extortion, (3) taxation, (4) exchange coerced by unequal power between the participants, (5) genuinely voluntary exchange, or (6) gift (ibid).

The process he designs as a formalisation of these ideas is a true interacting particle model: in a finite population, agents are randomly matched in pairs and try to catch part of the other’s wealth. A random toss $D_t \in 0, 1$ decides which of both agents is the winner of this conflict. Angle in various papers considers cases with equal winning probabilities 0.5 as well as others with probabilities being biased in favor of either the wealthier or poorer of both individuals. If the winner of this encounter is assumed to take away a fixed proportion of the other’s wealth, ω, the simplest version of the ”inequality process” leads to a stochastic evolution of wealth of individuals i and j who had bumped into each other according to:

$$w_{i,t} = w_{i,t-1} + D_t \omega w_{j,t-1} - (1 - D_t)\omega w_{i,t-1} \tag{1}$$

[1] It does not seem to occur to him nor his predecessors that at least some of the inequality could be due to different abilities and, therefore, different contributions to the generation of surplus. Note that this idea would explain wealth inequality by a standard economic argument à la wage being equal to marginal product of labor (which might differ over individuals). Of course, this would be an explanation in which interaction of agents and the organisation of modern economies would play only a minor part.

$$w_{j,t} = w_{j,t-1} + (1 - D_t)\omega w_{i,t-1} - D_t \omega w_{j,t-1}$$

Time t is measured in encounters and one pair of agents from the whole population is chosen for this interaction in each period. Angle (1986) shows via simulations that this dynamics leads to a stationary distribution which can be reasonally well fitted by a Gamma distribution. Angle (1993) provides a heuristic proof that the Gamma distribution is, in fact, the limiting distribution of this process. Later papers provide various extensions of the basic model. While the exponential decay of the Gamma distribution might not be in accordance with power law behavior at the high end of the richest individuals, Angle's model is the first agent-based approach matching several essential features of empirical wealth distributions which he carefully lists as desiderada (i.e. stylized facts) for a theory of inequality. Among other properties, he emphasizes the uni-modality with a mode above minimum income which could not be reproduced by a monotonic distribution function (this argument would also speak against the Boltzmann dynamics proposed by Drăgulescu and Yakovenko, 2000). Angle is also careful to point out that with binned data, realizations of his process would be hard to distinguish from realizations of Pareto random variables which he demonstrates via a few Monte Carlo runs.

Unfortunately, Angle's process might be hard to accept for economists as a theory of the emergence of inequality in market economies (as most of the economies have been for which accurate wealth measurements are available). First, a glance at the list of the six mechanisms for appropriation (or rather usurpation) of another agent's wealth might raise doubts about their relative importance in modern societies: for most countries of the world, "theft" should perhaps not be the most eminent mechanism for stratification of the wealth distribution. It is similarly remarkable that "genuinely voluntary exchange" is listed only at rank 5 and behind "exchange coerced by unequal power". However, voluntary exchange is at the heart of economic activity at all levels of development rather than being a minor facet. While theft is hopefully a minor distortion in most societies and purposeful redistribution policy had been restricted largely to welfare states of the twentieth century, voluntary exchange plays a major role in all relevant economies and, therefore, should be taken into account as a source of redistribution of wealth at a much more prominent rank.

However, despite its mentioning in the list of mechanisms of redistribution, voluntary exchange is not really considered in Angle's model in which an agent simply takes away part of the belongings of another. What is more, this kind of encounter would - in its literal sense - hardly be imaginable as both agents would rather prefer *not* to participate in this game of a burglar economy - at least if they possess a minimum degree of risk aversion. The model, thus, is not in harmony with the principle of voluntary participation of agents in the hypothesized process which economists would consider to be an important requirement for a valid theory of exchange activities. One should also note that another problem is the lack of consideration of the measurement of wealth (in terms of monetary units) and the influence of changes of the value of certain components of overall wealth.

Despite these problematic features from the viewpoint of economics, Angle's model deserves credit as the first contribution in which inequality resuslts as an emergent property of an agent-based approach. A glance at the recent econophysics literature shows that all relevant contributions share the structure of Angle's model. The inequality process depicted in eq. (1) is, for example, practically identical to the process proposed by Bouchaud and Mézard (2000) and isomorphic to almost all other models surveyed in Hayes (2002). This recent ??? of research on wealth dynamics is, therefore, almost exemplary for the lack of coordination among research pursued on the same topic in different disciplines and for the unfortunate duplication of effort that comes along with it. Interestingly, the criticism above had also been voiced in a review of monetary exchange models developed by physicists by Hayes (2002) who introduced the label of "theft and fraud" economies, but restricted it to variants in which the richer could lose more (in absolute value) than the poor. However, it is not clear why models which introduce a certain asymmetry to avoid this kind of exploitation should not also suffer from the lack of willingness of agents to participate in their exchange processes. It, therefore, appears that one might wish to reformulate the "burglar economies" in a way that brings elements of economic exchange processes into play. While the economics literature has not elaborated on wealth distributions emerging from exchange activities within a group of agents, a huge variety of approaches is available in economics that could be utilized for this purpose. An interesting start has been made in a recent paper by Silver, Slud and Takamoto (2002) which contains a two good general equilibrium model of an economy with heterogenous agents. Somewhat ironically, the overall outcome of this model is the same as with Angle's: the stationary wealth distribution turns out to be a Gamma distribution.

3 An Exchange Economy with Changing Preferences

Unlike the framework reviewed surveyed in the previous section, the setting of Silver et al. is an extremely familiar one for economists. Their economy consists of two goods, denoted x and y which necessitate the introduction of a relative price p being defined as the current value of a unit of good y in units of good x. Note that with this assumption, considerations of revaluation of wealth components come into play which are altogether neglected in the sociological/physical models. All agents of the economy have their preferences formalized by a so-called Cobb-Douglas utility function:

$$U_{i,t} = x_{i,t}^{f_{i,t}} \cdot y_{i,t}^{1-f_{i,t}} \tag{2}$$

Here, i and t are indices for the individuals and time, respectively. $x_{i,t}$ and $y_{i,t}$ are, therefore, the possessions of good x and y by individual i at time t and $f_{i,t} \in [0,1]$ is a preference parameter which might differ among individuals and, for one and the same individual, might also change over time. $U_{i,t}$, then, is utility gained by individual i at time t. Individuals start with a given endowment in $t = 0$ and try to maximize their utility via transactions in a competitive market where one good is exchanged against the other. Given their possessions of both goods at some time $t-1$, it is a simple exercise to compute their demands for goods x and y at time t given the current preference parameter $f_{i,t}$:

$$x_{i,t} = f_{i,t}(x_{i,t-1} + p_t y_{i,t-1}) \tag{3}$$

$$y_{i,t} = (1 - f_{i,t})(\frac{x_{i,t-1}}{p_t} + y_{i,t-1})$$

In (3), we have used the standard assumption that agents take the price as given in a competitive market. Note that this market, therefore, dispenses with any assumption of unequal exchange or even exploitation which is so central to the microscopic process of the previous chapter.
Summing up demand and supply by all our agents, we can easily calculate the equilibrium price which simultaneously clears both markets:

$$p_t = \frac{\sum_i (1 - f_{i,y})\, x_{i,t-1}}{\sum_i f_{i,t} y_{i,t-1}} \tag{4}$$

After meeting in the market, each agent possesses a different bundle of goods and his wealth can be evaluated as:

$$w_{i,t} = x_{i,t} + p_t y_{i,t} \tag{5}$$

The driving force of the dynamics of the model by Silver et al. is simply the assumption of stochastically changing preferences: all $f_{i,t}$ are drawn anew in each period independently for all individuals. In the baseline scenario, the $f_{i,t}$ are simply drawn from a uniform distribution over $[0,1]$, but other distributions lead to essentially the same results. The dynamics is, thus, generated via the agents' needs to rebalance their possessions in order to satisfy their new preference ordering. With all agents attempting to change the composition of their "wealth", price changes are triggered because of fluctuations in the overall demand for x and y. This leads to a revaluation of agents previous possessions, $x_{i,t-1}$ and $y_{i,t-1}$, and works like a capital gain or loss.

To summarize, we have a model in which all agents are identical except for their random preference shocks and no market or whatsoever power is attributed to anyone. The resulting inequality (illustrated in Fig. 1 for a typical simulation run) is, therefore, the mere consequence of the eventualities of the history of preference changes and ensuing exchanges of goods. We, therefore, do not have to impose any type of "power" in order to endogeneously generate a stratification of the wealth distribution that - like the model of sec. 2 - is able to capture all except the very end (the Pareto tail) of the empirical data.

We may note that our fits of the Gamma distribution are in very good harmony with heuristic arguments provided in Silver et al. according to which we shoult expect a Gamma $(2, \frac{2}{\omega})$ for monetary wealth with ω the units of a monetary numeraire of which a total of ωN units exists.

4 Some Extensions of the Monetary Exchange Model

The model by Silver et al. demonstrates that stratification of wealth can result from an innocuous exchange dynamics without agents robbing or fleecing each other. It should, therefore, be a promising avenue for the simpler dynamic models *à* la Angle's "inequality process". In some extensions, we, therefore, tried to explore the sensitivity of this approach to certain changes of its underlying assumptions. Among the many sensitivity tests we could imagine, we started with the following variations of the basic framework:

- replacement of market interaction by pairwise exchange,
- introduction of agents with higher bargaining power so that the outcome of pairwise matches could differ from a competitive framework,
- introduction of natural differences among agents of some kind: here we assumed that for part of the population, preference changes are less pronounced than for others,
- introduction of savings via a framework which allows for money as an additional component in the utility function.

Due to space limitations, we will not provide detailed results on all these experiments, but will rather confine ourselves to one particularly intersting variant: the introduction of market power.

The first variation consisted in replacing the aggregate market by bi-lateral exchanges: in each period, we randomly matched $\frac{N}{2}$ pairs of 2 agents, and determined the price for their bilateral exchange as the equilibrium price in a competetive market populated by only those two traders. Since in the aggregate market, the preference shocks over all the population will be averaged out to a large extent, we could assume that prices determined in this way might have higher volatility so that more extreme changes of wealth would occur. For example, an agent with a currently high preference for good x could meet another agent with a similar preference which would lead to a high relative price of x in their exchange. In the aggregate market setting, by contrast, each of these agents would face the overall market demand and supply as its "trading partner" which at the average over a large population would show only slight fluctuations. Wealth of each agent as specified in eq. (5) was computed by replacing the formerly unique market price by the weighted average of pairwise transaction prices with relative transaction volumes serving as the weights. The idea to be investigated would be wether this new set of eventualities would give a somewhat different direction to the process of wealth formation.

As it turns out, the exchange model is entirely robust with respect to this modification. Since the emergent wealth distribution practically collapses with that of the baseline case (shown in Fig. 1) we abstain from a detailed presentation of these results. However, we kept the setting of parwise encounters for the next set of experiments.

Introducing market power of some sort is certainly interesting in light of the focus of the sociological and physics-inspired literature on issues of power of some individuals over others. Different avenues for implementing market power seem possible. Here, for the sake of a first exploration of this issue, we chose a very simple and extreme one. We assume that part of the population can act as monopolists in pairwise encounters: if they are matched with an agent from the complementary subset of non-monopolists, they can demand the monopoly price. If two non-monopolists are matched, we compute the competitive solution. We do the same when two monopolists meet each other assuming that their potential monopolic power cancels out. It might be useful to present the monopoly scenario in more detail. What the monopolist, say agent i, does is maximizing his utility:

$$\begin{aligned} U_{i,t} &= x_{i,t}^{f_{i,t}} y_{i,t}^{1-f_{i,1}} \\ &= (x_{i,t-1} + x_{j,t-1} - x_{j,t})^{f_{i,t}} (y_{i,t-1} + y_{j,t-1} - y_{j,t})^{1-f_{i,t}} \end{aligned} \tag{6}$$

subject to the demand of his counterpart, agent j, which enters as a constraint in the maximization program of agent i.

Although this is an almost trivial insight in economics, it should be noted that the monopolist is not entirely free in dictating any price/transaction combination, but has to observe the constraint that the other agent has to voluntarily participate in the transaction. Since the option to not agree on the transaction would leave the monopolist with a zero gain as well, even in this extreme market scenario "exploitation" is much more limited than in a world of "theft and fraud" economics. Note also that although one could perhaps speak of exploitation (when comparing the monopoly setting with the competitive price), no *expropriation* is involved whatsoever since even the non-monopolist will still increase his utility by his transaction with the more "powerful" monopolist.

Since the choice variable of the monopolist is the price, we have to solve for this variable, say $p_{m,t}$, in order to determine the outcome of the encounter.

The reader might check that this monopoly price is the second root of the somewhat unhandy equation:

$$\begin{aligned}&-f_{i,t}f_{j,t}y_{j,t-1}\left(f_{j,t}y_{j,t-1}+y_{i,t-1}\right)p_{m,t}^{2}\\&+\left(2f_{i,t}-1\right)f_{j,t}\left(1-f_{j,t}\right)y_{j,t-1}x_{j,t-1}p_{m,t}\\&+\left(1-f_{i,t}\right)\left(1-f_{j,t}\right)x_{j,t-1}\left(x_{i,t-1}+\left(1-f_{j,t-1}\right)x_{j,t-1}\right)=0\end{aligned} \quad (7)$$

As it turns out, allowing few monopoly power indeed changes the resulting wealth distribution. Figs. 2 shows the pdf for (fixed) fractions[2] of monopolists. Varying the proportion of monopolists from 0 (the former competitive scenario with pair-wise transactions) to 0.4 we see a slight change in the shape of the distribution. As it happens all distributions still show pronounced exponential decline and can be well fitted by Gamma distributions. However, the estimated parameters of the Gamma distribution show a systematic variation. In particular, the slope parameter decreases with the fraction of monopolists, P_{mo} (despite the apparently different tendency at $P_{mo} = 0.1$. A closer look at the simulation results also shows that the average wealth of monopolists exceeds that of other agents but the difference decreases with increasing P_{mo}, indicating an increase in inequality with more agents being able to exert monopoly power over others. Note that the Gini dispersion ratio (G) is a negative function of λ for the Gamma distribution: $G = \frac{\Gamma(\alpha+0.5)}{\pi^2 \Gamma(\alpha+1)}$, so that the increasing inequality would also be indicated by this popular statistics.

Fig. 2: Kernel estimates of statistical wealth distributions with different fractions of monopolistic agents P_m. Results are from simulations with 10,000 agents recorded after $5 * 10_5$ trading results.

The result that monopoly power is not neutral with respect to the destribution of wealth is certainly reassuring. However, we may also note that its introduction in the present framework does not lead to a dramatic change of the shape of the distribution. In particular, it does not seem to lead to anything like a Pareto tail in place of the exponential tail of the more competitive society. Since we have already chosen the most extreme form of market power in the above setting it seems also unlikely that one could obtain widely different results with milder forms of bargaining power. [3]

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[2] We have also tried a model with assignment of monopoly power to randomly chosen agents in each period. However, this case was once more indistinguishable in its results from the original model.

[3] One could, for example use a standard bargaining approach in which the outcome of pairwise meetings is determined via the maximisation of:

$$V = (\triangle U_i)^{\alpha} (\triangle U_j)^{1-\alpha}$$

with $\triangle U_i$ and $\triangle U_j$ the attainable utility improvement of both agents and α symbolizing the bargaining power of i vis-à-vis j.

It might be interesting and constructive to contrast with the above results another scenario in which agents differ by nature. To this end, we assume that for some agent's preference parameters are drawn from a smaller set. This implies that their preferred x-y bundle is less volatile than that of an agent whose preference parameter can assume any value between 0 and 1. All simulations along this line showed that this scenario leads to much more dramatic results than monopoly power. In all cases, a bi-modal distribution of wealth emerged in which the agents with restricted set of preferences occupied the higher mode (cf. Fg. 4 and 5). Obviously, the unrestricted agents suffer from their sometimes more drastic preference changes, and would therefore move often agree to abnormally high prices for their preferred good. Due to this disadvantage they would on average loose wealth to the members of the new group. Again, however, this seeming exploitation is perfectly voluntary and the dynamic development is only caused by the unfortunate difference in preferences which works against one group.
One might also note that the shift of wealth towards the more fortunate agents does not continue without limit. As can be seen from Fig. 4, emergent distributions seem to converge to a stationary state so that bi-modality characterizes the statistical equilibrium. Note also that in the above simulations, we have assumed a competitive scenario, but additional experiments suggest that additionally introducing bargaining power is not a sufficiently strong mechanism to alter the tendency towards bi-modality.

5 Conclusions and Outlook to Future Research

What kind of conclusions can be drawn from this review of different approaches to agent-based models of wealth stratification? First, it is perhaps obvious that this author would like to advocate an approach in line with standard principles of economic modelling. If one is not willing to follow the emphasis of the sociological literature on all types of exertion of power, and if one tends to the view that wealth is influenced more by legal economic activity than by illegal theft and fraud, economic exchange should be explicitely incorporated in such models. This would also help to identify more clearly the sources of the changes of wealth. Note that despite the voluntary participation of agents in the exchange economy and the utility-improving nature of each trade, a change in the distribution of wealth comes with it. The difference to earlier models is that the changes in wealth are explained by deeper, underlying economic forces while they are simply introduced as such in the models reviewed in sec. 3. Exchange models also allow to consider changes of monetary evaluation of goods and assets as a potentially important source of changes in individual's nominal wealth.

Unfortunately, monetary exchange so far does not provide an explanation of the power-law characterizing the far end of the distribution. As we have shown above, even an unequal distribution of market power within the population seems not sufficient to replicate this important empirical feature. Following recent proposals in the literature one could try additional positive feedback effects that give agents with an already high level of wealth an additional advantage (West, 2005; Sinha, 2005).

In the above model, one could argue that the more wealthy agents would also acquire more bargaining power together with their higher rank in the wealth hierarchy. Whether this would help to explain the outer region, remains to be analyzed. However, there are perhaps reasons to doubt that the Pareto feature might be the mere result of clever bargaining. A glance at the Forbes list of richest individuals[4] (analyzed statistically by Levy and Solomon, 1997, and Milakovic, 2005) reveals that the upper end of the distribution is not populated by smart dealers who in a myriad of small deals succeeded to outwit their counterparts. Rather, it is the founders and inheritants of industrial dynasties and successful companies operating in new branches of economic activity. The conjecture based on this anecdotical evidence would be that the upper end of the spectrum has its roots in risky innovative investments. Few of these succeed but the owners behind the succeeding ones receive an overwhelming reward. This would suggest that models without savings and investments should lack a mechanism for a power law tail. One would, therefore, have to go beyond such conservative models and combine their exchange mechanism (which works well for the greater part of the distribution) with an economically plausible process for the emergence of very big fortunes.

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[4] While the majority of entrants in the Forbes list might fall into that category, a few are, in fact, rather suggestive of "theft and fraud" avenues to big fortunes.

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