

More Aspects of Polya Tree Distributions for Statistical Modelling

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Abstract: The definition and elementary properties of Polya tree distributions are reviewed. Two theorems are presented showing that Polya trees can be constructed to concentrate arbitrarily closely about any desired pdf, and that Polya tree priors can put positive mass in every relative entropy neighborhood of every positive density with finite entropy, thereby satisfying a consistency condition. Such theorems are false for Dirichlet processes. Models are constructed combining partially specified Polya trees with other information like monotonicity or unimodality. It is shown how to compute bounds on posterior expectations over the class of all priors with the given specifications. A numerical example is given. A theorem of Diaconis and Freedman about Dirichlet processes is generalized to Polya trees, allowing Polya trees to be the models for errors in regression problems. Finally, empirical Bayes models using Dirichlet processes are generalized to Polya trees. An example from Berry and Christensen is reanalyzed with a Polya tree model.

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1. Introduction

Polya trees form a class of distributions for a random probability measure \mathcal{P} intermediate between Dirichlet processes (Ferguson 1973) and tailfree processes (Freedman 1963 and Fabius 1964). Their advantage over Dirichlet processes is that they can be constructed to give probability one to the set of continuous or absolutely continuous probability measures, while their advantage over more general tailfree processes is their much greater tractability. The basic ideas of Polya trees can be found in Ferguson (1974), Mauldin, Sudderth and Williams (1992) and Lavine (1992). The rest of the introduction reviews the definitions and elementary properties of Polya trees. Section 2 contains two theorems showing the suitability of Polya trees for statistical modelling. Section 3 combines robust and nonparametric Bayes ideas by showing the feasibility of modelling with partially specified Polya trees and incorporating other information such as shape constraints. A numerical example is given. Section 4 proposes models in which Polya trees are used to represent the errors in regression settings. The main result is a generalization of a theorem of Diaconis and Freedman (1986). Finally, Section 5 addresses the empirical Bayes problem, using Polya trees in place of Dirichlet processes. See Antoniak (1974), Berry and Christensen (1979), Escobar (1990), Escobar and West (1990), Ferguson (1983), Kuo (1986), Lo (1984), and West (1990) for the use of Dirichlet processes. Section 5 contains a reanalysis using Polya trees of a problem from Berry and Christensen (1979).

Let $E = \{0, 1\}$, $E^0 = \emptyset$, E^m be the m -fold product $E \times E \times \dots \times E$, $E^* = \bigcup_0^\infty E^m$, and E^N be the set of infinite sequences of elements of E . Let Ω be a separable measurable space, $\pi_0 = \{\Omega\}$ and $\Pi = \{\pi_m; m = 0, 1, \dots\}$ be a separating binary tree of partitions of Ω ; that is, let π_0, π_1, \dots be a sequence of partitions such that $\bigcup_0^\infty \pi_m$ generates the measurable sets and such that every $B \in \pi_{m+1}$ is obtained by splitting some $B' \in \pi_m$ into two pieces. Let $B_\emptyset = \Omega$ and, for all $\epsilon = \epsilon_1 \dots \epsilon_m \in E^*$, let B_{ϵ_0} and B_{ϵ_1} be the two pieces into which B_ϵ is split. Degenerate splits are permitted; e.g., $B_\epsilon = B_{\epsilon_0} \cup \emptyset$. For every $\theta \in \Omega$, let $\epsilon^m(\theta)$ be the element of E^m such that $\theta \in B_{\epsilon^m(\theta)}$. Note therefore that $\theta = \bigcap B_{\epsilon^m(\theta)}$.

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Definition 1. A random probability measure \mathcal{P} on Ω is said to have a Polya tree distribution, or a Polya tree prior, with parameter (Π, \mathcal{A}) , written $\mathcal{P} \sim \text{PT}(\Pi, \mathcal{A})$, if there exist nonnegative numbers $\mathcal{A} = \{\alpha_\epsilon : \epsilon \in E^*\}$ and random variables $\mathcal{Y} = \{Y_\epsilon : \epsilon \in E^*\}$ such that

1. all the random variables in \mathcal{Y} are independent,
2. for every $\epsilon \in E^*$, Y_ϵ has a Beta distribution with parameters α_{ϵ_0} and α_{ϵ_1} and
3. for every $m = 1, 2, \dots$ and every $\epsilon \in E^m$,

$$\mathcal{P}(B_{\epsilon_1 \dots \epsilon_m}) = \left(\prod_{j=1; \epsilon_j=0}^m Y_{\epsilon_1 \dots \epsilon_{j-1}} \right) \left(\prod_{j=1; \epsilon_j=1}^m (1 - Y_{\epsilon_1 \dots \epsilon_{j-1}}) \right),$$

where the first term in the products above is interpreted as Y_\emptyset or $1 - Y_\emptyset$.

Degenerate Beta distributions are permitted; e.g., $\alpha_{\epsilon_0} = 0$ making the distribution degenerate at 0. The random variables $\Theta_1, \Theta_2, \dots$ are said to be a sample from \mathcal{P} if, given \mathcal{P} , they are i.i.d. with distribution \mathcal{P} . The Y_ϵ 's have the following interpretation: Y_\emptyset and $1 - Y_\emptyset$ are respectively the probabilities that $\Theta_i \in B_0$ and $\Theta_i \in B_1$, and Y_ϵ and $1 - Y_\epsilon$ are the conditional probabilities that $\Theta_i \in B_{\epsilon_0}$ and $\Theta_i \in B_{\epsilon_1}$ given that $\Theta_i \in B_\epsilon$.

Three important facts about Polya trees are:

- (1) They are conjugate. If \mathcal{P} has a Polya tree distribution and $\Theta | \mathcal{P} \sim \mathcal{P}$ then $\mathcal{P} | \Theta$ has a Polya tree distribution (Ferguson 1974, Mauldin Sudderth and Williams 1992). Updating a Polya tree after observing Θ_i is simple; for every ϵ such that $\Theta_i \in B_\epsilon$, add 1 to α_ϵ . We call the new parameters $\mathcal{A} | \Theta$. Sometimes we will not have observed Θ_i exactly but will only know that Θ_i belongs to some set. If that set happens to be B_δ for some $\delta \in E^*$ then again the updating follows the same rule. The difference is that when Θ_i is observed exactly there are infinitely many α_ϵ 's to update; when we see $\Theta_i \in B_\epsilon$, there are only finitely many.
- (2) Dirichlet processes are special cases of Polya trees. A Polya tree is a Dirichlet process if, for every $\epsilon \in E^*$, $\alpha_\epsilon = \alpha_{\epsilon_0} + \alpha_{\epsilon_1}$ (Ferguson 1974). The parameter of the Dirichlet process is $\alpha = mG_0$ where $m = \alpha_\emptyset$ and G_0 is determined by $G_0(B_\epsilon) = \mathbb{E}[\mathcal{P}(B_\epsilon)] = \Pr[\Theta_i \in B_\epsilon]$.
- (3) Some Polya trees assign probability 1 to the set of continuous distributions, for example when $\alpha_{\epsilon_1, \dots, \epsilon_m} = m^2$ as in Example 2 of Section 5.

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2. Suitability of Polya Trees for Statistical Modelling

One occasionally wants to construct a nonparametric prior centered at and concentrated near a given distribution. Dalal and Hall (1980) show that Dirichlet processes can be so constructed if “near” is interpreted in the sense of weak convergence. Lavine (1992) gives a theorem showing that the cdf of \mathcal{P} , with a Polya tree distribution, can be made uniformly close to a given cdf, with arbitrarily high probability. The next theorem says that using Polya trees the pdf of \mathcal{P} can be made close to a given pdf q . Such a theorem is impossible for Dirichlet processes because, under a Dirichlet process, \mathcal{P} does not have a pdf; whereas, Kraft (1964) and Metivier (1971) show that a Polya tree can be constructed so that \mathcal{P} is almost surely absolutely continuous with respect to Lebesgue measure. See also Ferguson (1974) and Lavine (1992). Note that if degenerate Beta distributions are allowed then Theorem 1 is true trivially: take the Polya tree to be degenerate at Q . The point of Theorem 1 is that the construction with nondegenerate Beta distributions still has full support in the set of all probability measures. Such constructions may be useful as priors in Bayesian analyses. Let p be the random density of \mathcal{P} . Let ess sup denote the essential supremum, the supremum except perhaps on a null set.

Theorem 1. *For a given probability measure Q with density q , any positive number k and any $\eta \in (0, 1)$ there exists a Polya tree distribution for \mathcal{P} such that \mathcal{P} has a density p satisfying $\Pr[\text{ess sup}_\theta |\log(p(\theta)/q(\theta))| < k] > \eta$.*

Proof. Construct the Polya tree so that $E[\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]] = Q(B_{\epsilon^m(\theta)})/Q(B_{\epsilon^{m-1}(\theta)})$.

Then,

$$\begin{aligned} \Pr[\text{ess sup}_\theta \left| \log \frac{p(\theta)}{q(\theta)} \right| < k] &= \Pr[\text{ess sup}_\theta \left| \sum_{m=1}^{\infty} \log \frac{\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]}{E[\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]]} \right| < k] \\ &\geq \Pr[\text{ess sup}_\theta \sum_{m=1}^{\infty} \left| \log \frac{\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]}{E[\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]]} \right| < k] \\ &\geq \Pr[\sum_{m=1}^{\infty} \text{ess sup}_\theta \left| \log \frac{\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]}{E[\mathcal{P}[B_{\epsilon^m(\theta)}|B_{\epsilon^{m-1}(\theta)}]]} \right| < k] \\ &= \Pr[\sum_{m=0}^{\infty} \sup_{\epsilon \in E^m} \left(\left| \log \frac{Y_\epsilon}{E[Y_\epsilon]} \right| \vee \left| \log \frac{1 - Y_\epsilon}{E[1 - Y_\epsilon]} \right| \right) < k]. \end{aligned}$$

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There are only finitely many elements in E^m so, by choosing the α_ϵ 's sufficiently large each summand can be made arbitrarily small with arbitrarily large probability and the probability that the sum is less than k can be made greater than η . ■

For distributions F_* and F with densities f_* and f , let $D(F_*, F)$ denote the relative entropy or Kullback-Leibler divergence $\int \log(f_*(\theta)/f(\theta)) f_*(\theta) d\theta$ and let $N_\delta(F_*) = \{F : D(F_*, F) < \delta\}$ denote a relative entropy neighborhood. Barron (1986) remarks that if a prior μ satisfies $\mu(N_\delta(F_*)) > 0$ for all $\delta > 0$, then μ satisfies a consistency condition at F_* . It may therefore be desirable for a prior to put positive mass in every relative entropy neighborhood of a wide class of probability measures to ensure consistency over that class. The next theorem says that Polya trees can be constructed with positive mass in every relative entropy neighborhood of every distribution with an essentially positive density. Such a theorem is impossible for Dirichlet processes because they give probability one to the set of discrete distributions.

Theorem 2. *A Polya tree prior μ can be constructed so that, for any distribution F_* with density f_* and having finite entropy, $\mu(N_\delta(F_*)) > 0$ for all $\delta > 0$.*

Proof. We suppose that the μ to be constructed below almost surely has a density. Let δ be given and let λ denote Lebesgue measure.

$$\begin{aligned}
 D(F_*, F) &= \int \log\left(\lim_{m \rightarrow \infty} \left(\frac{F_*(B_{\epsilon^m}(\theta))/\lambda(B_{\epsilon^m}(\theta))}{F(B_{\epsilon^m}(\theta))/\lambda(B_{\epsilon^m}(\theta))}\right)\right) f_*(\theta) d\theta \\
 &= \int \log\left(\lim_{m \rightarrow \infty} \left(\prod_{j=1}^m \frac{F_*(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))/\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}{F(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))/\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}\right)\right) f_*(\theta) d\theta \\
 &= \int \sum_1^\infty \log\left(\frac{F_*(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}{\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}\right) f_*(\theta) d\theta \\
 &\quad - \int \sum_1^\infty \log\left(\frac{F(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}{\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}\right) f_*(\theta) d\theta. \tag{1}
 \end{aligned}$$

The first equality above is implicit in either Theorem 35.8 of Billingsley (1986) or Eq. 1 of Lavine (1992). The first integral of the RHS of Eq. 1 converges by assumption. The second integral is

$$\int_{B_0} \sum_1^\infty \log\left(\frac{F(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}{\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}\right) f_*(\theta) d\theta + \int_{B_1} \sum_1^\infty \log\left(\frac{F(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}{\lambda(B_{\epsilon^j}(\theta)|B_{\epsilon^{j-1}}(\theta))}\right) f_*(\theta) d\theta =$$

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$$\begin{aligned}
& F_*(B_0) \log\left(\frac{F(B_0)}{\lambda(B_0)}\right) + \int_{B_0} \sum_2^\infty \log\left(\frac{F(B_{\epsilon^j(\theta)}|B_{\epsilon^{j-1}(\theta)})}{\lambda(B_{\epsilon^j(\theta)}|B_{\epsilon^{j-1}(\theta)})}\right) f_*(\theta) d\theta \\
& + F_*(B_1) \log\left(\frac{F(B_1)}{\lambda(B_1)}\right) + \int_{B_1} \sum_2^\infty \log\left(\frac{F(B_{\epsilon^j(\theta)}|B_{\epsilon^{j-1}(\theta)})}{\lambda(B_{\epsilon^j(\theta)}|B_{\epsilon^{j-1}(\theta)})}\right) f_*(\theta) d\theta.
\end{aligned}$$

Similarly, by dividing the integral over B_0 into the sum of two integrals over B_{00} and B_{01} and likewise for the integral over B_1 , and by continuing to subdivide integrals indefinitely, the previous expression becomes

$$\begin{aligned}
& \sum_{m=0}^\infty \sum_{\epsilon \in E^m} F_*(B_{\epsilon 0}) \log\left(\frac{F(B_{\epsilon 0}|B_\epsilon)}{\lambda(B_{\epsilon 0}|B_\epsilon)}\right) + F_*(B_{\epsilon 1}) \log\left(\frac{F(B_{\epsilon 1}|B_\epsilon)}{\lambda(B_{\epsilon 1}|B_\epsilon)}\right) \\
& = \sum_{m=0}^\infty \sum_{\epsilon \in E^m} F_*(B_{\epsilon 0}) \log\left(\frac{Y_\epsilon}{\lambda(B_{\epsilon 0}|B_\epsilon)}\right) + F_*(B_{\epsilon 1}) \log\left(\frac{1-Y_\epsilon}{\lambda(B_{\epsilon 1}|B_\epsilon)}\right) \\
& \leq \sum_{m=0}^\infty \max_{\epsilon \in E^m} \left(\log\left(\frac{Y_\epsilon}{\lambda(B_{\epsilon 0}|B_\epsilon)}\right) \vee \log\left(\frac{1-Y_\epsilon}{\lambda(B_{\epsilon 1}|B_\epsilon)}\right) \right). \tag{2}
\end{aligned}$$

The Polya tree parameter \mathcal{A} can be chosen so that $\alpha_{\epsilon 0}/(\alpha_{\epsilon 0} + \alpha_{\epsilon 1}) \equiv \mathbb{E}[Y_\epsilon] = \lambda(B_{\epsilon 0}|B_\epsilon)$ and with the α_ϵ 's increasing sufficiently rapidly with m so that the sum in (2) converges with positive probability. But if the sum in (2) converges then the tail sum, $\sum_M^\infty \dots$ must be less than $\delta/2$ for some M . Thus,

$$0 < \Pr[(2) \text{ converges}] \leq \Pr\left[\sum_1^\infty \dots < \delta/2\right] + \Pr\left[\sum_2^\infty \dots < \delta/2\right] + \dots$$

which implies the existence of an M such that $\Pr[\sum_M^\infty \dots < \delta/2] > 0$. Finally, because the Y_ϵ 's have Beta distributions and hence have full support, there is positive probability that the sum $\sum_1^{M-1} \dots$ differs from the first integral of Eq. 1 by no more than $\delta/2$. Therefore, there is positive probability that Eq. 1 is less than δ . ■

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3. Partially Specified Polya Trees

It may appear initially that calculations with Polya tree models are impossible to perform exactly because of the need to update infinitely many parameters. That this is not necessarily so is demonstrated in Lavine (1992). Nonetheless, calculations and computer programs may be simplified if the Polya trees are updated only as far as a predetermined level. This section presents two scenarios under which it is sensible to stop updating below a given predetermined level, and under which the error of approximation may be either estimated or bounded.

To give probability one to the set of absolutely continuous distributions, to model other beliefs in smoothness, or to satisfy Theorems 1 and 2, Polya trees may be constructed so that the α 's increase rapidly toward the bottom of the tree. A sample of n observations then cannot affect the tree very strongly below the level at which the α 's become large relative to n . To achieve, therefore, a specified accuracy in the computation of the predictive distribution, it is only necessary to update finitely many levels of the Polya tree. For a specific example, suppose that for $\epsilon \in E^{m-1}$, $\alpha_{\epsilon 0} = \alpha_{\epsilon 1} \geq m^2$, and that a sample of size n has been observed. The updated predictive density g at a point θ can be bounded above by

$$\begin{aligned} \log g(\theta|\Theta_1, \dots, \Theta_n) &\leq \log g(\theta) + \sum_{m=1}^{\infty} \log \frac{2(\alpha_{\epsilon^m(\theta)} + n)}{2\alpha_{\epsilon^m(\theta)} + n} \\ &\leq \log g(\theta) + \sum_{m=1}^{\infty} \log\left(1 + \frac{n}{2\alpha_{\epsilon^m(\theta)}}\right) \\ &\leq \log g(\theta) + \frac{n}{2} \sum_{m=1}^{\infty} \frac{1}{\alpha_{\epsilon^m(\theta)}} \\ &\leq \log g(\theta) + \frac{n}{2} \sum_{m=1}^{\infty} \frac{1}{m^2} \end{aligned}$$

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and below by

$$\begin{aligned}
 \log g(\theta|\Theta_1, \dots, \Theta_n) &\geq \log g(\theta) + \sum_{m=1}^{\infty} \log \frac{2(\alpha_{\epsilon^m(\theta)})}{2\alpha_{\epsilon^m(\theta)} + n} \\
 &\geq \log g(\theta) + \sum_{m=1}^{\infty} \log\left(1 - \frac{n}{2\alpha_{\epsilon^m(\theta)}}\right) \\
 &\geq \log g(\theta) - \frac{n}{2} \sum_{m=1}^{\infty} \frac{1}{\alpha_{\epsilon^m(\theta)}} \\
 &\geq \log g(\theta) - \frac{n}{2} \sum_{m=1}^{\infty} \frac{1}{m^2}.
 \end{aligned}$$

Therefore, $g(\theta|\Theta_1, \dots, \Theta_n)$ can be evaluated to within a factor of δ by updating the Polya tree as far as level M , where $\log \delta \geq (n/2) \sum_M^{\infty} m^{-2}$.

Another argument for considering only finitely many levels of a Polya tree arises out of robust Bayesian considerations. It is unreasonable to expect an elicitee to specify a single Polya tree, or even a mixture of Polya trees, that is the only reasonable representation of prior beliefs. But it might be possible for the elicitee to specify finitely many parameters near the top of a tree, possibly along with other beliefs, such as shape constraints, about \mathcal{P} . There will then be a class Γ of prior distributions consistent with the prior information, and one can search for upper and lower bounds on prior and posterior quantities of interest, over Γ .

Let S be a finite subset of E^* such that for any $\epsilon = \epsilon_1 \cdots \epsilon_m$ in S , the initial sequence $\epsilon_1 \cdots \epsilon_j$ for each $j < m$ is also in S and suppose that the elicitee is willing to specify the parameters $\{B_{\epsilon_0}, B_{\epsilon_1}, \alpha_{\epsilon_0}, \alpha_{\epsilon_1} : \epsilon \in S\}$. Let $T_1 = \{\mathcal{P}(B_{\epsilon_0}), \mathcal{P}(B_{\epsilon_1}) : \epsilon \in S\}$ be the random probabilities assigned by the partially specified Polya tree and let T_2 be the mass distribution of \mathcal{P} conditional on T_1 . Thus, $\mathcal{P} = (T_1, T_2)$ and $\mathcal{L}(\mathcal{P}) = \mathcal{L}(T_1) \times \mathcal{L}(T_2|T_1)$.

Definition 2. *The random variable T_1 is said to have a finite Polya tree distribution with parameter $(\mathcal{B}^S, \mathcal{A}^S)$, written $T_1 \sim \text{PT}(\mathcal{B}^S, \mathcal{A}^S)$, if there exist sets $\mathcal{B}^S = \{B_{\epsilon_0}, B_{\epsilon_1} : \epsilon \in S\}$, numbers $\mathcal{A}^S = \{\alpha_{\epsilon_0}, \alpha_{\epsilon_1} : \epsilon \in S\}$ and random variables $\mathcal{Y}^S = \{Y_{\epsilon} : \epsilon \in S\}$ such that*

1. *all the random variables in \mathcal{Y}^S are independent,*
2. *for every $\epsilon \in S$, Y_{ϵ} has a Beta distribution with parameters α_{ϵ_0} and α_{ϵ_1} and*
3. *for every $\epsilon = \epsilon_1 \cdots \epsilon_m \in S$,*

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$$\mathcal{P}(B_{\epsilon_1 \dots \epsilon_m 0}) = \left(\prod_{j=1; \epsilon_j=0}^{m+1} Y_{\epsilon_1 \dots \epsilon_{j-1}} \right) \left(\prod_{j=1; \epsilon_j=1}^{m+1} (1 - Y_{\epsilon_1 \dots \epsilon_{j-1}}) \right).$$

Mauldin Sudderth and Williams (1992) define finite Polya trees using a special case of the preceding definition. We suppose that the elicitee will be able to give a marginal prior for T_1 , but not a conditional prior for T_2 given T_1 . We suppose also that the elicitee will be able to name a set \mathbf{P} , a subset of the set of all possible distributions which will receive mass one under any reasonable prior. For example, \mathbf{P} may be the set of all unimodal distributions. Let E be the event $(T_1, T_2) \in \mathbf{P}$ and let ν be a measure on Ω . The class of priors Γ is the class of all priors, conditioned on E , such that $T_1 \sim \text{PT}(\mathcal{B}^S, \mathcal{A}^S)$, such that $\Pr[\mathcal{P} \ll \nu] = \Pr[d\mathcal{P}/d\nu \text{ is a.e. positive}] = 1$ and such that $\mathcal{L}(T_2|T_1)$ is otherwise arbitrary. That is, probabilities of the form $\Pr[\mathbf{P} \cap \{\mathcal{P} = (T_1, T_2) : T_1 = t\}]$ are governed by the finite Polya tree.

It is the unconditional distribution of T_1 that has the Polya tree distribution. Conditional on E , the distribution of T_1 might not be Polya tree. For example, let $\Omega = (0, 1]$, $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ where $B_i = ((i-1)/4, i/4]$ and \mathbf{P} be the set of unimodal distributions. The distribution of T_1 given E cannot be Polya tree because the event $[\mathcal{P}(B_1) > \mathcal{P}(B_2); \mathcal{P}(B_4) > \mathcal{P}(B_3)]$ has positive probability under any Polya tree but probability 0 conditional on E .

We want bounds on posterior expectations of interesting functions of \mathcal{P} . Two useful examples are the posterior predictive mean, $\mathbb{E}[f \theta d\mathcal{P}]$ and the posterior predictive probability of a set, $\mathbb{E}[\mathcal{P}(C)]$. We concentrate on finding the upper bound, as finding the lower bound is similar.

The following theorem says that the class of posteriors, Γ_Θ , has the same form as the class of priors. Therefore finding bounds on posterior expectations can be reduced to the problem of finding bounds on prior expectations.

Theorem 3. *Let ν be a measure on Ω , $\pi = \{B_1, \dots, B_k\}$ be a partition of Ω , $T_1 = (\mathcal{P}(B_1), \dots, \mathcal{P}(B_k))$, $T_2 = (\mathcal{P}|T_1)$, the set of priors Γ be the set of all conditional distributions for $\mathcal{P} = (T_1, T_2)$ given E where $\mathcal{L}(T_1) = \text{PT}(\mathcal{B}^S, \mathcal{A}^S)$ is specified, $\Pr[\mathcal{P} \ll \nu] =$*

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$\Pr[d\mathcal{P}/d\nu$ is a.e. positive] = 1 and where $\mathcal{L}(T_2|T_1)$ is otherwise arbitrary. Let Θ be an observation from \mathcal{P} and J be defined by $\Theta \in B_J$. Then the corresponding set of posteriors Γ_Θ is the set of all conditional distributions for (T_1, T_2) given E where $\mathcal{L}(T_1|\Theta) = \text{PT}(\mathcal{B}^S, \mathcal{A}^S|\Theta \in J)$, $\Pr[\mathcal{P} \ll \nu] = \Pr[d\mathcal{P}/d\nu$ is everywhere positive] = 1 and where $\mathcal{L}(T_2|T_1)$ is otherwise arbitrary.

Proof. We prove Theorem 3 without restriction to E , then note that the set of posteriors corresponding to priors restricted to E can be obtained by restricting to E the set of posteriors of unrestricted priors. Note first that J is sufficient for T_1 , so $\mathcal{L}(T_1|\Theta) = \text{PT}(\mathcal{B}^S, \mathcal{A}^S|\Theta \in J)$ follows easily. Now we want to show that any probability measure Q for $\mathcal{L}(T_2|T_1)$ that is absolutely continuous with an almost everywhere positive density is the posterior $\mathcal{L}(T_2|T_1, \Theta)$ for some prior Q_0 . But let Q_0 be proportional to $Q/(d\mathcal{P}/d\nu(\Theta))$ and the result follows. ■

Remark 1. The class Γ changes into the class Γ_Θ only by updating the distribution for T_1 , a finite set of probabilities and a more easily understood object than \mathcal{P} .

Remark 2. Although Theorem 3 is proven for general distributions for T_1 , we intend to use finite Polya tree distributions for T_1 , because they are conjugate and easily updated.

Remark 3. Given T_1 , the classes Γ and Γ_Θ are ordinary quantile classes, possibly with restrictions, so that known results for quantile classes may apply.

Example 1. Let $\Omega = [0, 4)$. We present a class of finite Polya tree priors with shape constraints to model an opinion that the law of the data has a nonincreasing density that is approximately exponential with parameter 1, and show how to bound expectations over that class. Let $S = \{\emptyset, 1, 11\}$. Define $B_0 = [0, \ln 4/3)$, $B_{10} = [\ln 4/3, \ln 2)$ and $B_{110} = [\ln 2, \ln 4)$, so that Ω is the disjoint union $B_0 \cup B_{10} \cup B_{110} \cup B_{111}$. Let $\alpha_0 = 20$, $\alpha_1 = 60$, $\alpha_{10} = 80/3$, $\alpha_{11} = 160/3$ and $\alpha_{110} = \alpha_{111} = 40$. The opinion that Θ_1 has roughly an exponential (1) distribution is modelled as $Y = \mathcal{P}(B_0) \sim \text{Beta}(20, 60)$, $Y_1 = \mathcal{P}(B_{10})/\mathcal{P}(B_1) \sim \text{Beta}(80/3, 160/3)$ and $Y_{11} = \mathcal{P}(B_{110})/\mathcal{P}(B_{11}) \sim \text{Beta}(40, 40)$. Figure 1 shows the densities of Y , Y_1 and Y_{11} .

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For any $Q \in \Gamma$ the expectation of Θ is $\int \int E[\Theta|T_1, T_2] dQ(T_2|T_1) dQ(T_1)$, which is maximized over Γ by determining $Q(T_2|T_1)$ separately for each value of T_1 to maximize $E[\Theta]$ subject to monotonicity. The upper and lower bounds can be estimated by:

- Step 1 Generate a sample t_1, \dots, t_N from the known distribution of T_1 ,
- Step 2 For each t_j in the sample, let $q_{ji} = p_{ji}[\phi_i, \phi_{i+1}) / (\phi_{i+1} - \phi_i)$, for $i \in \{0, \dots, 3\}$ be the average density assigned by t_j to the interval $[\phi_i, \phi_{i+1})$,
- Step 3 For each j , a nonincreasing density consistent with t_j exists if and only if $q_{j0} \geq \dots \geq q_{j3}$.
If these inequalities do not hold then drop the j th point from the sample. Let N^* be the number of points remaining.
- Step 4 Let u_j and ℓ_j be the maximum and minimum expected values of $\Theta|t_j$ respectively,
- Step 5 Let $\hat{u} = \sum u_j / N^*$ and $\hat{\ell} = \sum \ell_j / N^*$ be the estimates.

For Step 4 it is easy to see that \hat{u} is given by the density that is uniform and equal to q_{ji} over the interval $[\phi_i, \phi_{i+1})$. The minimum, $\hat{\ell}$, can be found by an optimization problem in a small number of variables. See Berger and O'Hagan (1988) for details of a similar optimization problem. O'Hagan and Berger (1988) point out that the minimum can be bounded below using a density that is a step function with at most two values in each interval; the value of the bounding density in the interval $[\phi_i, \phi_{i+1})$ is $q_{j\ i-1}$ on the left side of the interval and $q_{j\ i+1}$ on the right side of the interval.

A sample of size $N = 1000$ was generated, of which $N^* = 818$ were acceptable according to Step 3. The estimates are $\hat{u} = 1.09$ and $\hat{\ell} = .80$. Figures 2 and 3 show how closely the model mimics the $\exp(1)$ distribution. Figure 2 shows boxplots of all 4000 of the q_{ji} 's grouped by interval. The solid diamonds are .25 divided by the interval length — a typical value for q_{ji} according to the prior. Figure 3 shows boxplots of points on the 818 good cdf's. The solid diamonds are the values from the $\exp(1)$ cdf. ■

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4. Errors in Regression

This section describes the posterior distribution when Polya trees are used to model the errors in regression settings. The main result is a generalization of the following theorem which is implicit in Lemma 2.1 and Remark 1 of Diaconis and Freedman (1986, p71).

Theorem DF. *Let $Y_i = \theta + \epsilon_i$ where the ϵ_i are independent with unknown distribution \mathcal{P} . With respect to the prior Q , let θ and \mathcal{P} be independent, θ having density f and \mathcal{P} being Dirichlet with parameter measure α which is absolutely continuous; let $g = \alpha'/\|\alpha\|$, where $\|\alpha\|$ is the mass of α . The posterior Q_n can be characterized as follows:*

$$Q_n(d\theta) = C_n^{-1} f(\theta) \prod^* g(Y_i - \theta) d\theta,$$

$$Q_n\{d\mathcal{P}|\theta\} \text{ is } \mathcal{D}\left(\alpha + \sum_{i=1}^n \delta_{Y_i - \theta}\right),$$

where

$$C_n = \int_{-\infty}^{\infty} f(\theta) \prod^* g(Y_i - \theta) d\theta,$$

and where \prod^* is the product over distinct values in the sample.

The first generalization is to drop the requirement that θ and \mathcal{P} be independent with respect to the prior. Note that in the setting of Theorem DF, after a single observation Y_1 , θ and \mathcal{P} are no longer independent. The second generalization is from a single sample to a regression model: $\theta_i = r(X_i, \beta)$ where r is a known regression function, X_i is a known vector of covariates and β is an unknown vector of parameters, with prior density f . Finally, the third generalization is from Dirichlet processes to Polya trees: $\mathcal{P}|\beta \sim \text{PT}(\Pi_\beta, \mathcal{A}_\beta)$.

Theorem 4. *Let $Y_i = r(X_i, \beta) + \epsilon_i$ where the ϵ_i are independent with unknown distribution \mathcal{P} . With respect to the prior Q , let β have density f and $\mathcal{P}|\beta \sim \text{PT}(\Pi_\beta, \mathcal{A}_\beta)$. The posterior Q_n can be characterized as follows:*

$$Q_n(d\beta) = C_n^{-1} f(\beta) \prod_{i=1}^n q(Y_i|\beta, Y_1, \dots, Y_{i-1}) d\beta,$$

$$Q_n\{d\mathcal{P}|\beta\} \text{ is } \text{PT}(\Pi_\beta, \mathcal{A}_\beta | Y_1 - r(X_1, \beta), \dots, Y_n - r(X_n, \beta)),$$

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where

$$C_n = \int_B f(\beta) \prod_{i=1}^n q(Y_i|\beta, Y_1, \dots, Y_{i-1}) d\beta$$

and where $q(Y_i|\beta, Y_1, \dots, Y_{i-1})$ is the density of Y_i given $\beta, Y_1, \dots, Y_{i-1}$.

Proof. The theorem follows by applying Bayes Theorem and the definition of conditional distribution. ■

One reason for stating the theorem, apart from theoretical interest, is to point out that posterior densities for β and for Y_f , a future observation with known covariate X_f , are computable. It is only necessary to evaluate $q(Y_i|\beta, Y_1, \dots, Y_{i-1})$, which is explained in Lavine (1992).

5. Empirical Bayes Problems

The following empirical Bayes model was first stated by Antoniak (1974): \mathcal{P} , an unknown probability measure, has a prior that is a mixture of Dirichlet processes; conditional on \mathcal{P} , $\Theta_1, \dots, \Theta_n$ is a sample of size n from \mathcal{P} ; for each $i \in \{1, \dots, n\}$, conditional on $\mathcal{P}, \Theta_1, \dots, \Theta_n$, X_i is a sample of size 1 from F_{Θ_i} , independent of \mathcal{P} and $\{X_j, \Theta_j : j \neq i\}$. When $\vec{X} = (X_1, \dots, X_n)$ is observed, but not $\vec{\Theta} = (\Theta_1, \dots, \Theta_n)$, we may wish to estimate or find the posterior distributions of $\mathcal{P}, \Theta_i, \Theta_{n+1}$, or X_{n+1} .

The problem was further studied by Berry and Christensen (1979) in the case where F_{Θ_i} is the binomial distribution with parameter Θ_i , by Ferguson (1983) in the case where F_{Θ_i} is the normal distribution with parameter Θ_i , and by Lo (1984) in the general case. Kuo (1986) proposes a Monte Carlo method for computing the estimates. More recently, the model has appeared (Escobar 1990, West 1990, Escobar and West 1990) with computations done by means of the Gibbs sampler (Gelfand and Smith 1990), an algorithm for drawing an approximate sample from a multivariate distribution when the conditional distribution of each variable given all the others can be sampled. Here we generalize the model so that \mathcal{P} has a Polya tree prior, show how posteriors can be computed via the Gibbs sampler, and discuss advantages provided by mixtures of Polya trees over mixtures of Dirichlet processes.

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To generate a sample from the posterior distribution of $(\vec{\Theta})$ via the Gibbs sampler it is required to generate an observation from the distribution of, say, $(\Theta_n | \vec{X}, \Theta_1, \dots, \Theta_{n-1})$, which is the same as the distribution of $(\Theta_n | X_n, \Theta_1, \dots, \Theta_{n-1})$. But, as described in Lavine (1992), the density of $(\Theta_n | \Theta_1, \dots, \Theta_{n-1})$ is a piecewise rescaled version of the prior density of Θ_n , so the density of $(\Theta_n | X_n, \Theta_1, \dots, \Theta_{n-1})$ is a piecewise rescaled version of the density of $(\Theta_n | X_n)$. Therefore, if the density of $(\Theta_n | X_n)$ is available for sampling then so is the density of $(\Theta_n | X_n, \Theta_1, \dots, \Theta_{n-1})$, at least to arbitrary accuracy. The sampling algorithm is particularly easy if only finitely many of the Polya tree parameters need be updated.

Algorithmically, the difference between a Dirichlet process prior and a Polya tree prior is that for the Dirichlet process the distribution of $(\Theta_n | X_n, \Theta_1, \dots, \Theta_{n-1})$ is a mixture of the distribution of $(\Theta_n | X_n)$ and the degenerate distributions $\delta_{\Theta_1}, \dots, \delta_{\Theta_{n-1}}$. Sampling from this mixture would typically be just as easy as the sampling required for the Polya tree model. But the Dirichlet process model imposes features that may be deemed undesirable. For example, when \mathcal{P} has a Dirichlet process prior then with positive probability some of the Θ_i 's are equal (Ferguson 1973), and the law of the pattern of multiplicities has a specific form (Antoniak 1974). In contrast, the Polya tree prior can be constructed so that $\Theta_1, \dots, \Theta_n$ are distinct with probability one.

Example 2. An example from Berry and Christensen (1979) uses data from Martz and Lian (1974), who are quoted as saying “The Portsmouth Naval Shipyard, Portsmouth, N.H. routinely must assess the quality of submitted lots of vendor produced material. The following data consist of the number of defects x_i of a specified type in samples of size $n = 5$ from past lots of welding material. The past data are $(0, 1, 0, 0, 5)$ and in the current, i.e., sixth, lot, $x = 0$.”

Berry and Christensen (1979) use a model in which x_1, \dots, x_6 are observations from binomial distributions with $n = 5$ and success probabilities $\Theta_1, \dots, \Theta_6$. The Θ 's in turn are a sample from \mathcal{P} which has a Dirichlet process prior with base measure Beta(1,1). They estimate $\hat{\Theta}_1 = \hat{\Theta}_3 = \hat{\Theta}_4 = \hat{\Theta}_6 = .1143$, $\hat{\Theta}_2 = .2178$ and $\hat{\Theta}_5 = .7795$, the points shown as

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diamonds in Figure 4.

The data were reanalyzed with a similar model, but with \mathcal{P} having a Polya tree prior where π_m is the collection of intervals $[i/2^m, (i+1)/2^m)$ for $i = 0, \dots, 2^m - 1$, and where $\alpha_{\epsilon_1 \dots \epsilon_{m-1} 0} = \alpha_{\epsilon_1 \dots \epsilon_{m-1} 1} = m^2$ to ensure the absolute continuity of \mathcal{P} . A sample of size 1000 from the posterior distribution of $\Theta_1, \dots, \Theta_6$ was obtained via the Gibbs sampler. Figure 4 shows univariate posterior density estimates. Figure 5 indicates the bivariate posterior distributions with plots of the sampled points. By symmetry, $\theta_1, \theta_3, \theta_4$ and θ_6 have the same marginal posterior. However, because the Polya tree prior puts its mass on continuous \mathcal{P} , $\theta_1, \theta_3, \theta_4$ and θ_6 are distinct with probability 1. ■

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Figure 1. Densities of Y , Y_1 and Y_{11} .

Figure 2. Boxplots of the q_{ji} from Example 1. Typical values according to the prior are indicated by diamonds.

Figure 3. Boxplots of points on the random cdf's from Example 1. Values from the $\exp(1)$ distribution are indicated by diamonds.

Figure 4. Density estimates for the Θ_i 's from Example 2. The left curve is for Θ_1 , Θ_3 , Θ_4 and Θ_6 , the middle curve is for Θ_2 , and the right curve is for Θ_5 .

Figure 5. Pairwise plots of the Gibbs sampled points from Example 2.