Actuarial Calculations using a Markov Model

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1994

Abstract

This paper develops a general approach to actuarial calculations in applications which can be modeled as multi-state processes. Such situations arise when benefits are payable upon a change in the status of the insured, or while the insured maintains a given status. Examples include life insurance, annuities, pensions, disability income insurance and certain types of long-term care insurance.

The method is based on convenient matrix results which are available when we assume a continuous-time Markov model with constant forces of transition. In this case probabilities are easily obtained regardless of the number of states. This is of considerable benefit as it allows us to deal with very complicated actuarial problems.

Section 1 provides some background on the kinds of problems for which the approach is suitable. The basic properties of the Markov process are presented in Section 2. We consider some useful results which hold under the assumption of constant forces of transition, and explain how the results can be used in the case of piecewise constant forces. In Section 3, we address the situation in which the Markov assumption is inappropriate. Here we find that, rather than using a more general semi-Markov model, we can reflect duration dependence by increasing the number of states in the model. This is justified by a limiting result and demonstrated by an example which applies the approach to select and ultimate mortality.

1 INTRODUCTION

1.1 Background

Some of the traditional problems dealt with in actuarial mathematics are conveniently viewed in terms of multi-state processes. We assume that, at any point in time, an individual is in one of a number of states. The individual's presence in a given state, or movement (transition) from one state to another, may have some financial impact. Our task then is to quantify this impact, usually by estimating the expected value of future cash flows.

The simplest situation involves only two states: "alive" and "dead." As shown in Figure 1, an individual may make only one transition. For a simple life annuity, benefits are payable while the annuitant is in state 1, and cease upon transition to state 2. In the case of a whole life insurance policy, premiums are payable while the insured is in state 1, and the death benefit is paid at the time of transition to state 2. Approaches to calculating actuarial values in these cases are simple and well known (see Bowers et al [1]).

A more complicated situation arises when we have additional states. Figure 2 illustrates the three-state process commonly used to describe the state of an individual insured under a disability income policy. In this case, premiums are payable while the insured is in state

Figure 1: **Example of Two-State Model**

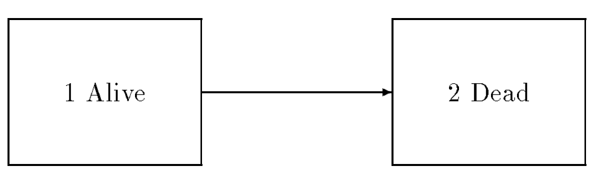
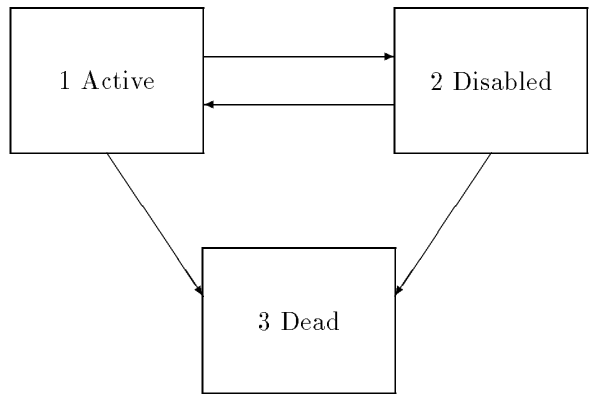


Figure 2: **Example of Three-State Model**



1, and benefits are payable while the insured is in state 2 (usually after a waiting period). Actuarial calculations for this example are more difficult since the individual can make repeat visits to each of states 1 and 2. For this reason it is often assumed that transitions from state 2 to state 1 are not possible.

There are numerous other areas in which a multi-state model provides an intuitively pleasing description of the possible outcomes. In examining a long-term care system, we may represent the several levels of care available as states of a multi-state model. Ongoing costs could then be associated with each state. We could use a multi-state process in a life insurance context to describe the movement of individuals amongst various risk categories such as smoking status and blood pressure grouping. This was discussed by Tolley and Manton [23]. Pension plans can also be modeled within a multi-state framework. In the simplest case, we would require states for working plan members, retirees, and those who have died. A more complicated model might require a disabled state and three retired states reflecting the status of a joint and last survivor annuity.

Many authors have used multi-state models to analyze actuarial problems. Much of this work has drawn on the theory of stochastic processes to obtain new results of interest and to generalize results found using more traditional methods. Such models are most tractable when it is assumed that the process satisfies the Markov property. Under this assumption, Hoem [5, 7] generalized a number of standard results from life contingencies. Wolthuis and van Hoek [27] considered the expected value and variance of the loss function in a Markov model setting. The stochastic properties of the profit earned on an insurance policy were examined by Ramlau-Hansen [17, 18], who also analyzed the distribution of surplus in [19]. Tolley and Manton [23] proposed models for morbidity and mortality which include various risk factors in the model state space. In modeling the mortality of individuals infected with the HIV virus, Panjer [12] and Ramsay [16] used a Markov process with states which represent the stages of infection. Waters [24] discussed the development of formulas for probabilities and the estimation of parameters in a Markov model. The use of more general stochastic models has been considered by Hoem [6], Hoem and Aalen [8], Ramsay [15], Seal [21] and Waters [25, 26].

1.2 Actuarial Calculations

Premiums and reserves for insurance and annuity contracts which are long term in nature are usually based on the present value of payments to be made under the contract (both premiums and benefits). Typically, the occurrence, timing and/or amount of each payment is not known exactly in advance; it depends on some random outcome. Thus, it is customary to calculate the expectation of this present value. Such expected present values are easily obtained in the two-state case described earlier since the random outcome can be represented as a single random variable, $T(x)$, measuring the time until death of an individual presently age x. For example, the expected present value of a continuous annuity paying 1 per annum for the remaining lifetime of an individual aged x is

$$E\left[\bar{a}_{\overline{T(x)|}}\right] = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu_{x+t}dt = \int_0^\infty v^t\,{}_tp_x dt.$$

The expected present value of 1 payable upon the death of an individual aged x is

$$E\left[v^{T(x)}\right] = \int_0^\infty v^t\,{}_tp_x\mu_{x+t}dt.$$

In the three-state case shown in Figure 2, the expected present value calculation is more difficult. Suppose we seek the expected present value of 1 payable continuously while in state 2 to an individual presently age x in state 1. This may be written

$$\int_0^\infty v^t p_{12}(x, x+t)dt,$$

where $p_{ij}(x, x+t)$ is the probability that an individual presently age x in state i will be in state j at age $x+t$. Unfortunately, this probability is not easy to obtain since it must allow for the possibility that the individual returns to state 1 one or more times between ages x and $x+t$. If we make the simplifying assumption that transitions from state 2 to state 1 cannot occur, then

$$p_{12}(x, x+t) = \int_0^t e^{-\int_0^s (\mu^{12}_{x+u}+\mu^{13}_{x+u})du}\mu^{12}_{x+s}e^{-\int_s^t \mu^{23}_{x+u}du}ds,$$

where μ_y^{ij} is the force of transition from state i to state j at age y. The integrand in this expression may be interpreted as the probability that an individual in state 1 at age x moves to state 2 between age $x+s$ and $x+s+ds$, and remains in state 2 until age $x+t$. This "no recovery" assumption is often made in analyzing long term disability insurance. However, for coverages in which transitions occur more frequently, such an assumption is inappropriate. We therefore need a general method of finding the probabilities $p_{ij}(t, x+t)$ in multi-state models with three or more states.

Keyfitz and Rogers [9] provided a method determining transition probabilities under a Markov process. The approach was developed assuming that forces of transition are constant within age intervals of a fixed length. Transition probability matrices can then be calculated recursively for time periods which are multiples of this age interval. Our method leads to expressions for the transition probabilities which are more convenient for certain types of calculations. We also provide a strategy for dealing with duration dependence, an issue which is not considered by Keyfitz and Rogers.

1.3 Outline of Paper

The main purpose of this paper is to present a method of finding probabilities needed for actuarial calculations in applications which can be represented as multi-state processes. Much of the research cited earlier focuses on finding theoretical results under various multi-state model assumptions. This paper provides techniques for calculating numerical results which are easily applied to a wide variety of problems. The approach is suitable for situations involving an arbitrary, but finite number of states. Thus, it may be a useful tool in analyzing complicated actuarial problems such as those presented by disability income insurance and long-term care.

Section 2 begins with a brief review of the properties of the Markov process. We then present a key result which exploits the mathematical tractability of Markov processes with constant forces of transition. We find that a decomposition of the force of transition matrix leads to a convenient representation of the transition probability matrix. The latter is expressed explicitly in terms of the time interval of interest. Furthermore, the probabilites are linear combinations of exponential functions. Therefore, the integration needed to compute expected sojourn times in the various states as well as actuarial values can be carried out analytically. We demonstrate that this is also true when the forces of transition are piecewise constant. Thus, our approach avoids the numerical integration required by the method of Keyfitz and Rogers [9].

The idea of "duration dependence" is discussed in Section 3. Frequently we encounter applications in which the forces of transition should depend on the time since entry to the current state. Such a process is called "semi-Markov." Unfortunately, the probabilities we seek are not easily obtained using a semi-Markov model. We find, however, that by creating additional states we can construct a Markov model which approximates the semi-Markov model. This approximation is justified by a limiting result which illustrates the convergence of the approximating Markov process to the semi-Markov process. We also provide a numerical example which allows for the duration dependence involved with select and ultimate mortality by including a third state. The two "alive" states are interpreted as "select" and "ultimate." This three-state Markov model with piecewise constant forces of transition yields probabilities which are very close to those obtained directly from the mortality table upon which parameter values were based.

Section 4 closes the paper with a brief summary.

2 THE MARKOV PROCESS

2.1 Basic Properties

As discussed in the previous section, we shall consider actuarial problems in which the cash flows depend on the outcome of a multi-state process. To begin, let $X(t)$ represent the state of an individual at time (age) $t \geq 0$. We then denote the stochastic process by $\{X(t), t \geq 0\}$. It will be assumed that there are a finite number of states labeled $1, 2, \ldots, k$. That is, the process has state space $\{1, 2, \ldots, k\}$. Now, as defined by Ross [20, ch. 5], $\{X(t), t \geq 0\}$ is a Markov process if, for all $s, t \geq 0$ and $i, j, x(u) \in \{1, 2, \ldots, k\}$,

$$\begin{aligned} & \Pr\{X(s+t) = j | X(s) = i, X(u) = x(u), 0 \leq u < s\} \\ = & \Pr\{X(s+t) = j | X(s) = i\}. \end{aligned}$$

Thus, the future of the process (after time s) depends only on the state at time s and not on the history of the process up to time s.

The reasonableness of the Markov assumption depends somewhat on the level of detail in the state description. For example, consider the three-state process shown earlier in Figure 2. In this case, the Markov assumption may be inappropriate. The future health of a recently disabled individual is likely to differ from that of someone of the same age who has been disabled for a long period of time. This will be discussed further in Section 3.

We define the transition probability function

$$p_{ij}(s, s+t) \equiv \Pr\{X(s+t) = j | X(s) = i\},\ i, j \in \{1, 2, \ldots, k\},$$

and assume that

$$\sum_{j=1}^{k} p_{ij}(s, s+t) = 1 \text{ for all } t \geq 0.$$

We also assume the existence of the limits

$$\mu_{ij}(t) = \lim_{h \to 0^+} \frac{p_{ij}(t, t+h) - \delta_{ij}}{h},\ i, j \in \{1, 2, \ldots, k\},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

If $i \neq j$, $\mu_{ij}(t)$ represents the force of transition from state i to state j. It is easily seen that, for $s, t, u \geq 0$,

$$p_{ij}(s, s+t+u) = \sum_{l=1}^{k} p_{il}(s, s+t) p_{lj}(s+t, s+t+u),\ i, j \in \{1, 2, \ldots, k\}. \tag{1}$$

These are known as the Chapman-Kolmogorov equations.

The transition probability function $p_{ij}(s, s+t)$ corresponds to the probability ${}_t p_s^{ij}$ introduced in Section 1. As discussed, these probabilities are needed in the calculation of actuarial values. The forces of transition and the transition probability functions are related by the Kolmogorov forward and backward equations. These are

$$\frac{\partial}{\partial t} p_{ij}(s, s+t) = \sum_{l=1}^{k} p_{il}(s, s+t)\mu_{lj}(s+t), \tag{2}$$

and

$$\frac{\partial}{\partial s} p_{ij}(s, s+t) = -\sum_{l=1}^{k} \mu_{il}(s) p_{lj}(s, s+t), \tag{3}$$

respectively, with boundary conditions $p_{ij}(s, s) = \delta_{ij}$. In general, these systems of differential equations must be solved numerically to obtain the transition probability functions—a very tedious task.

2.2 Constant Forces of Transition

Explicit expressions for the transition probability functions are available when we assume that $\mu_{ij}(t) = \mu_{ij}$ for all t. Such a Markov process is referred to as time-homogeneous or stationary. The assumption of constant forces of transition implies that the time spent in each state is exponentially distributed . Also, the functions $p_{ij}(s, s+t)$ are the same for all $s \geq 0$, and may therefore be written $p_{ij}(t)$.

It is convenient to express the forces of transition and transition probability functions in matrix form. Let Q be the $k \times k$ matrix with (i, j) entry μ_{ij} and $P(t)$ be the $k \times k$ matrix with (i, j) entry $p_{ij}(t)$. Corresponding to (1), the Chapman-Kolmogorov equations are given by

$$P(t+u) = P(t)P(u). \tag{4}$$

Also, corresponding to (2) and (3), the Kolmogorov differential equations may be written

$$P'(t) = P(t)Q \tag{5}$$

and

$$P'(t) = QP(t), \tag{6}$$

with boundary condition $P(0) = I$. Equations (5) and (6) have the solution

$$\begin{aligned} P(t) &= e^{Qt} \\ &= I + Qt + \frac{Q^2t^2}{2!} + \cdots \end{aligned}$$

This is of limited use since the series may converge rather slowly. However, as noted by Cox and Miller [3], if Q has distinct eigenvalues, $d_1, d_2, \ldots, d_k$, then $Q = ADC$ where $C = A^{-1}$, $D = \mathrm{diag}(d_1, \ldots, d_k)$ and the ith column of A is the right-eigenvector associated with d_i. Furthermore,

$$P(t) = A\,\mathrm{diag}(e^{d_1 t}, \ldots, e^{d_k t})\,C. \tag{7}$$

Therefore, the problem of finding the transition probability functions is reduced to a problem of determining the eigenvalues and eigenvectors of the force of transition matrix Q. Software to perform this task is readily available. The requirement that Q have distinct eigenvalues imposes no practical restriction. In the situations we consider, this will be the case for almost all parameter values.

We can illustrate this in the case of the three-state model shown in Figure 2. We have

$$Q = \begin{bmatrix} -(\mu_{12}+\mu_{13}) & \mu_{12} & \mu_{13} \\ \mu_{21} & -(\mu_{21}+\mu_{23}) & \mu_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of Q are the solutions of

$$d[d^2 + (\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23})d + \mu_{12}\mu_{23} + \mu_{13}\mu_{21} + \mu_{13}\mu_{23}] = 0.$$

Clearly, 0 is an eigenvalue. Neither of the other two eigenvalues can be 0 since this would require at least one force of mortality to be 0. Thus, if the three eigenvalues are not distinct, the quadratic in square brackets must have only one root. That is,

$$(\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23})^2 - 4(\mu_{12}\mu_{23} + \mu_{13}\mu_{21} + \mu_{13}\mu_{23}) = 0.$$

This implies that, for any choice of three parameter values, the fourth must satisfy a quadratic equation. Therefore, at most two values of the fourth parameter will result in eigenvalues which are not distinct. It is then quite unlikely that the parameter estimates will result in non-distinct eigenvalues. In the event that this occurs, a slight change in one parameter will eliminate the problem.

If, in dealing with more complicated models, distinct eigenvalues cannot be achieved, an analogous decomposition to Jordan canonical form is possible (see Cox and Miller [3]).

From (7), we may now write

$$p_{ij}(t) = \sum_{n=1}^{k} a_{in} c_{nj} e^{d_n t}, \tag{8}$$

where a_{ij} and c_{ij} are the (i,j) entries of A and C respectively. Thus, we can express the transition probability functions explicitly as simple functions of t.

Note that in the two-state case shown in Figure 1, we have

$$Q = \begin{bmatrix} -\mu_{12} & \mu_{12} \\ 0 & 0 \end{bmatrix}.$$

Then $d_1 = -\mu_{12}$ and $d_2 = 0$. Corresponding eigenvectors are $(1,0)'$ and $(1,1)'$. Thus,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$C = A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

From (7), we then have

$$\begin{aligned} P(t) &= A\text{diag}(e^{d_1 t}, e^{d_2 t})C \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-\mu_{12} t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\mu_{12} t} & 1 - e^{-\mu_{12} t} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence, as expected, $p_{11}(t) = e^{-\mu_{12} t}$, $p_{12}(t) = 1 - e^{-\mu_{12} t}$, $p_{22}(t) = 1$ and $p_{21}(t) = 0$.

2.3 Piecewise Constant Forces

We assumed above that forces of transition were constant with respect to time. This permits convenient representation of the transition probability functions. Unfortunately, in many actuarial applications, this is impractical. We require forces which vary with age. In the two-state example shown in Figure 1, we clearly need a force of mortality which varies with the age of the individual. We can accomplish this, while preserving the tractability of constant forces, by using force of transition functions which are piecewise constant. We may wish to use forces of transition which vary with each year of age. In some instances, though, it will be reasonable to use broader age groups.

Let $\mu_{ij}(t) = \mu_{ij}^{(m)}$ if $t \in [t_{m-1}, t_m)$, for $m = 1, 2, \ldots$, where $t_0 = 0$. Also, let $p_{ij}^{(m)}(t)$ be the transition probability function associated with time intervals $[u, u+t)$ contained in $[t_{m-1}, t_m)$. In matrix form, we have $Q^{(m)}$ and $P^{(m)}(t)$. Now define m_t to be the integer such that $t_{m_t-1} \leq t < t_{m_t}$. Then from (4), we have

$$P(s,t) = P^{(m_s)}(t_{m_s} - s)P^{(m_s+1)}(t_{m_s+1} - t_{m_s}) \cdots P^{(m_t)}(t - t_{m_t-1}). \tag{9}$$

Thus, given s and t, the transition probability matrix can be computed. We first determine $A^{(m)}$, $D^{(m)}$ and $C^{(m)} = (A^{(m)})^{-1}$ from $Q^{(m)}$ for each m, as described in Subsection 2.2. $P^{(m)}(t)$ is then obtained using (7). Finally, $P(s,t)$ can be found using (9).

As mentioned earlier, an advantage of the approach described in this paper is that transition probability functions are expressed in a very convenient form. In order to obtain quantities such as the expected time spent in a given state or the expected present value of payments made continuously while in a given state, the required integration can be performed analytically.

To illustrate this, consider a Markov process with forces of transition which are constant within each year of age. Let $\mu_{ij}^{(x)}$ be the force of transition from state i to state j for an individual between age x and age $x+1$ where x is a non-negative integer and $i, j \in \{1, 2, \ldots, k\}$. These forces of transition can be used to construct the matrix $Q^{(x)}$, from which we can determine $A^{(x)}$, $C^{(x)}$ and $D^{(x)}$. Let $p_{ij}^{(x)}(t)$ be the i to j transition probability function associated with the age interval from x to $x+1$. Suppose we wish to determine the expected time spent in state j between ages 30 and 40 by an individual in state i at age 30. This quantity is given by

$$\begin{aligned}
\int_{30}^{40} p_{ij}(30,t)dt &= \sum_{x=30}^{39} \int_{x}^{x+1} p_{ij}(30,t)dt \\
&= \sum_{x=30}^{39} \int_{x}^{x+1} \sum_{h=1}^{k} p_{ih}(30,x) p_{hj}^{(x)}(t-x)dt \\
&\quad \text{from (1)} \\
&= \sum_{x=30}^{39} \sum_{h=1}^{k} p_{ih}(30,x) \int_{x}^{x+1} \sum_{n=1}^{k} a_{hn}^{(x)} c_{nj}^{(x)} e^{d_n^{(x)}(t-x)} dt \\
&\quad \text{from (8)} \\
&= \sum_{x=30}^{39} \sum_{h=1}^{k} p_{ih}(30,x) \sum_{n=1}^{k} a_{hn}^{(x)} c_{nj}^{(x)} \int_{x}^{x+1} e^{d_n^{(x)}(t-x)} dt \\
&= \sum_{x=30}^{39} \sum_{h=1}^{k} p_{ih}(30,x) \sum_{n=1}^{k} a_{hn}^{(x)} c_{nj}^{(x)} \frac{e^{d_n^{(x)}} - 1}{d_n^{(x)}}, \qquad (10)
\end{aligned}$$

where $p_{ih}(30,x)$ is the (i,h) entry of

$$P(30,x) = \prod_{y=30}^{x-1} P^{(y)}(1)$$

$$= \prod_{y=30}^{x-1} A^{(y)} \mathrm{diag}(e^{d_1^{(y)}}, \ldots, e^{d_k^{(y)}}) C^{(y)},$$

$a_{hn}^{(x)}$ is the (h,n) entry of $A^{(x)}$, $c_{nj}^{(x)}$ is the (n,j) entry of $C^{(x)}$ and $d_n^{(x)}$ is the (n,n) entry of $D^{(x)}$. In matrix form, (10) can be written as

$$\int_{30}^{40} P(30,t)dt = \sum_{x=30}^{39} P(30,x) A^{(x)} \mathrm{diag}\left(\frac{e^{d_1^{(x)}} - 1}{d_1^{(x)}}, \ldots, \frac{e^{d_k^{(x)}} - 1}{d_k^{(x)}} \right) C^{(x)}.$$

We also find that the expected present value of 1 payable continuously while in state j between ages 30 and 40 to an individual now in state i at age 30 is

$$\int_{30}^{40} e^{-\delta(t-30)} p_{ij}(30,t)dt = \sum_{x=30}^{39} e^{-\delta(x-30)} \sum_{h=1}^{k} p_{ih}(30,x) \sum_{n=1}^{k} a_{hn}^{(x)} c_{nj}^{(x)} \frac{e^{d_n^{(x)} - \delta} - 1}{d_n^{(x)} - \delta},$$

where δ is the force of interest. In matrix form,

$$\int_{30}^{40} e^{-\delta(t-30)} P(30,t)dt = \sum_{x=30}^{39} e^{-\delta(x-30)} P(30,x) A^{(x)} \mathrm{diag}\left(\frac{e^{d_1^{(x)} - \delta} - 1}{d_1^{(x)} - \delta}, \ldots, \frac{e^{d_k^{(x)} - \delta} - 1}{d_k^{(x)} - \delta} \right) C^{(x)}.$$

3 DURATION DEPENDENCE

3.1 Failure of the Markov Assumption

In the previous section we mentioned that, for some actuarial applications of multi-state models, the Markov assumption is unsuitable. We cited the three-state disability case as a situation in which the probability of transition out of the disabled state may be influenced by the time since disablement as well as the age of the individual. Another example in which duration dependence arises is select mortality. In life insurance, it is generally assumed that mortality rates depend on the time since the individual became insured in addition to the individual's attained age. Lapse rates are also heavily dependent on the duration of the insurance policy.

A stochastic model in which the future of the process depends on the time since transition to the current state is referred to as semi-Markov. Hoem [6] discussed a number of demographic and actuarial applications of semi-Markov models. Some aspects of the use of such models in sickness insurance were considered by Seal [21], Ramsay [15] and Waters [25, 26].

3.2 The General Semi-Markov Model

We can describe the general semi-Markov model in terms of the forces of transition. Let $\mu_{ij}(t, u)$ be the force of transition from state i to state j at time (age) t for an individual who has been in state i for a period of time u. We then require a more complicated definition of the transition probability functions, also involving the time since entry to state i. Hoem [6] defined these functions and pointed out a number of useful relationships.

Unfortunately, such a complicated stochastic model does not lead to convenient expressions for the probabilities needed to obtain actuarial values. As this is the objective of the present paper, we must seek some simplification. Seal [21] achieved such simplification in modeling the time spent in sickness of young and middle aged individuals by restricting the model to two states. Since mortality rates are low at these ages, Seal assumed that mortality transitions could be ignored. It was further assumed that the forces of transition to and from the sickness state are independent of attained age, and hence, depend only on the time since entry to the current state. The resulting stochastic process is called an alternating renewal process. The same model was used by Ramsay [15].

We wish to deal with more general multi-state situations, and therefore require some other form of simplification of the model. We suggest an approach which allows us to use the results for the Markov process discussed in the previous section. In particular, we propose the acceptance of a more complicated state space in exchange for the simpler Markov process.

3.3 Approximation by a Markov Model

An alternative approach to reflecting the duration dependence often present in actuarial applications is to treat each state as a collection of one or more substates. We then assume that future transitions are independent of the time of entry to the current substate. For example, we might assume that the "insured" state in a life insurance situation consists of two substates: "select" and "ultimate." This was suggested by Norberg [11] as a way of explaining select mortality. Norberg showed that, if

1. only select lives may enter the insured state
2. select lives may move to the ultimate state
3. ultimate lives may not return to the select state
4. the force of mortality for select lives is less than that for ultimate lives

then, for a fixed attained age, the force of mortality increases with duration since becoming insured. Møller [10] showed that the result also holds if assumption 3 is relaxed, and explored the selection effect using more than one ultimate state.

The same approach could be used in the disability insurance model illustrated in Figure 2. Here we must allow the force of transition from the disabled state to depend on the time since disablement. To accomplish this, we can represent the disabled state by two substates. These might be interpreted as "unstable" and "stable." The unstable state would be the state entered upon disablement and would have fairly high forces of transition both to the active state and to the dead state. An individual could also move from unstable to stable, a state with lower forces of recovery and mortality. If necessary, the disabled state could consist of more than two substates.

The approximation of a semi-Markov process by a Markov process was discussed by Cox and Miller [3]. In particular, the method of stages allows one to approximate any distribution by a combination of stages in series or parallel, where the time spent in each stage is exponentially distributed. Below we take such an approach in developing a limiting result which justifies the approximation in the context of select and ultimate mortality. The result can be generalized to more complicated processes.

Let $\mu(t)$, $t \geq 0$ be a bounded continuous function representing the force of mortality for a given age group, where t is the time since policy issue. Also, let $S(t) = e^{-\int_0^t \mu(s)ds}$ be the corresponding survival function. We can construct a time-homogeneous Markov process with a survival function which converges to $S(t)$ as the rate of transition from each state approaches infinity. To begin, suppose that an individual moves through a number of states labeled $1, 2, \ldots$ in sequence until death occurs. Let the total force of transition from each state (mortality and movement to next state) be given by λ at each point in time. Also, let the force of mortality in state i be $\mu(i/\lambda)$. Then we must have $\lambda > \mu(i/\lambda)$ for all i. Furthermore, $\lambda - \mu(i/\lambda)$ is the force of transition from state i to $i+1$. Note that $\mu(i/\lambda)$ equals the true force of mortality at a duration equal to the expected time of transition from state i.

Since the total force of transition from each state is λ, the Markov process can be thought of as a Poisson process where, at each event time, the individual either dies or moves to the next state. Let $N(t)$ represent the number of events in $(0, t]$ and $I(t)$ be 1 if the individual is alive at duration t and 0 otherwise. Then

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

Also,

$$\Pr\{I(t) = 1 | N(t) = n\} = \begin{cases} 1 & n = 0 \\ \prod_{i=1}^{n} \frac{\lambda - \mu(i/\lambda)}{\lambda} & n = 1, 2, \ldots \end{cases}.$$

Therefore,

$$\Pr\{I(t) = 1, N(t) = n\} = \begin{cases} e^{-\lambda t} & n = 0 \\ \frac{(\lambda t)^n e^{-\lambda t}}{n!} \prod_{i=1}^{n} \frac{\lambda - \mu(i/\lambda)}{\lambda} & n = 1, 2, \ldots \end{cases}$$

and

$$\Pr\{I(t) = 1\} = e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \prod_{i=1}^{n} \frac{\lambda - \mu(i/\lambda)}{\lambda}. \tag{11}$$

We now note that

$$\begin{aligned}
\log\left\{\prod_{i=1}^{n}\frac{\lambda-\mu(i/\lambda)}{\lambda}\right\} &= \sum_{i=1}^{n}\log\left\{\frac{\lambda-\mu(i/\lambda)}{\lambda}\right\} \\
&= -\sum_{i=1}^{n}\mu(i/\lambda)/\lambda+o(1/\lambda) \\
&= -\int_0^{n/\lambda}\mu(s)ds+o(1/\lambda) \\
&\quad \text{by the definition of a Riemann integral} \\
&= \log\{S(n/\lambda)\}+o(1/\lambda)
\end{aligned}$$

It follows that

$$\begin{aligned}
\prod_{i=1}^{n}\frac{\lambda-\mu(i/\lambda)}{\lambda} &= e^{\log\{S(n/\lambda)\}+o(1/\lambda)} \\
&= S(n/\lambda)e^{o(1/\lambda)} \\
&= S(n/\lambda)\{1+o(1/\lambda)\} \\
&= S(n/\lambda)+o(1/\lambda). \qquad (12)
\end{aligned}$$

This is a special case of a more general result given by Pólya and Szegö [14, p. 47]. Now since $\sum_{n=0}^{\infty}\frac{(\lambda t)^n e^{-\lambda t}}{n!}$ is bounded for all λ, from (11) and (12) we have

$$\begin{aligned}
\lim_{\lambda\to\infty}\Pr\{I(t)=1\} &= \lim_{\lambda\to\infty}\sum_{n=0}^{\infty}\frac{(\lambda t)^n e^{-\lambda t}}{n!}S(n/\lambda) \\
&= \lim_{\lambda\to\infty}E[S(N(t)/\lambda)].
\end{aligned}$$

Now consider the sequence $\{\lambda_r; r=1,2,\ldots\}$ where $\lambda_r = r/t$. For a given r, $N(t)/(\lambda_r t)$ can be viewed as the sample mean of r independent Poisson random variables with mean 1. By the weak law of large numbers it follows that $N(t)/(\lambda t) \to 1$ in probability as

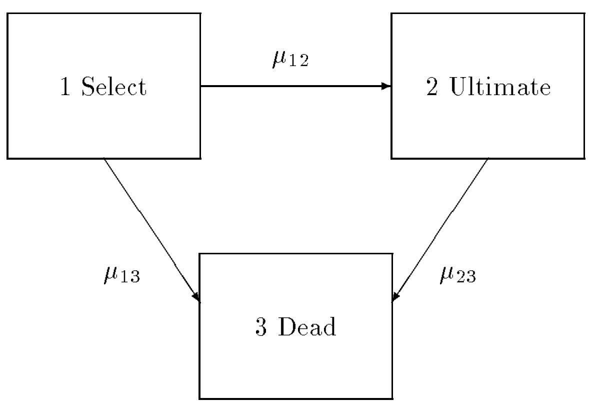
$\lambda \to \infty$. Hence, $N(t)/\lambda \to t$ in probability as $\lambda \to \infty$. Furthermore, since $S(\cdot)$ is continuous, $S(N(t)/\lambda) \to S(t)$ in probability as $\lambda \to \infty$. Finally, since $S(\cdot)$ is bounded, this implies that $E[|S(N(t)/\lambda) - S(t)|] \to 0$ as $\lambda \to \infty$. Therefore,

$$\lim_{\lambda \to \infty} \Pr\{I(t) = 1\} = S(t).$$

By a similar construction, it is possible to approximate a more general process involving transitions other than death. In this case we have $\mu(t) = \sum_j \mu_j(t)$, where $\mu_j(t)$ is the force of transition to state j at duration t.

To examine the select and ultimate mortality model more closely, consider the setup shown in Figure 3. Suppose that the μ_{ij} represent forces of transition for some (attained)

Figure 3: **Model for Select Mortality**



age group. According to this setup, the probability of surviving a period of time, t, is

$$\begin{aligned} p_{11}(t) + p_{12}(t) &= e^{-(\mu_{12}+\mu_{13})t} + \int_0^t e^{-(\mu_{12}+\mu_{13})x} \mu_{12} e^{-\mu_{23}(t-x)} dx \\ &= e^{-(\mu_{12}+\mu_{13})t} + \frac{\mu_{12} e^{-\mu_{23} t} \left[1 - e^{-(\mu_{12}+\mu_{13}-\mu_{23})t}\right]}{\mu_{12}+\mu_{13}-\mu_{23}} \end{aligned}$$

$$= \frac{(\mu_{13} - \mu_{23})e^{-(\mu_{12}+\mu_{13})t} + \mu_{12}e^{-\mu_{23}t}}{\mu_{12} + \mu_{13} - \mu_{23}}.$$

This is clearly a weighted average of the survival functions associated with two exponential distributions. The corresponding force of mortality is

$$\begin{aligned} \mu(t) &= \frac{\frac{d}{dt}[p_{11}(t) + p_{12}(t)]}{p_{11}(t) + p_{12}(t)} \\ &= \frac{(\mu_{23} - \mu_{13})(\mu_{12} + \mu_{13})e^{-(\mu_{12}+\mu_{13})t} - \mu_{12}\mu_{23}e^{-\mu_{23}t}}{(\mu_{23} - \mu_{13})e^{-(\mu_{12}+\mu_{13})t} - \mu_{12}e^{-\mu_{23}t}} \end{aligned} \tag{13}$$

Thus, μ_{12}, μ_{13} and μ_{23} should be chosen so that (13) best represents the selection effect for this age group.

We find that, for any choice of the three parameter values, there is a second choice which produces exactly the same $\mu(t)$. That is, if $\mu_{12} = \hat{\mu}_{12}$, $\mu_{13} = \hat{\mu}_{13}$, and $\mu_{23} = \hat{\mu}_{23}$, then we can achieve the same $\mu(t)$ by letting $\tilde{\mu}_{12} = \hat{\mu}_{23} - \hat{\mu}_{13}$, $\tilde{\mu}_{13} = \hat{\mu}_{13}$, and $\tilde{\mu}_{23} = \hat{\mu}_{12} + \hat{\mu}_{13}$. If we restrict our attention to the subset of the parameter space for which $\mu_{23} > \mu_{12} + \mu_{13}$ or $\mu_{23} < \mu_{12} + \mu_{13}$, then, for each $\mu(t)$, the parameterization is unique. In the absence of prior information about the parameter values, an arbitrary choice of subset may be made. Our objective is simply to find the best $\mu(t)$ based on this three-state setup. Ordinarily, we have no data on the three transitions shown in Figure 3, but only on transitions from states 1 and 2 combined to state 3.

Since the force of transition from state 1 to state 2 is likely to be quite large relative to the forces of mortality, it seems reasonable to assume that $\mu_{23} < \mu_{12} + \mu_{13}$. It follows from (13) that $\lim_{t\to\infty} \mu(t) = \mu_{23}$. We also have $\mu(0) = \mu_{13}$. For $0 < t < \infty$, $\mu(t)$ is a weighted average of the select force, μ_{13}, and the ultimate force, μ_{23}. The weights are the conditional probabilities of being in the select and ultimate states at duration t given survival to duration t. Tenenbein and Vanderhoof [22] also modeled the force of mortality as an average of select and ultimate forces. However, in their models, the select and ultimate proportions at each duration are deterministic.

3.4 Numerical Example

To illustrate the procedure described above, we shall use the three-state model shown in Figure 3 to describe select mortality. Parameter values are obtained so that the model reflects the male aggregate mortality in the 1982-1988 Individual Ordinary Mortality Table published by the Canadian Institute of Actuaries [2]. The development of this table was described by Panjer and Russo [13].

We assumed that the forces of transition are constant within each year of age. Thus, for the age range $[x, x+1)$, we used the various tabular mortality rates for attained age x to determine estimates of $\mu_{12}^{(x)}$, $\mu_{13}^{(x)}$ and $\mu_{23}^{(x)}$. We first calculated tabular forces of mortality from the mortality rates by assuming that, for $k = 0, 1, \ldots, 14$,

$$\mu^T_{[x-k]+k+1/2} = -\log(1 - q^T_{[x-k]+k})$$

and

$$\mu^T_{x+1/2} = -\log(1 - q^T_x).$$

The T denotes a tabular value. Although the forces of mortality do not appear in the published table, we use the superscript to distinguish them from the forces of mortality which result from our Markov model.

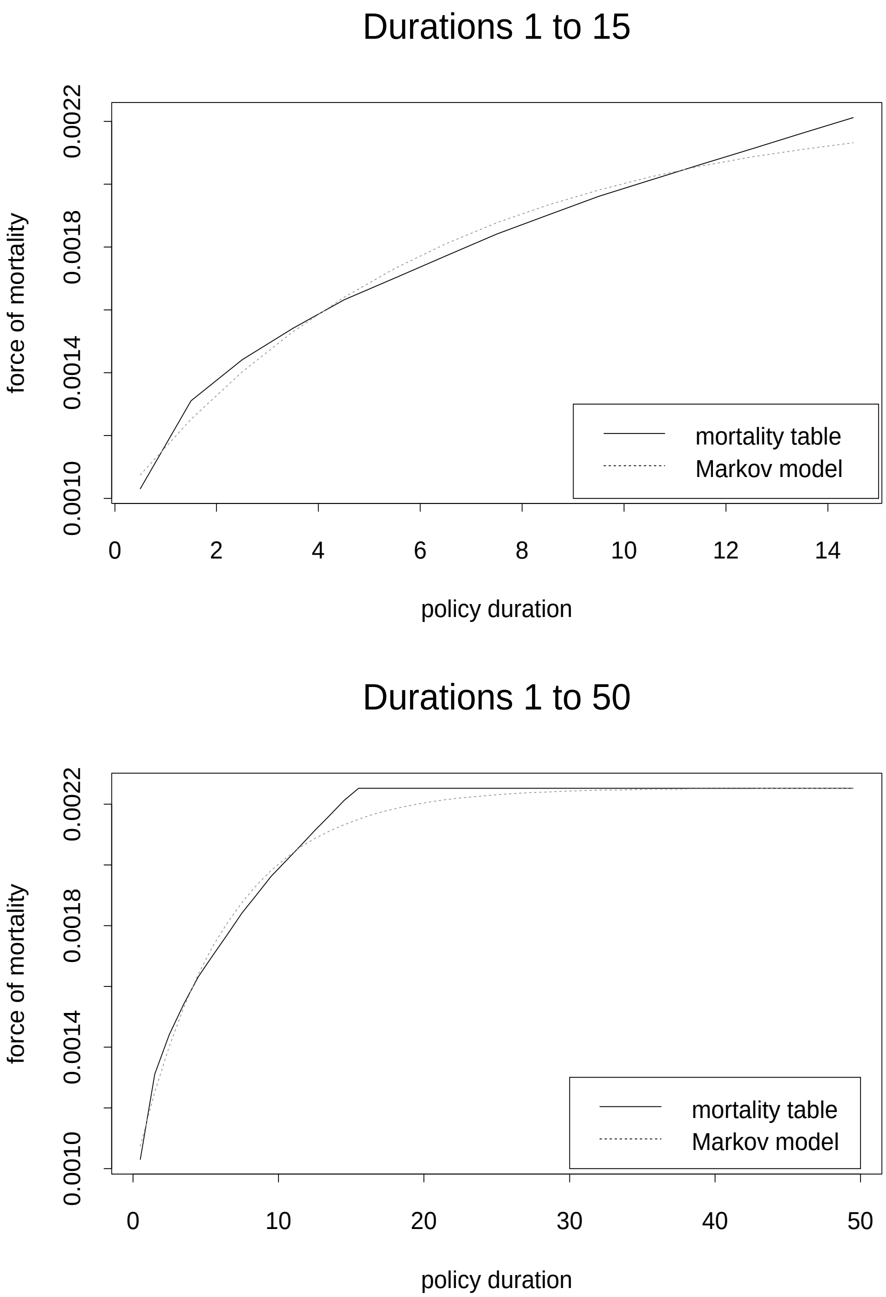
As a function of policy duration (with attained age fixed), the tabular force of mortality increases during the 15-year select period, and remains constant thereafter. Our corresponding "fitted" force of mortality, $\mu^{(x)}(t)$, determined by (13), does not exhibit this behavior. It increases toward its limit as the policy duration approaches infinity. We let our estimate of $\mu_{23}^{(x)}$ equal the tabular ultimate force of mortality for age $x+1/2$. The estimates of $\mu_{12}^{(x)}$ and $\mu_{13}^{(x)}$ were obtained to minimize the squared deviations of $\mu^{(x)}(t)$ from the tabular forces of mortality at durations $0.5, 1.5, 2.5, \ldots, 14.5$. That is, we minimized

$$\sum_{k=0}^{14} \left[\mu^{T}_{[x-k]+k+1/2} - \mu^{(x)}(k+1/2)\right]^2 .$$

The resulting $\mu^{(45)}(t)$ is plotted in Figure 4 along with $\mu_{[45-t]+t}$. The first graph shows only the select period. It demonstrates that, although we have used a model with just three parameters, our resulting force of mortality function is quite close to that based on the table at nearly all select durations. It is only near the end of the select period that our force of mortality appears to be significantly lower. This is quite noticeable in the second graph. The reason is that our fitted force of mortality must have a much smoother path toward the ultimate level. We also see from the second graph that the two curves are very close after 30 years and almost indistinguishable after 40. The results are similar for other attained ages.

The parameter estimates obtained for ages 45 through 70 are given in Table 1. As we might expect, there is very little variation in the estimates of $\mu_{12}^{(x)}$; all lie in the range from 0.163 to 0.174. The estimates of $\mu_{13}^{(x)}$ are slightly lower than the duration 1 tabular forces. As

Figure 4: **Comparison of Force of Mortality (Age 45)**



indicated above, the $\mu_{23}^{(x)}$ estimates are equal to the corresponding tabular ultimate forces.

Table 1: **Forces of Transition**

x	$\mu_{12}^{(x)}$	$\mu_{13}^{(x)}$	$\mu_{23}^{(x)}$	x	$\mu_{12}^{(x)}$	$\mu_{13}^{(x)}$	$\mu_{23}^{(x)}$
45	0.164	0.00097	0.00225	58	0.165	0.00304	0.00933
46	0.164	0.00107	0.00251	59	0.165	0.00329	0.01036
47	0.163	0.00117	0.00280	60	0.167	0.00352	0.01150
48	0.163	0.00128	0.00313	61	0.167	0.00380	0.01274
49	0.163	0.00140	0.00350	62	0.167	0.00410	0.01411
50	0.163	0.00154	0.00391	63	0.167	0.00442	0.01560
51	0.163	0.00168	0.00437	64	0.168	0.00472	0.01725
52	0.164	0.00183	0.00488	65	0.169	0.00503	0.01905
53	0.164	0.00201	0.00544	66	0.169	0.00540	0.02102
54	0.164	0.00218	0.00608	67	0.170	0.00574	0.02318
55	0.164	0.00238	0.00677	68	0.171	0.00609	0.02553
56	0.164	0.00259	0.00755	69	0.173	0.00638	0.02811
57	0.164	0.00282	0.00840	70	0.174	0.00674	0.03093

The parameter estimates shown in Table 1 can be used to find various probabilities of interest. For the age range $[x, x+1)$, The force of transition matrix is

$$\begin{bmatrix} -(\mu_{12}^{(x)} + \mu_{13}^{(x)}) & \mu_{12}^{(x)} & \mu_{13}^{(x)} \\ 0 & -\mu_{23}^{(x)} & \mu_{23}^{(x)} \\ 0 & 0 & 0 \end{bmatrix}.$$

Upon finding the corresponding eigenvalues and right-eigenvectors, the transition probability matrix for time intervals within this age range can be obtained using (7). If this procedure is repeated for all x, then we can use (9) to determine the transition probability matrix for any age range.

Table 2 shows survival probabilities which were obtained in this manner. The numbers

Table 2: **Comparison of Survival Probabilities**

t	${}_tp^T_{[45]}$	$1 - p_{13}(45, 45+t)$	t	${}_tp^T_{[45]}$	$1 - p_{13}(45, 45+t)$
1	0.9989700	0.9989312	14	0.9436032	0.9437097
2	0.9975215	0.9975511	15	0.9341106	0.9346207
3	0.9957659	0.9958410	16	0.9234337	0.9245411
4	0.9936747	0.9937680	17	0.9117430	0.9134101
5	0.9912204	0.9912964	18	0.8989695	0.9011527
6	0.9883657	0.9883858	19	0.8850534	0.8877119
7	0.9850546	0.9849978	20	0.8699190	0.8730093
8	0.9812228	0.9810886	21	0.8535037	0.8569861
9	0.9768171	0.9766096	22	0.8357508	0.8395813
10	0.9717767	0.9715020	23	0.8166037	0.8207379
11	0.9660141	0.9657210	24	0.7960171	0.8004175
12	0.9594548	0.9591973	25	0.7739516	0.7785747
13	0.9520191	0.9518781	26	0.75037699	0.75518104

represent the probability that an individual issued insurance at age 45 survives each of the next 50 years, that is, $1 - p_{13}(45, 45+t)$. These are compared to the probabilities ${}_tp^T_{[45]}$, determined from the mortality table. Table 2 indicates that the survival probabilities obtained using a simple three-state Markov model are very close to those obtained directly from the mortality table.

We close this section by noting that many other techniques are available for modeling select and ultimate mortality. Examples include those discussed by Currie and Waters [4], Panjer and Russo [12] and Tenenbein and Vanderhoof [22]. Select and ultimate mortality was discussed in this section because it is a simple case involving duration dependence. The techniques developed in this paper are most useful in dealing with applications requiring a greater number of states.

4 SUMMARY

This paper has described an approach whereby we can determine probabilities required for actuarial calculations in applications which are represented as multi-state processes. We have drawn on some very convenient mathematical results which are available when the process is assumed to be Markov with constant forces of transition (i.e. a time-homogeneous Markov process). The extension to piecewise constant forces is straight-forward. In cases involving duration dependence we find that, rather than using the less tractable semi-Markov process, it is possible to approximate the impact of duration by the inclusion of additional states in the model.

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