Modeling of Impact Dynamics: A Literature Survey

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Abstract
The purpose of this paper is to present a general overview of impact analysis and some of the most important approaches in this area. It is not intended to provide a tutorial on impact with all the required mathematical developments. The paper is a preliminary literature survey by the authors whose goal is to develop a sound and a practical methodology to analyze impact and implement it within the ADAMS program.

1. Introduction

The dynamic analysis of multibody systems with kinematic constraints is a well-established area of mechanics. To make mathematical modeling possible, bodies are assumed to be perfectly rigid and joints to have no clearance. Many computer programs have been built on these assumptions although they include enhancements to model flexible bodies, friction and non-linear springs and dampers. One of the toughest physical phenomena to model is the impact between two bodies. Impact may be defined as a sudden change in the momentum of each contacting body, without a corresponding change in position. Impact is inherent to unilateral constraints, i.e. a constraint that acts at a given instant only in one direction of the common normal of contacting surfaces.

The subject of impact attracts the interest of scientists and engineers from different areas of knowledge from astrophysics to robotics. The common goal is to develop theories that can predict the behavior of colliding objects. Our focus in this paper, however, will be mainly on impact modeling as it relates to rigid bodies. The mechanical engineer's interest in impact problems is motivated by the desire to develop valid models for mechanical systems where impact is inherent to their function (e.g. crushers, circuit breakers, presses). Other issues such as play in the joints and damage due to accidental or functional impact are also important to understand.

In the evolution of impact theory four major aspects emerged as distinct (but not unrelated) subjects of interest. Depending on impact characteristics (velocity, materials), the assumptions made and the results sought, one aspect will become more predominant than the others thus leading to a solution approach to impact analysis. These four aspects are:

- *Classical mechanics*
- *Elastic stress wave propagation*
- *Contact mechanics*
- *Plastic deformation*

Below is an overview of these four aspects.

2. The four major aspects of impact theory

Classical mechanics: This involves the application of the fundamental laws of mechanics to predict the velocities after impact. The impulse-momentum law forms the core of this approach. Goldsmith, in his classic work **[4]** devotes a chapter to the application of this theory to several problems. Brach uses this approach exclusively in **[1]** to model numerous problems of practical value. The algebraic nature of this method makes the mathematical development easy and accessible to most engineers as can be seen in Brach's work. The loss of energy inherent to any real impact process is taken into account by the means of the coefficient of restitution. The accuracy of this coefficient is crucial to obtaining sufficiently good results. Unfortunately, this approach is unable to predict the contact force between bodies or the stresses in them.

Elastic wave propagation: Impact is accompanied by a stress wave that propagates in the impacting bodies away from the region of impact. If the energy thus transformed into vibrations becomes an important fraction of the total energy, then the classical approach becomes insufficient to examine an impact problem. Goldsmith **[4, ch. 3]** applies this approach to many problems, some of which are: longitudinal impact of two rods, impact of a mass on a rod, tensile impact of a mass on a rod and the effect of viscoelasticity on impact behavior. The wave propagation approach is also covered extensively by Zukas *et al*. **[2]**.

Contact mechanics: The contact stresses resulting from the impact of two bodies are another area of interest in the study of impact. Conventional contact mechanics is mainly concerned with static contact although it has been extended to approximate solutions when impact is involved. For spheroidal surfaces, Hertz theory is used to obtain the force deformation relation needed to calculate the duration of impact and the maximum indentation. This approach has been extended to the cases where contained plastic deformation occurs, generally with the assumption of a material having a yield point. Numerical models of the contact zone are also used when Hertz theory is not applicable. The force-deformation equation is often augmented with a damping term to reflect dissipation in the contact area, thus allowing us to effectively model the contact area as a spring-damper system.

Plastic deformation: When plastic strains go beyond the scale of contained deformation, the elastic wave propagation model can no longer be applied to analyze impact problems. This is the domain of high velocity impact generally associated with explosives and projectiles. Goldsmith **[4, ch. 5]** presents an extended study of the subject using two approaches: the hydrodynamic theory of the behavior of solid bodies and the theory of plastic wave propagation. In the hydrodynamic theory, permanent deformation is considered to be a result of a change in the body's density. An equation of state for the material that relates pressure to density changes and temperature or entropy is used together with the laws of conservation of momentum, energy and mass. In the theory of plastic strain propagation, the material is considered to be incompressible in the plastic domain. As well, the state equation relating stress, strain and strain rate is assumed to be independent of temperature. Maugin **[21]** and Lubliner **[20]** postulate that where ductile materials are used, the loading is applied over a long period of time, high temperatures are involved or high strain rates occur, rate dependence cannot be ignored in describing the plastic behavior of materials. Zukas *et al* **[5]** present an extensive treatment of plastic wave propagation using both rate-dependent and rate-independent theories.

How do these four aspects relate to the needs of the mechanical engineer?
In most problems, the mechanical engineer seeks to answer two fundamental questions:
1) What is the relationship between the velocities before and after impact?
2) What is the force at the impact point?

The impulse momentum approach can address the first question adequately, given the knowledge of the coefficient of restitution. As mentioned earlier, it is incapable of answering the second question. The wave propagation theory is necessary to predict stresses inside the solid but its integration to the dynamic analysis of multibody mechanical systems can be an overwhelming task. The contact mechanics approach offers the possibility of treating the contact region as a spring damper system, making it possible to treat the impact as a continuous-time dynamic phenomenon. The large plastic strain theory is most useful in the domain of perforation by projectiles, as in ballistics. This area shall be considered out of the scope of this survey. Therefore, we shall focus the rest of this paper on reviewing the work of different researchers who use either the restitution coefficient model or the contact force approach.

3. The restitution coefficient model

According to Kozlov **[3]**, the first investigation of impact goes back to 1668 and was carried out by Wallis, Wren and Huygens. Newton later referred to Wren's work in his famous work *Mathematical Foundations of Natural Philosophy* published in 1687. The result of Huygens' work was the derivation of the law of conservation of momentum, which is fundamental in impact theory. The major assumption in this theory is that the colliding bodies are perfectly rigid. Consequently, the impact duration must be zero. The conservation of momentum law alone is not always sufficient to predict the velocities of the two colliding bodies after impact. Two limit cases are considered in the elementary theory of impact: a perfectly elastic impact, and a perfectly inelastic impact. The former case implies that the kinetic energy of the system is conserved. The latter case assumes that the two bodies coalesce, to move as a single mass, after impact. The velocity of the combined mass can then be predicted using only the conservation of momentum. However, most impacts are neither fully elastic nor fully inelastic. This partial loss of the initial kinetic energy is expressed in terms of the restitution coefficient e first introduced by Newton. This coefficient relates the relative velocities before and after impact according to the equation:

$$v_{1f} - v_{2f} = -e(v_{1i} - v_{2i}) \qquad (1)$$

where the subscripts 1 and 2 refer to the two bodies and i and f stand for initial and final states. This equation provides the additional information needed to complete the resolution of a basic impact problem.

The quantity e is a dimensionless coefficient between 0 and 1 where 0 corresponds to a totally inelastic impact and 1 to a perfectly elastic impact. The restitution coefficient is a global measure of the energy loss during impact and may incorporate different forms of dissipation such as viscoelastic work performed on the materials of the impacting bodies, plastic deformation of contact surfaces and vibration in the two bodies. The restitution coefficient is not an intrinsic material property. It depends on the materials of the bodies, their surface geometry and the impact velocity **[4, p. 262]**. The major advantage of the restitution coefficient concept is its mathematical simplicity. The simple algebraic relation between velocities before and after impact makes it easy to determine the velocities after impact. However, the engineering value of the information thus obtained remains highly dependent on the knowledge of the restitution coefficient. Determining its value accurately requires experimental work in general. Moreover, the contact force at the impact point cannot be determined from this model. In spite of its questionable fundamental significance, the restitution coefficient remains a useful tool for analyzing many impact problems. Brach **[1]** solves numerous practical problems using the coefficient of restitution. He also introduces μ as the ratio of the tangential and the normal impulse components. This coefficient, termed the impulse ratio, is necessary for treating oblique impact problems, that is those with tangential relative velocity. If a Coulomb model is used for friction between surfaces then μ can be related to the dynamic coefficient of friction. As well, μ can be positive or negative to ensure that the tangential force is dissipative, as is usually the case. Brach also notes that the restitution coefficient can be allowed to take negative values between 0 and -1. That means that some energy has been lost during impact but without velocity reversal. As an example, consider a projectile penetrating through a barrier. The penetration work reduces the velocity of the projectile without reversing it. For oblique contacts, Brach proposes the use of the tangential restitution coefficient e_t to relate tangential velocities before and after impact. He also shows **[1, p. 30]** that μ and e_t are related. Therefore, only two independent coefficients e and μ are required to solve impact problems. An important result comes out of Brach's analysis: The final system energy cannot be zero for a perfectly inelastic and frictionless impact (e=0 and μ=0).

Variation of restitution coefficient with velocity

The variation of the coefficient of restitution with respect to the relative initial velocity of the contacting bodies has been examined in several papers. There is general agreement on the theoretical models of the interplay between the velocity of contact and the coefficient of restitution. Most references **[8, 10, 12]** discuss a relationship where

$$e(v) = 1 - f\left(v^{\frac{1}{5}}\right) \tag{2}$$

Through the inverse relationship, it may be inferred that at higher impact velocities, the coefficient of restitution is lower, meaning that more energy is dissipated when the colliding bodies are moving faster. The relationship above is derived from models of purely viscoelastic behavior. In reality, there are often other mechanisms of dissipation to consider as well. At high impact velocities, the energy dissipated in the form of elastic waves increases, as does the loss of energy due to plastic deformation **[10]**. At low impact velocities, the effects of phenomena such as adhesion **[8]** and gravity become significant **[10]**.

4. The contact force-indentation model

The elementary theory of impact reviewed above is based on the simplifying assumption of perfectly rigid bodies. Actual physical objects are compliant and hence the impact duration is strictly greater than zero. This more realistic view of impact phenomena led many researchers to consider the continuous-dynamics models of collision where bodies deform during impact and the collision dynamics are treated as continuous-time dynamic phenomena. The success of this approach as an analysis tool relies heavily on a sound mathematical model for the force-indentation and indentation rate relationship. Also, a clear and a practical way to identify any needed parameters is a must. In its general form, the force-indentation relationship may look like:

$$F = F_c(\delta) + F_v(\delta, \dot{\delta}) + F_p(\delta, \dot{\delta}) \qquad (3)$$

where, F_c is the elastic (conservative) part of the contact force F, F_v the viscous damping part and F_p the dissipative part due to plastic deformation. Viscous dissipation can be related to the rheological properties of the materials. Plastic dissipation can be determined from the materials' stress-strain curve. In the following, we first review the contact mechanics work leading to the development of the compliance relation $F_c(\delta)$.

Force-indentation relationship

Perhaps the biggest landmark in contact mechanics was the work of Hertz on the elastic contact of semi-infinite solids, published in 1882. Johnson **[6, ch. 4]** gives an excellent coverage of this theory with a formula summary in appendix. Hertz theory predicts the stress distribution in the contact zone between two bodies having a surface of revolution. It also allows us to calculate the normal and shear stress distribution inside the solid. This reveals some interesting and important facts. For example, the maximum shear stress, which is directly related to material failure, occurs below the contact surface, potentially causing undetected plastic yielding. A very commonly used result is the force-indentation relation for sphere to sphere contact: **[6, p. 93]**:

$$F = K\delta^{3/2} \qquad (4)$$

where,
F = Normal force pressing the solids together
δ = approach of the two spheres, i.e. total of deformation of both surfaces
K= constant depending on the sphere radii and elastic properties of the sphere materials.

This equation was combined with the equations of motion by Timoshenko **[7, pp. 420-422]** to treat the impact of two spheres. The maximum indentation and the impact duration were calculated. A similar treatment is also found in Goldsmith **[4, pp. 83-91]** and Johnson **[6, pp. 351-354]**. This analysis constitutes what is known as *Hertz theory of impact*. Eq. (4) above is also valid for any 3D contact of solids. For cylinders in contact with parallel axes, the approach distance will depend on the selected datum **[6, p. 90]**. Thus an equation similar to (4) cannot be found for cylinders. Young **[14, p. 651]** gives a formula for the approach of the cylinder centers when the length/diameter ratio is large.

It is important to note that Hertz formulas are only applicable to non-conformal contacts. In other words, they cannot be used when the radii of curvature at the contact point are too close, such as in journal bearings or plane to plane contact. The contact area must also remain small compared to the bodies' dimensions and the radii of curvature at the contact point. Consequently, Hertz formulas should be used with caution on materials with large elastic strains such as rubber. Two other classical contact problems, line loading and point loading of an elastic half space, are analyzed by Johnson **[6, ch. 2,3]** along with indentation by a rigid flat punch. He also treats conformal contacts such as those produced by wedges and cones in **[6, ch. 5]**. For journal bearings and other joints used in machinery, Rivin **[5, p. 106]** gives a detailed analysis of their compliance supported with experimental data and practical examples. For the case of a cylindrical joint, he bases his analysis on a sinusoidal distribution of contact between shaft and bearing. To calculate the pressure distribution, he considers two possible models for the force-deformation equation, linear and quadratic:

$$F = k\delta \tag{5a}$$

and,

$$F = k'\delta^2 \tag{5b}$$

Results of experiments carried out with cast iron and hardened steel showed that the quadratic model offered a better correlation of the force deflection in a journal bearing when a high load per unit length was applied. For low loads, the linear model represented a better fit **[5, p.111]**.

Initiation of plastic yield in contacts

An understanding of how yield occurs in contacts is needed to develop the appropriate expression for the force indentation relation and to predict any damage due to plastic deformation. Beyond the elastic loading we consider two stages, elastic-plastic and fully plastic. In the elastic-plastic stage, the plastic deformation is small enough to be accommodated by an expansion of the surrounding area. As the load increases, the plastic zone grows and the displaced material flows to the sides of the indenter. For this analysis, the rigid-perfectly-plastic material model is commonly used. It assumes that the elastic deformation is small enough to be negligible and the material flows plastically at a constant stress Y in tension. For sphere-sphere contact, Johnson **[6, ch. 6]** shows that, under those assumptions, yield will initiate when the mean contact pressure p_m is $1.1Y$ and the flow will become fully plastic at about $p_m = 3.0Y$. Stronge **[13],** takes the same approach to derive an expression for the restitution coefficient that reflects the dissipation due to plastic work under different conditions of friction. Based on the rigid-perfectly-plastic model and Hertz theory of impact, Johnson **[6, p.361]** calculates the velocity V_Y necessary to initiate yield. For a sphere striking the plane surface of a massive body he shows that:

$$\rho \frac{V_Y^2}{Y} = 26 \left(\frac{Y}{E^*} \right)^4 \qquad (6)$$

where ρ is the sphere material density and E^* is an equivalent elastic modulus. For example, for a medium hard steel, Y= 1000 N/mm^2 and V_Y =0.14 m/s.

Naturally, this velocity is quite low and one should expect that most impacts between metallic bodies will involve some plastic deformation. To study impact in the plastic range, Johnson proposes using stresses calculated under static conditions at moderate velocities (up to 500 m/s as a first approximation). Under fully plastic deformation, the mean contact pressure p_m is constant and equal to 3.0 Y. By considering energy during compression and rebound, Johnson derives an expression for the restitution coefficient e. This is found to be proportional to $V^{-1/4}$. This result is confirmed by measurements from Goldsmith **[4, p. 262]** performed with various material combinations. Another important result is that e is proportional to $Y_d^{5/8}$ and thus depends on the material hardness through the dynamic yield strength Y_d. For this simplified theory to be valid, Johnson re-examines the initial hypothesis of shallow indentation which implies that:

$$\delta = \frac{a^2}{2R} \qquad (7)$$

where,
a= radius of contact area
R= sphere radius
The expression of δ above is a good approximation as long as (a/R) <0.5. Johnson further shows that this condition will require that:

$$\frac{1}{2} m V^2 \Big/ p_d R^3 < 0.05 \qquad (8)$$

For a steel sphere on a steel plate, it translates to V<100 m/s. In other words, large plastic deformation starts at about 100 m/s impact velocity.

Impact regime of behavior
Johnson **[15]** suggested the non-dimensional parameter ($\rho V^2/Y_d$) as a way to characterize impact between two metallic bodies. The following table from Johnson **[6, p. 366]** provides a preliminary guide:

Table 1: Impact regimes as function of $\rho V^2/Y_d$.

Regime	$\rho V^2/Y_d$	Approximate velocity (m/s)
Elastic	$<10^{-6}$	<0.1
Fully plastic	$\sim 10^{-3}$	~ 5
Limits of shallow indentation	$\sim 10^{-1}$	~ 100
Extensive plastic flow, beginning of hydrodynamic behavior (e.g. bullets)	~ 10	~ 1000
Hypervelocity (e.g. Laser beams, meteorites)	$\sim 10^3$	~ 10000

Physically, ρV^2 can be interpreted as the stagnation pressure of the projectile seen as a fluid jet and Y_d as the strength of the target. When the ratio $\rho V^2/Y_d$ exceeds 1, the inertia of the deforming material becomes predominant over yield strength.

Viscous dissipation

To complete the description of the force deformation model, we need to define the viscous term $F_v(\delta, \dot{\delta})$ in Eq. (3) above. The simplest model used in literature is the linear damper defined by:

$$F_v = c\dot{\delta} \tag{9}$$

where c is the damping factor. This is used in the ADAMS impact statement with a slight difference **[16]**. Namely, the coefficient c is increased from zero at the beginning of impact to c at a certain penetration δ specified by the user. This prevents the viscous force from being discontinuous. Brach **[1, p. 54]** treats the collision of two masses with the contact area modeled as a linear spring damper. The solution is identical to that of the classic problem of damped vibration of a mass spring system. From the ratio of the initial and final relative velocities, the restitution coefficient is found to be:

$$e = \exp\frac{-\pi\xi}{\sqrt{1-\xi^2}} \tag{10}$$

with,

$$\xi = \frac{c}{2\overline{m}\,\omega_n} \quad \text{and} \quad \omega_n^2 = \frac{k}{\overline{m}}$$

Note that e thus found is independent of the initial velocities. As expected, for an elastic collision, ξ=0 and e=1. For an inelastic collision, ξ =1 and e=0. In this case c=c_r where c_r represents the critical damping ratio $2\sqrt{k\overline{m}}$. The usefulness of such a model is limited to those contacts that can be approximated reasonably well by a linear spring damper model. A non linear expression for the visco-elastic force was originally proposed by Hunt and Crossley in **[17]**. It has the following general form:

$$F_v = \beta\delta^\gamma\dot{\delta} \tag{11}$$

where β and γ are constants. The force deformation curve then takes the shape of a hysteresis curve as shown in Figure 1.

Kuwabara and Kono **[12]** extend Hertz theory and find that γ =1/2 for sphere to sphere contact. Experiments by Falcon *et al*. **[10]** show that γ =1/4 is a better fit. The value of the parameter β can also be determined from the viscosity coefficients associated with shear and volume deformation **[10,12]**. Unfortunately, these coefficients are neither easy to find in the literature nor easy to measure. Landau and Lifchitz **[19]** derive equations that relate viscosity coefficients and other thermomechanical constants to the attenuation constants of transverse and longitudinal sound waves in isotropic solids.

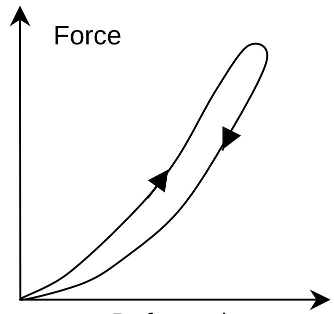

Figure 1: Hysteresis curve for non-linear viscous damping

Friction in impact

Friction affects the reaction force at the contact point and may increase or decrease the coefficient of restitution depending on whether sliding persists in the same direction or reverses after the moment of impact. Stronge **[13]** developed an expression for the coefficient of restitution that reflects this effect. He concludes, however, that the effect of friction on the coefficient of restitution is insignificant if the coefficient of friction is less than 1. The other effect of friction to consider in impact analysis is energy dissipation due to sliding. Ivanov **[22]** develops a visco-elastic model to treat impact of rigid bodies with friction. He uses a potential function from which the force at the contact point is derived. The first term of this function represents surface roughness and the second one is quadratic in relative velocities, that is, a Rayleigh's dissipative function. Practical determination of the two parameters of the potential function remains unexplored.

Relationship between restitution coefficient and dissipation in contacts

The major difficulty in implementing an approach based on a contact force model is identifying parameters such as β in the force indentation relationship. One possible solution is to relate these unknown parameters to the restitution coefficient which is much easier to measure. Naturally, this will work best if there is a predominant and known dissipation mechanism in the impact problem at hand. Lankarani and Nikravesh **[18]** applied this approach and treated two ideal cases of impact between spheres, one with viscous dissipation only and the other with plastic dissipation. Their contact force model was based on the equation by Hunt and Crossley **[17]**:

$$F = K\delta^{\gamma} + \beta\delta^{\gamma}\dot{\delta} \tag{12}$$

By equating the energy loss derived from the momentum impulse approach and the one derived from hysteretic damping they established a relation between the hysteretic damping factor β and the restitution coefficient e:

$$\beta = \frac{3K(1-e^2)}{4(\Delta V)} \tag{13}$$

where ΔV is the relative velocity of the spheres before impact.

Note that the exponent γ is the one found from Hertz theory to be 1.5. The fact that the same exponent is also used for the viscous term $\beta\delta^{\gamma}\dot{\delta}$ is somewhat perplexing. Recall that work cited above **[10,12]** suggests that a reasonable value of that exponent was found to lie between 0.25 and 0.5.

Applying a similar approach for plastic dissipation, Lankarani and Nikravesh **[18]** further introduce an expression for the contact force F_u during the unloading phase of impact.

$$F_u = F_m \left(\frac{\delta - \delta_p}{\delta_m - \delta_p} \right)^{\gamma} \tag{14}$$

where the maximum force $F_m = K\delta_m^{\gamma}$ and δ_m is determined from Hertz theory of impact.

The energy dissipated by plastic deformation is the area of the hysteresis loop. By equating it with energy loss expressed as a function of the restitution coefficient, they relate the permanent deformation δ_p to e by the equation:

$$\delta_p = \frac{(\gamma + 1)\overline{m}(\Delta V)^2}{2F_m}\left(1 - e^2\right) \tag{15}$$

where $\overline{m} = \dfrac{m_1 m_2}{m_1 + m_2}$

Their comparison with experimental data from Goldsmith **[4, p. 260]** shows a good correlation, confirming that energy dissipation in impact between metallic spheres is mainly due to plastic deformation.

5. Conclusion

In order to analyze impact successfully, the mechanical engineer must first determine whether plastic deformation may occur in the impact problem at hand. In the case of metal to metal contact, viscous dissipation is very small and can be neglected, especially when plastic dissipation is present. If the contact force at the impact point is not needed for a preliminary analysis, the restitution coefficient model should be considered. It will give a good approximation provided that e is known to a reasonable degree of accuracy. A more complete analysis is possible only with a contact force model. This can be included into the equations of motion to analyze multibody dynamics. Unfortunately, there are two major difficulties with this model. First, one has to use the proper form of the contact force equation, and second, the equation's parameters must be identified. Although Hertz theory can help in this regard, many contacts that occur in practice are of the non-hertzian type. Another factor that should not be overlooked is the dissipation in other parts of the system. Indeed, dissipation at the impact point may not be dominant in the system as a whole. The authors intend to continue their research in this area to come up with a methodology to treat impact based on the contact force model and using known geometry, material properties and impact velocity to determine the needed parameters.

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