# No Polynomial Bound for the Period of the Parallel Chip Firing Game on Graphs

M. Kiwi * MIT & U. Chile

R. Ndoundam † U. Yaoundè-I

M. Tchuente ‡ U. Yaoundè-I

E. Goles § U. Chile

May 2, 1994

**Abstract**

The following (solitaire) game is considered: Initially each node of a simple, connected, finite graph contains a finite number of chips. A move consists in firing all nodes with at least as many chips as their degree, where firing a node corresponds to sending one of the node's chips to each one of the node's neighbors.

In [Bi89] it was conjectured that every parallel chip firing game played on an $N$ node connected and undirected graph finally evolves into a steady phase with a period that does not exceed $N$. It was later conjectured in [Pr93] that over every strongly connected Eulerian multidigraph there is a parallel chip firing game that evolves into a steady state with period equal to the length of the longest dicycle of the underlying digraph. In this work we disprove both these conjectures by exhibiting a parallel chip firing game on an $N$ node connected and undirected graph with period $e^{\Omega\left(\sqrt{N \log N}\right)}$.

## 1 Introduction

Consider the following game: Each node of a connected finite graph contains a pile of chips. A move in the game consists of selecting *one* node with at least as many chips as its degree, and *firing* that node by moving one chip of its pile to each one of its neighbors. This game is known as the *chip firing game*.

The *parallel chip firing game*, is similar to the chip firing game, except that now a move consists in firing simultaneously *all* nodes with at least as many chips as their respective degree. Since the total number of chips in a parallel chip firing game remains constant, these games eventually exhibit periodic behavior.

Chip firing games have been independently introduced and are related to many problems, some of which we briefly mention in this section.

*Dept. of Applied Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139. mkiwi@math.mit.edu. Supported by an AT&T Bell Laboratories PhD Scholarship. On leave of absence from Dept. de Ingeniería Matemática, U. de Chile.

†Dept. d'Informatique, Faculté des Sciences, Université de Yaoundè-I, BP 812, Yaoundè, Cameroon.

‡Dept. d'Informatique, Faculté des Sciences, Université de Yaoundè-I, BP 812, Yaoundè, Cameroon. tchuente@yaounde.orstorn.fr. Research supported by the Micro-Processors and Informatics Program of the United Nations U. and by the U. of Yaoundè-I through the research project N°591017. FAX 237-22-13-20.

§Dept. de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170-3, Correo 3, Santiago, Chile. egoles@uchcecvm.cec.uchile.cl. Research supported by FONDECYT 1940520-94 and Micro-Processors and Informatics Program of the United Nations U. Part of this work was done while the author was at the U. of Yaoundè-I

While analyzing a particular "balancing game", Spencer [Sp86] introduced a process which can be viewed as a chip firing game in an infinite undirected path. Anderson, Lovász, Shor, Spencer, Tardos and Winograd [ALSSTW] studied this process, which was later extended from paths to general graphs by Björner, Lovász and Shor [BLS91]. Tardos [Ta88] showed that if a chip firing game over an undirected graph terminates, then it terminates in a polynomial (in the number of nodes of the underlying graph) number of steps. Eriksson [Er91] showed that when the underlying graph is directed then a terminating game can be exponentially long. Chip firing games are also related to Petri nets, and, as pointed out in [GK93] to the sand-pile model analyzed in physics.

Parallel chip firing games where studied by Goles and Bitar in [BG92], where it is shown that these games converge towards periods of length at most 2 when played on an undirected finite connected acyclic graph. It was conjectured in [Bi89] that for arbitrary undirected connected graphs on $N$ nodes, parallel chip firing games have periods of length at most $N$. Prisner [Pr93] later showed that on every strongly connected Eulerian multidigraph there is a parallel chip firing game that evolves into a steady state with period equal to the length of the longest dicycle in the underlying digraph. In [Pr93] it was conjectured that this was the longest possible achievable period for a parallel chip firing on such digraphs. In this work we disprove both previously mentioned conjectures.

## 2 Preliminaries

We will say $\mathcal{G} = (G(\mathcal{G}), c_0(\mathcal{G}))$ is a *game*, if $G(\mathcal{G})$ is a graph with node set $V(\mathcal{G})$ and $c_0(\mathcal{G}) = (c_0(v, \mathcal{G}))_{v\in V(\mathcal{G})}$ is an initial distribution of chips on the nodes of $G(\mathcal{G})$. All graphs we will consider are finite connected and undirected. We will always assume that the total number of chips in the initial distribution is finite. For a non-negative integer $t$ we will denote by $c_t(\mathcal{G}) = (c_t(v, \mathcal{G}))_{v\in V(\mathcal{G})}$ the distribution of chips on the nodes of $G(\mathcal{G})$ obtained after $t$ moves of the parallel chip firing game played on the graph $G(\mathcal{G})$ with initial position $c_0(\mathcal{G})$. We shall denote by $period(\mathcal{G})$ the smallest positive integer $p$ such that $c_{t+p}(\mathcal{G}) = c_t(\mathcal{G})$ for some $t$. Furthermore, for every non-negative integer $k$, $t_k(v, \mathcal{G})$ will denote the $(k+1)$-th time step node $v$ of $G(\mathcal{G})$ fires, and $period(v, \mathcal{G})$ will denote the smallest positive integer $p$ such that $t_{k+p}(v, \mathcal{G}) = t_k(v, \mathcal{G})$ for all except finitely many $k$'s.

The following basic fact about parallel chip firing games was established in [Pr93]. For completeness we review its proof.

**Lemma 1 ([Pr93])** *For every game $\mathcal{G}$,*

$$period(\mathcal{G}) = lcm\,(period(v, \mathcal{G}))_{v\in V(\mathcal{G})}\,.$$

**Proof:** Let $p$ be $period(\mathcal{G})$, and $l$ be $lcm\,(period(v, \mathcal{G}))_{v\in V(\mathcal{G})}$. Clearly, $l$ is at most $p$. Moreover, for $t$ sufficiently large, it follows that for every node $v$:

$$0 = c_{t+p*l}(v, \mathcal{G}) - c_t(v, \mathcal{G}) = p * (c_{t+l}(v, \mathcal{G}) - c_t(v, \mathcal{G})),$$

thus, $c_{t+l}(\mathcal{G}) = c_t(\mathcal{G})$, implying that $p$ is at most $l$. □

## 3 The Non-polynomially Bounded Period Game

We start by defining the game $\mathcal{G}_n$. Let $G(\mathcal{G}_n)$ be the graph shown in Figure 1(a), with nodes $v_0(\mathcal{G}_n), \ldots, v_{n-1}(\mathcal{G}_n)$. Say that there is a *vehicle* in position $v_i(\mathcal{G}_n)$ at time step $t$

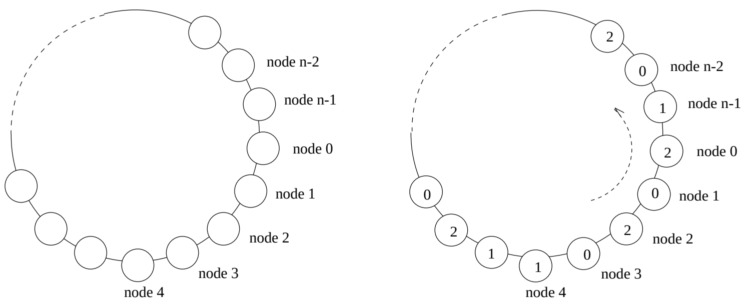


Figure 1: (a) The graph $G(\mathcal{G}_n)$. (b) The distribution $c_0(\mathcal{G}_n)$.

if, after $t$ moves, of game $\mathcal{G}_n$, there are two chips at node $v_i(\mathcal{G}_n)$ and no chips at node $v_{i+1 \pmod n}(\mathcal{G}_n)$. Say that $v_i(\mathcal{G}_n)$ is a *vehicle node* if there is a vehicle either at node $v_i(\mathcal{G}_n)$ or $v_{i-1 \pmod n}(\mathcal{G}_n)$. From now on, we will assume that $n \geq 6$ is a multiple of 3, and we will take as initial distribution of chips, $c_0(\mathcal{G}_n)$, one where there are vehicles at nodes $v_i(\mathcal{G}_n)$ if $i \equiv 0 \pmod 3$ and $i \neq 3$, another vehicle at node $v_2(\mathcal{G}_n)$, and where there is one chip in every non-vehicle node (see Figure 1(b)). It is not hard to see that with each move the vehicles turn counter-clockwise in $G(\mathcal{G}_n)$, in particular, the following holds:

$$t_k(v_0(\mathcal{G}_n), \mathcal{G}_n) = \begin{cases} 3k & \text{if } k \equiv 0 \pmod 3,\ k \neq 1, \text{ and } k < n/3 \\ 2 & \text{if } k = 1 \\ t_{k-n/3}(v_0(\mathcal{G}_n), \mathcal{G}_n) + n & \text{if } k \geq n/3. \end{cases}$$

Hence, by Lemma 1, $\mathit{period}(\mathcal{G}) \geq \mathit{period}(v_0(\mathcal{G}_n), \mathcal{G}_n) = n$.

Let $n_1, \ldots, n_l \geq 6$ be multiples of 3, and consider the game $\mathcal{G}$, obtained from the games $\mathcal{G}_{n_1}, \ldots, \mathcal{G}_{n_l}$ as follows:

- $G(\mathcal{G})$ has node set $\bigcup_{j=1}^{l} V(\mathcal{G}_{n_j}) \cup \{v\}$.
- The nodes in $V(\mathcal{G}_{n_j})$ are connected as in the graph $G(\mathcal{G}_{n_j})$, and $v$ is connected to $v_0(\mathcal{G}_{n_1}), \ldots, v_0(\mathcal{G}_{n_l})$ (see Figure 2).
- The initial distribution of chips on the nodes $V(\mathcal{G}_{n_1}), \ldots, V(\mathcal{G}_{n_l})$ is given by $c_0(\mathcal{G}_{n_1}), \ldots, c_0(\mathcal{G}_{n_l})$ respectively, except for the fact that one more chip is added to the nodes $v_0(\mathcal{G}_{n_1}), \ldots, v_0(\mathcal{G}_{n_l})$. Initially $v$ has no chips. (See Figure 2).

We want to lower bound the period of the game $\mathcal{G}$.

We claim that

$$\forall k \geq 0,\ \forall i \in \{1, \ldots, l\},\ \forall j \in \{0, \ldots, n_i - 1\} \quad t_k(v_j(\mathcal{G}_{n_i}), \mathcal{G}) = t_k(v_j(\mathcal{G}_{n_i}), \mathcal{G}_{n_i}) \tag{1}$$

The preceding claim can be demonstrated by showing, through induction on $t$, that $\forall u \in V(\mathcal{G})$ if $t_k(u, \mathcal{G}) \leq t$, then,

$$\text{if } u \in V(\mathcal{G}_{n_i}) \text{ then } t_k(u, \mathcal{G}) = t_k(u, \mathcal{G}_{n_i}),$$

$$\text{if } u = v \text{ then } t_k(u, \mathcal{G}) = \begin{cases} 3k + 1 & \text{if } k \equiv 0 \pmod 3,\ k \neq 1, \text{ and } k < p/3 \\ 3 & \text{if } k = 1 \\ t_{k-p/3}(u, \mathcal{G}) + p + 1 & \text{if } k \geq p/3, \end{cases}$$

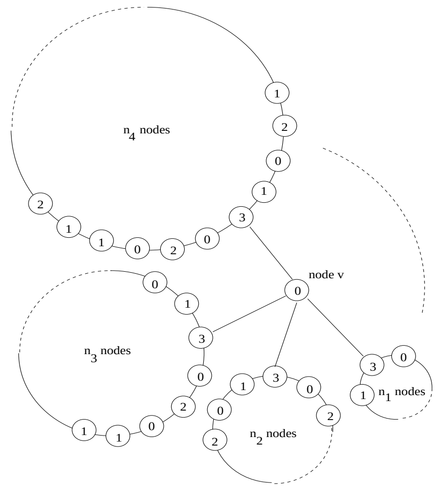


Figure 2: The initial position of game $\mathcal{G}$.

where $p = lcm\,(n_1, \ldots, n_l)$ (note that $p$ is a multiple of 3).

We now choose $n_i = 3p_i$, where $p_i$ is the $i$-th smallest prime, and let $l = \pi(m)$ where $\pi(m)$ denotes the number of primes not exceeding $m$. It follows that:

- $\mathcal{G}$ is a parallel chip firing game where,

$$|V(\mathcal{G})| = 3\sum_{i=1}^{l} p_i + 1 \leq 3m\pi(m) + 1. \tag{2}$$

- $\mathcal{G}$ is such that

$$period(\mathcal{G}) \geq lcm\,(period(\mathcal{G}_{n_i}))_{i=1,\ldots,l} = 3\prod_{i=1}^{l} p_i = 3e^{\theta(m)}, \tag{3}$$

  where the first inequality follows from Lemma 1 and (1), and where $\theta(m) = \sum_{i=1}^{\pi(m)} \log p_i$.

Finally, recall that the Prime Number Theorem (see [HW, Theorem 6]) says that $\pi(m) = \Theta(m/\log m)$, furthermore, $\theta(m) = \Theta(\pi(m)\log m)$ (see [HW, Theorem 420]), which together with (2) and (3) imply that

$$period(\mathcal{G}) \geq e^{\Omega\left(\sqrt{|V(\mathcal{G})|\log|V(\mathcal{G})|}\right)},$$

clearly showing that the period of $\mathcal{G}$ is not bounded by any polynomial in $|V(\mathcal{G})|$.

## References

[ALSSTW] R.J. Anderson, L. Lovász, P.W. Shor, J. Spencer, É. Tardos and S. Winograd, "Disks balls and walls: Analysis of a combinatorial game", *Amer. Math. Monthly* **96** (1989), 481–493.

[BG92] J. Bitar and E. Goles, "Parallel chip firing games on graphs", *Theor. Comp. Sci.* **92** (1992), 291–300.

[Bi89] J. Bitar, "Juegos combinatoriales en redes de autómatas", Tésis de Ingeniero Matemático, Fac. de Cs. Físicas y Matemáticas, U. de Chile, Santiago, Chile, 1989.

[BLS91] A. Björner, L. Lovász and P.W. Shor, "Chip-firing games on graphs", *European J. Combin.* **12** (1991), 283–291.

[GK93] E. Goles and M. Kiwi, "Games on line graphs and sand piles", *Theor. Comp. Sci.* **115** (1993), 321–349.

[Er91] K. Eriksson, "No polynomial bound for the chip-firing game on directed graphs", *Proc. Amer. Math. Soc.* **112** (1991), 1203–1205.

[HW] G.H. Hardy and E.M. Wright, "An Introduction to the Theory of Numbers", Oxford University Press, New York, fifth edition (1979).

[Pr93] E. Prisner, "Parallel chip firing on digraphs", Technical Report Mathematisches Seminar, Universität Hamburg, Germany (1993).

[Sp86] J. Spencer, "Balancing vectors in the max norm", *Combinatorica* **6** (1986), 55–66.

[Ta88] G. Tardos, "Polynomial bounds for a chip-firing game on graphs", *SIAM J. Discrete Math* **1** (1988), 397–398.