THE DISCRETE VARIATIONAL PROBLEM WITH RIGHT FOCAL CONSTRAINTS

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Abstract

We consider the discrete variational problem for which f(t,u,v) is a real-valued C2 function of u and v for each fixed t in the discrete interval [a+2,b+4], and is a functional whose domain consists of real-valued functions on the discrete interval [a,b+4]. Specifically, we consider right focal boundary conditions, i.e., we fix y(a), (y(a), (2y(b+2), and (3y(b+1). We prove several necessary and sufficient conditions.

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1. Introduction.

This paper seeks to continue the development of the theory for the discrete calculus of variations which was tackled by Fort [?] as early as 1948, by Cadzow [?] and Logan [?, ?, ?] in the 1970's, and which has received increasing attention in the last few years. Throughout this paper a discrete interval will be denoted by $[m, n] = [m,\mathrm{m+1}, ..., n]$. In particular we will deal with the interval $[a, b + 2]$, the motivation for which is found in Hartman's seminal paper on nth order linear difference equations [?]. In 1985 Ahlbrandt and Hooker published a paper dealing with discrete variational theory [?], and in 1992 Ahlbrandt published a paper on discrete variational inequalities [?]. These papers concerned conjugate boundary conditions, in which $y(a)$ and $y(b + 2)$ are fixed. These ideas were summarized in a recent textbook by Kelley and Peterson [?]. This paper will deal with analogous results using right focal boundary values, in which $y(a)$ and $\Delta y(b + 1)$ are fixed. I take this opportunity to thank my thesis advisor, Professor Allan Peterson, for his help in putting this paper together, Professor Roger Wiegand, for his supervision of much of my earlier training in mathematics, and my husband Gary, for his support, patience and love.

2. The Euler-Lagrange Necessary Condition.

For the conjugate problem, the discrete Euler-Lagrange equation holds on $[a+1, b+1]$. (see Chapter 8 in [?].) For the right focal problem, however, it is true only for $t \in [a+1, b]$. A different, but related condition, is satisfied for $t = b + 1$.

Definition 1 *Let* $f : [a, b+2] \times \Re^2 \to \Re$ *be a function of the variables* $t, u,$ *and* $v,$ *and assume that for each fixed* $t \in [a, b+2]$, $f \in C^2$ *with respect to* u *and* v. *Let* $A, B \in \Re$ *and define*

$$\mathcal{F} = \{y : [a, b+2] \to \Re; y(a) = A,\ \Delta y(b+1) = B\}.$$

Define J *on* $\mathcal{F}$ *by*

$$J[y] = \sum_{t=a+1}^{b+2} f(t, y(t), \Delta y(t-1)).$$

We call $\mathcal{F}$ *the* set of admissible functions for J.

The decision of what to use as the discrete replacement for y' in the expression $f(t, y(t), y'(t))$ has varied over the years. Fort [?] used $f(x_i, y(x_i), \Delta y(x_i))$. Cadzow [?] used $f(j, r_{j+1}, r_j)$. Logan used $f(n, r_n, r_{n-1})$ in two papers [?, ?] and $f(n, r_n+1, \Delta r_n)$ in a third paper [?]. This paper uses the formulation which Kelley and Peterson [?], Ahlbrandt [?] and Alhlbrandt and Hooker [?] used $f(t, y(t), \Delta y(t-1))$. With this formulation the theory of the second variation generates a self-adjoint Jacobi equation.

Definition 2 *The* set of admissible variations for $\mathcal{F}$ *will be denoted by*

$$\mathcal{F}_0 = \{\eta : [a, b+2] \to \Re; \eta(a) = 0,\ \Delta\eta(b+1) = 0\}.$$

Throughout the paper, free use will be made of the fact that $\eta(b+2) = \eta(b+1)$ for all $\eta \in \mathcal{F}_0$.

Definition 3 *Let* $y \in \mathcal{F}$. *The* first variation of J along y *is defined for all* $\eta \in \mathcal{F}_0$ *by*

$$J_1[\eta, y] = \sum_{t=a+1}^{b+2} \{f_u(t, y(t), \Delta y(t-1))\eta(t) + f_v(t, y(t), \Delta y(t-1))\Delta\eta(t-1)\}.$$

Definition 4 *Let a norm for* y *be defined by*

$$\|y\| = \max_{t \in [a, b+2]} |y(t)|.$$

Let $y_0 \in \mathcal{F}$. *Then* J *has a* local maximum *at* y_0 *if there is a* $\delta > 0$ *such that* $J[y] \leq J[y_0]$ *for all* $y \in \mathcal{F}$ *with* $\|y - y_0\| < \delta$, *and* J *has a* local minimum *at* y_0 *if there is a* $\delta > 0$ *such that* $J[y] \geq J[y_0]$ *for all* $y \in \mathcal{F}$ *with* $\|y - y_0\| < \delta$.

Theorem 1 *(Euler-Lagrange Necessary Condition) If J has a local maximum or a local minimum at $y_0 \in \mathcal{F}$, then $y_0(t)$ satisfies*

$$Ey(t) = \begin{cases} 0 \quad \textit{for } t \in [a+1, b] \\ -f_u(b+2, y(b+2), \Delta y(b+1)) \\ \qquad -f_v(b+2, y(b+2), \Delta y(b+1)) \ \textit{for} \ \ t = b+1. \end{cases} \tag{1}$$

where $Ey(t) = f_u(t, y(t), \Delta y(t-1)) - \Delta f_v(t, y(t), \Delta y(t-1))$.

Proof: Let $\eta \in \mathcal{F}_0$. Define $\Phi : \Re \to \Re$ by $\Phi(\epsilon) = J[y_0 + \epsilon\eta]$. Since y_0 is a local maximum or a local minimum for J, it follows that 0 is a local extremum for Φ. Hence $\Phi'(0) = 0$. We have

$$\begin{aligned} \Phi'(\epsilon) = \sum_{t=a+1}^{b+2} \{ & f_u(t, y_0(t) + \epsilon\eta(t), \Delta y_0(t-1) + \epsilon\Delta\eta(t-1))\eta(t) \\ & + f_v(t, y_0(t) + \epsilon\eta(t), \Delta y_0(t-1) + \epsilon\Delta\eta(t-1))\Delta\eta(t-1)\}. \end{aligned} \tag{2}$$

Thus

$$\Phi'(0) = J_1[\eta, y_0], \tag{3}$$

from which it follows that $J_1[\eta, y_0] = 0$ for all $\eta \in \mathcal{F}_0$. For any $y \in \mathcal{F}$ we can write $J_1[\eta, y]$ as

$$\begin{aligned} J_1[\eta, y] = \quad & \sum_{t=a+1}^{b+1} \{ f_u(t, y(t), \Delta y(t-1)) \ \eta(t) \\ & \quad + f_v(t, y(t), \Delta y(t-1)) \ \Delta\eta(t-1)\} \\ & + f_u(b+2, y(b+2), \Delta y(b+1))\eta(b+2). \end{aligned}$$

Summation by parts on the second term under the sum yields

$$\begin{aligned} J_1[\eta, y] \quad = \quad & \sum_{t=a+1}^{b+1} \{ f_u(t, y(t), \Delta y(t-1)) - \Delta f_v(t, y(t), \Delta y(t-1))\} \ \eta(t) \\ & + f_u(b+2, y(b+2), \Delta y(b+1))\eta(b+1) \\ & + f_v(b+2, y(b+2), \Delta y(b+1))\eta(b+2). \end{aligned} \tag{4}$$

Fix $s \in [a+1, b]$, and choose

$$\eta(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \in [a, b+2] \text{ but } t \neq s. \end{cases}$$

Then $\eta \in \mathcal{F}_0$. Since $J_1[\eta, y_0] = 0$, (**??**) yields

$$f_u(s, y_0(s), \Delta y_0(s-1)) - \Delta f_v(s, y_0(s), \Delta y_0(s-1)) = 0.$$

Thus the discrete Euler-Lagrange equation holds on $[a+1, b]$. Now choose

$$\eta(t) = \begin{cases} 1 & \text{if } t = b+1 \text{ or } t = b+2 \\ 0 & \text{if } t \in [a, b]. \end{cases}$$

Then $\eta \in \mathcal{F}_0$ and (**??**) yields

$$\begin{array}{l} f_u(b+1, y_0(b+1), \Delta y_0(b)) - \Delta f_v(b+1, y_0(b+1), \Delta y_0(b)) \\ + f_u(b+2, y_0(b+2), \Delta y_0(b+1)) + f_v(b+2, y_0(b+2), \Delta y_0(b+1)) = 0 \end{array}$$

from which the result follows. □

Corollary 1 *If J has a local maximum or a local minimum at $y_0 \in \mathcal{F}$, then*

$$\begin{aligned} f_u(b+1, y_0(b+1), \Delta y_0(b)) + f_u(b+2, y_0(b+2), \Delta y_0(b+1)) \\ + f_v(b+1, y_0(b+1), \Delta y_0(b)) = 0. \end{aligned}$$

3. The Second Variation as a Quadratic Form.

Definition 5 *Let $y \in \mathcal{F}$. The* second variation of J along y *is defined for all $\eta \in \mathcal{F}_0$ by*

$$\begin{aligned} J_2[\eta, y] = \sum_{t=a+1}^{b+2} \{ & f_{uu}(t, y(t), \Delta y(t-1))\eta^2(t) \\ & + 2f_{uv}(t, y(t), \Delta y(t-1))\eta(t)\Delta\eta(t-1) \\ & + f_{vv}(t, y(t), \Delta y(t-1))[\Delta\eta(t-1)]^2 \}. \end{aligned} \tag{5}$$

The second variation is a quadratic form in the variables $\eta(a+1), \ldots, \eta(b+1)$. It can be written in two ways which show more clearly its nature as a quadratic form. First, we introduce some notation.

Definition 6 *Let $y \in \mathcal{F}$. For $t \in [a+1, b+2]$, let*

$$\begin{array}{l} P(t) = P(t; y) = f_{uu}(t, y(t), \Delta y(t-1)) \\ Q(t) = Q(t; y) = f_{uv}(t, y(t), \Delta y(t-1)) \\ R(t) = R(t; y) = f_{vv}(t, y(t), \Delta y(t-1)) \\ p(t-1) = p(t-1; y) = Q(t; y) + R(t; y). \end{array}$$

For $t \in [a+1, b+1]$, let

$$\begin{array}{l} q(t) = q(t; y) = \Delta Q(t; y) - P(t; y) \\ c(t) = c(t; y) = q(t; y) - p(t; y) - p(t-1; y). \end{array}$$

Now we can write the second variation along y as follows:

$$J_2[\eta, y] = \sum_{t=a+1}^{b+2} \{P(t)\ \eta^2(t) + 2Q(t)\ \eta(t)\ \Delta\eta(t-1) + R(t)[\Delta\eta(t-1)]^2\}.$$

Since

$$2Q(t)\eta(t)\Delta\eta(t-1) = Q(t)\eta^2(t) + Q(t)[\Delta\eta(t-1)]^2 - Q(t)\eta^2(t-1),$$

we have

$$\begin{aligned} J_2[\eta, y] &= \sum_{t=a+1}^{b+2} \{P(t)\eta^2(t) + Q(t)\eta^2(t) - Q(t)\eta^2(t-1) \\ &\quad + [Q(t) + R(t)]\,[\Delta\eta(t-1)]^2\}. \end{aligned}$$

Using $\eta(a) = 0$, and $\Delta\eta(b+1) = 0$, and the fact that

$$-\sum_{t=a+1}^{b+2} Q(t)\eta^2(t-1) = -\sum_{t=a+1}^{b+1} Q(t+1)\eta^2(t),$$

we have

$$\begin{aligned} J_2[\eta, y] &= \sum_{t=a+1}^{b+1} [Q(t) + R(t)][\Delta\eta(t-1)]^2 \\ &\quad + \sum_{t=a+1}^{b+1} [P(t) + Q(t) - Q(t+1)]\eta^2(t) \\ &\quad + [P(b+2) + Q(b+2)]\eta^2(b+2). \end{aligned}$$

Recalling the definitions for p and q, we have the following form for the second variation:

$$\begin{aligned} J_2[\eta, y] &= \sum_{t=a+1}^{b+1} \Big\{p(t-1)[\Delta\eta(t-1)]^2 - q(t)\eta^2(t)\Big\} \\ &\quad + [P(b+2) + Q(b+2)]\eta^2(b+2). \end{aligned} \tag{6}$$

To derive another form for $J_2[\eta, y]$, we continue. Squaring out the $[\Delta\eta(t-1)]^2$ term and using $\eta(b+2) = \eta(b+1)$, we have

$$\begin{aligned} J_2[\eta, y] &= \sum_{t=a+1}^{b+1} \{p(t-1)\eta^2(t) - 2p(t-1)\eta(t)\eta(t-1) \\ &\quad + p(t-1)\eta^2(t-1) - q(t)\eta^2(t)\} \\ &\quad + [P(b+2) + Q(b+2)]\eta^2(b+1). \end{aligned}$$

Rewriting

$$\sum_{t=a+1}^{b+1} p(t-1)\eta^2(t-1) \quad = \quad \sum_{t=a+1}^{b+1} p(t)\eta^2(t) - p(b+1)\eta^2(b+1),$$

and

$$2\sum_{t=a+1}^{b+1} p(t-1)\eta(t)\eta(t-1) = 2\sum_{t=a+1}^{b} p(t)\eta(t+1)\eta(t),$$

we see that

$$\begin{aligned} J_2[\eta, y] \quad = \quad & \sum_{t=a+1}^{b+1} \{p(t-1) + p(t) - q(t)\}\eta^2(t) \\ & -2\sum_{t=a+1}^{b} p(t)\eta(t+1)\eta(t) \\ & +[-p(b+1) + P(b+2) + Q(b+2)]\eta^2(b+1). \end{aligned}$$

Rewriting $-p(b+1)+P(b+2)+Q(b+2) = P(b+2)-R(b+2)$ and recalling the definition of $c(t)$, we obtain a another useful form,

$$\begin{aligned} J_2[\eta, y] = -\sum_{t=a+1}^{b+1} c(t)\eta^2(t) - 2\sum_{t=a+1}^{b} p(t)\eta(t+1)\eta(t) \\ +[P(b+2) - R(b+2)]\eta^2(b+1). \end{aligned} \qquad (7)$$

We can also write (**??**) as follows: Let $\eta \in \mathcal{F}_0$, and let $v^T = [\eta(a+1)\ldots\eta(b+1)]$. For each $y \in \mathcal{F}$ let $S = c(b+1) + R(b+2) - P(b+2)$, and define the tridiagonal matrix $A(y)$ by

$$A(y) = \begin{bmatrix} c(a+1) & p(a+1) & & & & & \\ p(a+1) & c(a+2) & p(a+2) & & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & p(b-1) & c(b) & p(b) \\ & & & & & p(b) & S \end{bmatrix},$$

where $c(t), p(t)$, and S depend on y. Then

$$J_2[\eta, y] = -v^T A(y)\ v. \qquad (8)$$

(See Chapter 8 in [**?**] for a similar procedure for the discrete conjugate problem. The matrix, however, differs from that used here.)

4. The Legendre Necessary Condition.

As with the discrete Euler-Lagrange equation, the discrete Legendre inequality holds only for $t \in [a+1, b]$ for the right focal problem, with a different but related condition at $t = b+1$. The Legendre inequality concerns $c(t)$ which appeared in (**??**).

Theorem 2 *(Legendre Necessary Condition)*

a) If J has a local minimum at $y_0 \in \mathcal{F}$, then

$$c(t; y_0) \le \begin{cases} 0 \ \textit{if} \ \ t \in [a+1, b] \\ P(b+2; y_0) - R(b+2; y_0) \ \textit{if} \ t = b+1. \end{cases}$$

b) If J has a local maximum at $y_0 \in \mathcal{F}$, then

$$c(t; y_0) \ge \begin{cases} 0 \ \textit{if} \ \ t \in [a+1, b] \\ P(b+2; y_0) - R(b+2; y_0) \ \textit{if} \ t = b+1. \end{cases}$$

Proof: Let $y_0 \in \mathcal{F}$ be a local minimum and let $\eta \in \mathcal{F}_0$. As in Theorem **??**, define $\Phi(\epsilon) = J[y_0 + \epsilon\eta]$. Since y_0 is a local minimum for J, 0 is a local minimum for Φ. Thus $\Phi''(0) \ge 0$. Using the expression for $\Phi'(\epsilon)$ given by (**??**) we see that

$$\begin{aligned} \Phi''(\epsilon) \;\; = \;\; & \sum_{t=a+1}^{b+2} \Big\{ f_{uu}(t, y_0(t) + \epsilon\eta(t), \Delta y_0(t-1) + \epsilon\Delta\eta(t-1))\eta^2(t) \\ & + 2 f_{uv}(t, y_0(t) + \epsilon\eta(t), \Delta y_0(t-1) + \epsilon\Delta\eta(t-1))\eta(t)\Delta\eta(t-1) \\ & + f_{vv}(t, y_0(t) + \epsilon\eta(t), \Delta y_0(t-1) + \epsilon\Delta\eta(t-1))[\Delta\eta(t-1]^2 \Big\}. \end{aligned}$$

By (**??**)

$$\Phi''(\epsilon) = J_2[\eta, y_0 + \epsilon\eta], \tag{9}$$

from which it follows that $\Phi''(0) = J_2[\eta, y_0]$. Therefore

$$J_2[\eta, y_0] \ge 0$$

for all $\eta \in \mathcal{F}_0$. Fix $s \in [a+1, b]$ and set

$$\eta(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \ne s \text{ for } t \in [a, b+2]. \end{cases}$$

Then $\eta \in \mathcal{F}_0$. By substituting $\eta(t)$ and $y_0(t)$ into (**??**) we see that

$$J_2[\eta, y_0] = -c(s; y_0).$$

Therefore $c(s; y_0) \le 0$ for all $s \in [a+1, b]$. Let $s = b+1$ and choose

$$\eta(t) = \begin{cases} 0 \text{ if } t \in [a,b] \\ 1 \text{ if } t = b+1 \text{ or } t = b+2. \end{cases}$$

Then $\eta \in \mathcal{F}_0$. By substituting $\eta(t)$ and $y_0(t)$ into (**??**) we get

$$J_2[\eta, y_0] = -c(b+1; y_0) + P(b+2; y_0) - R(b+2; y_0).$$

from which a) follows. If y_0 is a local maximum, the proof is similar. □

5. The First Sufficient Condition.

If $y_0 \in \mathcal{F}_0$ satisfies (**??**) and, in addition, the second variation along y_0 can be determined to be either positive definite or negative definite on $\mathcal{F}_0$, then we know that J has a proper local maximum or a proper local minimum at y_0. Under certain circumstances, this maximum or minimum can be shown to be global.

Theorem 3 *(First Sufficient Condition) Suppose $y_0 \in \mathcal{F}$ satisfies (**??**). If $J_2[\eta, y_0]$ is positive(negative) definite on $\mathcal{F}_0$, then J has a proper local minimum(maximum) at y_0.*

Proof: Suppose $y_0 \in \mathcal{F}$ satisfies (**??**) and $J_2[\eta, y_0]$ is positive definite on $\mathcal{F}_0$. We want to show that if $y \neq y_0$ is close enough to y_0, then $J[y] > J[y_0]$. For the moment choose any $y \neq y_0 \in \mathcal{F}$ and let $\eta = y - y_0$. Then $\eta \in \mathcal{F}_0$ and $\eta \neq 0$. Consider $\Phi(\epsilon) = J[y_0 + \epsilon\eta]$. By Taylor's theorem,

$$\Phi(1) = \Phi(0) + \Phi'(0)/1! + \Phi''(\zeta)/2!$$

for some $\zeta \in (0,1)$. By the derivation of (**??**) and (**??**) we obtain

$$\begin{aligned} \Phi'(0) = \sum_{t=a+1}^{b+1} \{f_u(t, y_0(t), \Delta y_0(t-1)) - \Delta f_v(t, y_0(t), \Delta y_0(t-1))\}\eta(t) \\ +\{f_u(b+2, y_0(b+2), \Delta y_0(b+1)) \\ +f_v(b+2), y_0(b+2), \Delta y_0(b+1))\}\eta(b+1). \end{aligned}$$

Applying (**??**) to the right hand side of this equation, we see that $\Phi'(0) = 0$. It is clear that $\Phi(1) = J[y]$, and $\Phi(0) = J[y_0]$. By the derivation of (**??**), we have $\Phi''(\zeta) = J_2[\eta, y_0 + \zeta\eta]$. Hence Taylor's Theorem yields

$$J[y] - J[y_0] = \frac{\Phi''(\zeta)}{2} = \frac{J_2[\eta, y_0 + \zeta\eta]}{2}. \tag{10}$$

Since $J_2[\eta, y_0]$ is positive definite on $\mathcal{F}_0$, and $J_2[\eta, y_0] = -v^T A(y_0)v$, it follows that the matrix $A(y_0)$ is negative definite. Hence all the eigenvalues of $A(y_0)$ are negative. In particular, the maximum eigenvalue $\lambda_{max}A(y_0) = \mu$ for some $\mu < 0$. Since f is a C^2 function of u and v, the entries of $A(y)$ are continuous functions of y. Thus, since the eigenvalues of a square matrix depend continuously upon its entries (See Horn and Johnson [?]) and the eigenvalues of a symmetric matrix are real, the maximum eigenvalue of $A(y)$ is a continuous real-valued function of y. Therefore there exists a $\delta > 0$ such that whenever $\|y - y_0\| < \delta$, then $\lambda_{max}A(y) \leq \mu/2 < 0$. Now

$$\|(y_0 + \zeta\eta) - y_0\| = \zeta \|\eta\| = \zeta \|y - y_0\| < \delta.$$

Thus, if the y chosen at the beginning of the proof satisfies $\|y - y_0\| < \delta$, it follows that $\lambda_{max}A(y_0 + \zeta\eta) < 0$. In that case all the eigenvalues of $A(y_0 + \zeta\eta)$ are negative, so $A(y_0 + \zeta\eta)$ is negative definite. It follows that $J_2[\eta, y_0 + \zeta\eta]$ is positive definite on $\mathcal{F}_0$. Thus by (**??**) $J[y] > J[y_0]$ for $\|y - y_0\| < \delta$, $y \neq y_0$, giving a proper local minimum at y_0. The proof for a proper local maximum is similar. □

The following corollary follows from a close look at the proof of Theorem **??** and the entries of the matrix A(y).

Corollary 2 *Suppose $y_0 \in \mathcal{F}$ satisfies (**??**) and $J_2[\eta, y_0]$ is positive (negative) definite on $\mathcal{F}_0$. If, in addition, f_{uu}, f_{uv}, and f_{vv} are functions of t only, then J has a proper global minimum(maximum) at y_0.*

By examining (**??**) we obtain another corollary.

Corollary 3 *Suppose $y_0 \in \mathcal{F}$ satisfies (**??**). If $J_2[\eta, y]$ is positive(negative) definite on $\mathcal{F}_0$ for all $y \in \mathcal{F}$, then J has a proper global minimum(maximum) at y_0.*

6. C-Disfocality and the Second Sufficient Condition.

Disfocality is a concept related to disconjugacy. The letter C in C-disfocal stands for Coppel and comes from his study in [?] of a similar definition of disfocality for the continuous case. The matrix version for the discrete case is studied in Peil and Peterson [?] and Peterson [?]. If we apply Theorem **??** to the second variation, we get the Jacobi equation. In the conjugate problem, this is a self-adjoint equation, where the discrete self-adjoint operator L is defined on the set of real-valued functions with domain $[a, b+2]$ by

$$Lu(t) = \Delta\left\{\rho(t-1)\Delta u(t-1)\right\} + \psi(t)u(t) \tag{11}$$

for $\rho(t)$ defined and nonzero for all $t \in [a, b+1]$, and $\psi(t)$ defined on $[a+1, b+1]$. (See [?] and in Chapter 6 of [?] for a discussion of the scalar self-adjoint equation, and [?, ?, ?, ?, ?] for the matrix case.) Since (**??**) differs from the Euler-Lagrange equation at $t = b+1$, we will define an operator similar to L but different at $t = b+1$.

Definition 7 *Fix $y \in \mathcal{F}$. Let H be a constant, and let ρ and ψ be as in (??). In addition, assume $\rho(b+1) - H \neq 0$. Define the operator $M = M(\rho, \psi, H)$ on the set of real-valued functions with domain $[a, b+2]$ by*

$$Mu(t) = \begin{cases} Lu(t) \text{ for } t \in [a+1, b] \\ Lu(b+1) - Hu(b+2) \text{ for } t = b+1. \end{cases} \tag{12}$$

Definition 8 *The second order difference equation*

$$Mu(t) = 0 \text{ for } t \in [a+1, b+1] \tag{13}$$

is C-disfocal *on $[a, b+2]$ if whenever $u(t)$ is a nontrivial solution of (??) such that $\Delta u(b+1) = 0$, then $\rho(t-1)u(t)u(t-1) > 0$ for all $t \in [a+1, b+1]$.*

Notice that by specifying $\rho(t) \neq 0$ for $t \in [a, b+1]$, and $\rho(b+1) - H \neq 0$ in the definition of the operator M, we have ensured that any solution of (**??**) is actually valid on $[a, b+2]$.

This definition for disfocality could also have be stated in terms of generalized zeros. (See Hartman [**?**].) There is also an alternate type of disfocality which is studied in Peil [**?**] and Peterson and Ridenhour [**?**] for nth order linear difference equations.

The following theorem is similar to that given by Peil and Peterson [**?**], and Peterson [**?**]. The quadratic form used in those papers, however, is based on the second variation for the conjugate problem, while this paper uses the second variation for the right focal problem. It is interesting that the theorem still holds. (See Reid [**?**] for a continuous version for matrix equations, and see Chapter 8 of [**?**] for a similar theorem relating to the disconjugacy of $Ly(t) = 0$.)

For the proof of the theorem we will use the discrete Riccati operator, $\mathcal{R}$, defined on the set of real-valued functions z with domain $[a+1, b+2]$, such that $z(t) + \rho(t-1) \neq 0$ on $[a+1, b+1]$ by

$$\mathcal{R}z(t) = \Delta z(t) + \psi(t) + \frac{z^2(t)}{z(t) + \rho(t-1)} \tag{14}$$

for $\rho(t)$ defined and nonzero on $[a, b]$ and $\psi(t)$ defined on $[a+1, b+1]$.

Theorem 4 *Let*

$$\mathcal{Q}[\eta] = \sum_{t=a+1}^{b+1} \{\rho(t-1)[\Delta\eta(t-1)]^2 - \psi(t)\eta^2(t)\} + Hy^2(b+2).$$

Then $Mu(t) = 0$ is C-disfocal on $[a, b+2]$ if and only if $\mathcal{Q}[\eta]$ is positive definite on $\mathcal{F}_0$.

Proof: $\Longrightarrow$ Assume $Mu(t) = 0$ is C-disfocal on $[a, b+2]$. Let $u(t)$ be the solution of the initial value problem

$$\begin{aligned} Mu(t) &= 0 \text{ for } t \in [a+1, b+1] \\ u(b+1) &= u(b+2) = 1. \end{aligned}$$

By the definition of C-disfocal $\rho(t-1)u(t)u(t-1) > 0$ on $[a+1, b+1]$. Set

$$z(t) = \frac{\rho(t-1)\Delta u(t-1)}{u(t-1)} \text{ for } t \in [a+1, b+2].$$

Since $Mu(t) = 0$ is C-disfocal on $[a, b+2]$, we have $z(t) + \rho(t-1) > 0$ on $[a+1, b+2]$. Operating on $z(t)$ by the Riccati operator $\mathcal{R}$, simplifying, using (**??**) and the initial values, we have for all $t \in [a+1, b+1]$,

$$\mathcal{R}z(t) = \begin{cases} 0 \text{ for } t \in [a+1, b] \\ H \text{ for } \ t = b+1. \end{cases} \tag{15}$$

Let $\eta \in \mathcal{F}_0$. For all $t \in [a+1, b+1]$, the product rule gives

$$\Delta[z(t)\eta^2(t-1)] = \eta^2(t)\Delta z(t) + z(t)\eta(t)\Delta\eta(t-1) + z(t)\eta(t-1)\Delta\eta(t-1).$$

For $t \in [a+1, b]$, (**??**) and (**??**) give us a replacement for $\Delta z(t)$, so that

$$\begin{aligned} \Delta\left[z(t)\eta^2(t-1)\right] &= \eta^2(t)\left[-\psi(t) - \frac{z^2(t)}{z(t)+\rho(t-1)}\right] \\ &\quad + z(t)\eta(t)\Delta\eta(t-1) + z(t)\eta(t-1)\Delta\eta(t-1). \end{aligned}$$

Writing $z(t)\eta(t-1)\Delta\eta(t-1) = -z(t)[\Delta\eta(t-1)]^2 + z(t)\eta(t)\Delta\eta(t-1)$, and recalling that $z(t) + \rho(t-1) > 0$ on $[a+1, b+2]$, we have

$$\begin{aligned} \Delta[z(t)\eta^2(t-1)] &= \rho(t-1)[\Delta\eta(t-1)]^2 - \psi(t)\eta^2(t) \\ &\quad - \left(\frac{z(t)\eta(t)}{\sqrt{z(t)+\rho(t-1)}} - \Delta\eta(t-1)\sqrt{z(t)+\rho(t-1)}\right)^2. \end{aligned}$$

At $t = b+1$, (**??**) and (**??**) give us a replacement for $\Delta z(b+1)$, so that

$$\begin{aligned} \Delta[z(b+1)\eta^2(b)] &= \rho(b)[\Delta\eta(b)]^2 - \psi(b+1)\eta^2(b+1) + H\eta^2(b+1) \\ &\quad - \left(\frac{z(b+1)\eta(b+1)}{\sqrt{z(b+1)+\rho(b)}} - \Delta\eta(b)\sqrt{z(b+1)+\rho(b)}\right)^2. \end{aligned}$$

If we sum $\Delta[z(t)\eta^2(t-1)]$ from $a+1$ to $b+1$ we see that

$$\begin{aligned} z(t)\eta^2(t-1)\Big|_{t=a+1}^{b+2} &= \sum_{t=a+1}^{b+1}\left\{\rho(t-1)[\Delta\eta(t-1)]^2-\psi(t)\eta^2(t)\right\}+H\eta^2(b+1) \\ &\quad -\sum_{t=a+1}^{b+1}\left(\frac{z(t)\eta(t)}{\sqrt{z(t)+\rho(t-1)}}-\Delta\eta(t-1)\sqrt{z(t)+\rho(t-1)}\right)^2. \end{aligned}$$

Since $z(b+2)=0$, this simplifies to

$$\mathcal{Q}[\eta] = \sum_{t=a+1}^{b+1}\left(\frac{z(t)\eta(t)}{\sqrt{z(t)+\rho(t-1)}}-\Delta\eta(t-1)\sqrt{z(t)+\rho(t-1)}\right)^2.$$

Hence $\mathcal{Q}[\eta]\geq 0$ for all $\eta\in\mathcal{F}_0$.
If $\eta=0$, it is clear that $\mathcal{Q}[\eta]=0$. On the other hand, if $\mathcal{Q}[\eta]=0$, then for all $t\in[a+1,b+1]$,

$$\frac{z(t)\eta(t)}{\sqrt{z(t)+\rho(t-1)}}-\Delta\eta(t-1)\sqrt{z(t)+\rho(t-1)}=0,$$

from which it follows that

$$\eta(t)=\frac{z(t)+\rho(t-1)}{\rho(t-1)}\eta(t-1) \text{ for } t\in[a,b+1].$$

Since $\eta(a)=0$ and $\Delta\eta(b+1)=0$ we have $\eta(t)=0$ for $t\in[a,b+2]$. Hence $\mathcal{Q}$ is positive definite on $\mathcal{F}_0$.
$\Longleftarrow$ Assume $\mathcal{Q}$ is positive definite on $\mathcal{F}_0$. Suppose $Mu(t)=0$ is not C-disfocal on $[a,b+2]$. Then it has a nontrivial solution $u(t)$ such that $\Delta u(b+1)=0$ and such that for some $t_0\in[a+1,b+1]$, we have $\rho(t_0-1)u(t_0)u(t_0-1)\leq 0$. Choose

$$\eta(t)=\begin{cases} 0 & \text{for } t\in[a,t_0-1] \\ u(t) & \text{for } t\in[t_0,b+2]. \end{cases}$$

It is clear that $\eta\in\mathcal{F}_0$. Since u is nontrivial and $\Delta u(b+1)=0$, we can not have $u(b+1)=0$. Hence η is nontrivial. Summation by parts on the first term of

$$\mathcal{Q}[\eta] = \sum_{t=a+1}^{b+1}\left\{\rho(t-1)[\Delta\eta(t-1)]^2-\psi(t)\eta^2(t)\right\}+H\eta^2(b+2)$$

gives

$$\begin{aligned}\mathcal{Q}[\eta] &= \rho(t-1)\Delta\eta(t-1)\eta(t-1)\Big|_{a+1}^{b+2} \\ &\quad - \sum_{t=a+1}^{b+1}\left\{\Delta[\rho(t-1)\Delta\eta(t-1)] + \psi(t)\eta(t)\right\}\eta(t) + H\eta^2(b+2) \\ &= -\sum_{t=a+1}^{b}\left\{\Delta[\rho(t-1)\Delta\eta(t-1)] + \psi(t)\eta(t)\right\}\eta(t) \\ &\quad -\{\Delta[\rho(b)\Delta\eta(b)] + \psi(b+1)\eta(b+1) - H\eta(b+2)\}\eta(b+1) \\ &\quad = -\sum_{t=a+1}^{b+1} M\eta(t)\eta(t).\end{aligned}$$

Since $\eta(t) = 0$ if $t \in [a, t_0 - 1]$ and $M\eta(t) = Mu(t) = 0$ if $t \in [t_0 + 1, b + 1]$, we have

$$\mathcal{Q}[\eta] = -M\eta(t_0)\eta(t_0).$$

If $t_0 \leq b$, we expand $M\eta(t_0)$ and replace $\eta(t_0 - 1)$ by $u(t_0 - 1) - u(t_0 - 1)$, then use $Mu(t_0) = 0$ to get

$$\begin{aligned}\mathcal{Q}[\eta] &= -\{\Delta[\rho(t_0-1)\Delta\eta(t_0-1)] + \psi(t_0)\eta(t_0)\}u(t_0) \\ &= -\{\Delta[\rho(t_0-1)\Delta u(t_0-1)] + \psi(t_0)u(t_0)\}u(t_0) \\ &\quad +\rho(t_0-1)u(t_0-1)u(t_0) \\ &= \rho(t_0-1)u(t_0-1)u(t_0).\end{aligned}$$

Similiarly, if $t_0 = b + 1$, we expand $M\eta(b + 1)$, replace $\eta(b)$ by $u(b) - u(b)$, and $\eta(b + 2)$ by $u(b + 2)$, and use $Mu(b + 1) = 0$ to get

$$\begin{aligned}\mathcal{Q}[\eta] &= -\{\Delta[\rho(b)\Delta\eta(b)] + \psi(b+1)\eta(b+1) - H\eta(b+2)\}u(b+1) \\ &= -\{\Delta[\rho(b)\Delta u(b)] + \psi(b+1)u(b+1) - Hu(b+2)\}u(b+1) \\ &\quad +\rho(b)u(b)u(b+1) \\ &= \rho(b)u(b)u(b+1).\end{aligned}$$

By our supposition, this means $\mathcal{Q}[\eta] \leq 0$. Since η is nontrivial, this contradicts the hypothesis that $\mathcal{Q}[\eta]$ is positive definite on $\mathcal{F}_0$. Hence $Mu(t) = 0$ is C-disfocal on $[a, b+2]$.

□

For the second sufficient condition, we will specify ρ, ψ and H so that $\mathcal{Q}[\eta]$ becomes $J_2[\eta, y]$.

Theorem 5 *(Second Sufficient Condition) Suppose $y_0 \in \mathcal{F}$ satisfies (**??**). Let $P, Q, p,$ and q be defined as in Definition **??**. Let $H = P(b+2)+Q(b+2)$ and define $M_y = M(p, q, H)$. If $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b + 2]$, then J has a proper local minimum at y_0.*

Proof: If $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b+2]$, it follows from Theorem **??** that $J_2[\eta, y_0]$ is positive definite on $\mathcal{F}_0$. By Theorem **??**, J has a proper local minimum at y_0. □

The following corollaries to Theorem **??** make use of corollaries **??** and **??** respectively.

Corollary 4 *Suppose $y_0 \in \mathcal{F}$ satisfies (**??**), and $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b+2]$. If, in addition, f_{uu}, f_{uv}, and f_{vv} are functions of t only, then J has a proper global minimum at y_0.*

Corollary 5 *Suppose $y_0 \in \mathcal{F}$ satisfies (**??**). If $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b+2]$ for all $y \in \mathcal{F}$, then J has a proper global minimum at y_0.*

7. Quadratic Functionals and the Weierstrass Summation Formula.

We define the quadratic functional Q by

$$Q[y] = \sum_{t=a+1}^{b+2} \{\rho(t-1)[\Delta y(t-1)]^2 - \psi(t)y^2(t)\}, \tag{16}$$

subject to $y \in \mathcal{F}$. In particular, for any J, the second variation J_2 is a quadratic functional.

For the conjugate problem, the Jacobi equation for an arbitrary functional J is $L\eta(t) = 0$, and the Euler-Lagrange equation for the quadratic functional Q is $Ly(t) = 0$. This analogy does not carry over to the right focal problem. To see this, notice that if we apply (**??**) to the quadratic functional (**??**) and simplify we get

$$Ly(t) = \begin{cases} 0 \text{ for } t \in [a+1, b] \\ -\psi(b+2)y(b+2) + \rho(b+1)\Delta y(b+1) \text{ for } t = b+1, \end{cases} \tag{17}$$

while if we apply (**??**) to

$$Q_2[\eta, y] = \sum_{t=a+1}^{b+1} \{2\rho(t-1)[\Delta\eta(t-1)]^2 - 2\psi(t)\eta^2(t)\} - 2\psi(t)\eta^2(b+2)$$

and simplify, we get

$$L\eta(t) \quad = \quad \begin{cases} 0 \text{ for } t \in [a+1, b] \\ -\psi(t)\eta(b+2) \text{ for } t = b+1, \end{cases} \tag{18}$$

which is equivalent to $M_y u(t) = 0$, where M_y is defined as in Theorem **??**. Clearly, it is the boundary condition $\Delta\eta(b+1) = 0$ which causes these two equations to differ at $t = b+1$.

If $f(t, u, v) = \rho(t-1)v^2 - \psi(t)u^2$, the second partials with respect to u and v depend on t only. This gives us the following corollaries of Theorem **??** and Theorem **??** respectively.

Corollary 6 *Suppose Q is a quadratic functional. If y_0 satisfies (**??**) and if the second variation, $Q_2[\eta, y]$, of Q along y is positive(negative) definite on $\mathcal{F}_0$, then Q has a proper global minimum (maximum) at y_0.*

Corollary 7 *Suppose Q is a quadratic functional. If y_0 satisfies (**??**), and if $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b+2]$, then Q has a proper global minimum at y_0.*

A discrete version of the Weirstrass Integral Formula was proved in 1985 by Ahlbrandt and Hooker [**?**] for the conjugate problem, using the operator L. (See also [**?**].) The proof of the adaptation to the right focal problem is given here. Note that it concerns (**??**) rather than the operator M.

Theorem 6 *(Weierstrass Summation Formula) Let Q be a quadratic functional. Assume that $y \in \mathcal{F}$ satisfies (**??**). If $\eta \in \mathcal{F}_0$, then $Q[y+\eta] = Q[y] + Q[\eta]$.*

Proof: By the previous discussion, y satisfies (**??**). Since the derivation of (**??**) in Theorem **??** did not depend on y_0 being a minimum, we can apply it here to get

$$Q[y+\eta] - Q[y] \quad = \quad \frac{Q_2[\eta, y+\zeta\eta]}{2}$$

for some $\zeta \in (0,1)$. Comparing the second variation of Q with Q itself, we see that

$$\begin{aligned} Q_2[\eta, y+\zeta\eta] &= \sum_{t=a+1}^{b+1} \{2\rho(t-1)[\Delta\eta(t-1)]^2 - 2\psi(t)\eta^2(t)\} - 2\psi(t)\eta^2(b+2) \\ &= \sum_{t=a+1}^{b+2} \{2\rho(t-1)[\Delta\eta(t-1)]^2 - 2\psi(t)\eta^2(t)\} \\ &= 2Q[\eta]. \end{aligned}$$

Thus $Q[y+\eta] = Q[y] + Q[\eta]$. □**8.**

Comparison Theorems There are two comparison theorems which, although not used in the remainder of the paper, are such immediate consequences of Theorem **??** that they are stated here without proof.

Corollary 8 *(Sturm Comparison Theorem) Assume that $\rho_2(t) \geq \rho_1(t)$ and $\rho_1(t) \neq 0$ on $[a+1, b+1]$, $\psi_1(t) \geq \psi_2(t)$ on $[a, b+1]$, and $H_2 \geq H_1$. Let*

$$M_1 = M(H_1, \rho_1, \psi_1)$$
$$M_2 = M(H_2, \rho_2, \psi_2.$$

If $M_1u(t) = 0$ is C-disfocal on $[a, b+2]$ then $M_2u(t) = 0$ is C-disfocal on $[a, b+2]$.

Corollary 9 *Let* ρ_1 *and* ρ_2 *be defined and nonzero on* $[a, b+1]$, *and* ψ *be defined on* $[a+1, b+1]$. *Let* $M_1 = M(H_1, \rho_1, \psi_1)$, *and* $M_2 = M(H_2, \rho_2, \psi_2)$. *Also let* $H = H_1 + H_2$, $\rho(t) = \lambda_1\rho_1(t) + \lambda_2\rho_2(t)$, $\psi(t) = \lambda_1\psi_1(t) + \lambda_2\psi_2(t)$, *where* $\lambda_1 > 0,\ \lambda_2 > 0$, *and* $\rho(t) \neq 0$ *on* $[a, b+1]$. *If* M_1 *and* M_2 *are C-disfocal on* $[a, b+2]$, *and* $Mu(t) = M_1u(t) + M_2u(t)$, *then* $Mu(t) = 0$ *is C-disfocal on* $[a, b+2]$.

9. Examples.

First, we look at an example whose solution is intuitively clear.

Example 1 *Find the shortest polygonal path beginning at* (a, A) *where* A *is fixed, and ending at* $(b+2, y)$ *where the slope for the last segment of the path is fixed but* y *is free.*

Solution: Let the left endpoint of any path y be (a, A) and let $\Delta y(b+1) = m$. Let the horizontal lengths of the line segments of the path be given by $d(t) > 0$ for $t \in [a+1, b+2]$. To find the shortest path, we must minimize

$$J[y] = \sum_{t=a+1}^{b+2} \sqrt{[d(t)]^2 + [\Delta y(t-1)]^2} \text{ with } y \in \mathcal{F}, y(a) = A,\ \Delta y(b+1) = m.$$

Any solution $y_0(t)$ must satisfy (**??**), yielding

$$\begin{cases} \Delta\left[\dfrac{\Delta y_0(t-1)}{\sqrt{[d(t)]^2 + [\Delta y_0(t-1)]^2}}\right] & = \ 0 \text{ for } t \in [a+1, b] \\ \Delta y_0(b) & = \ 0. \end{cases}$$

Taking the indefinite sum of the first of these two equations, inverting the result, squaring, and simplifying, we obtain

$$\frac{d(t)}{\Delta y_0(t-1)} = C \text{ for all } t \in [a+1, b].$$

for some constant C. Since this is the slope of a line segment on the polygonal path, it follows that $y_0(t)$ is a straight line on $[a, b+1]$. This fact, together with $\Delta y_0(b) = 0$ and the boundary conditions, gives the following as the only possible candidate for a maximum or a minimum.

$$y_0(t) = \begin{cases} A \text{ for } t \in [a, b+1] \\ A + m \text{ for } t = b+2, \end{cases}$$

which is what we would expect. We see that $J_2[\eta, y]$ is positive definite for all $y \in \mathcal{F}$, so J has a proper global minimum at y_0 by Corollary **??**.

For our next example, we look at a quadratic functional.

Example 2 *Find the global minimum for*

$$J[y] = \sum_{t=a+1}^{b+2} \left\{(1/8)^{t-1}[\Delta y(t-1)]^2 - 3(1/8)^t y^2(t)\right\}$$

with $y(a) = A$ *and* $\Delta y(b+1) = m$.

Solution: Any solution y_0 must satisfy (**??**), yielding

$$\begin{cases} y_0(t+1) - 6y_0(t) + 8y_0(t-1) = 0 \text{ for } t \in [a+1, b] \\ 3\, y_0(b+2) - 40\, y_0(b+1) + 64\, y_0(b) = 0. \end{cases}$$

Using these equations and the boundary conditions we get the following as the only possible candidate for a maximum or a minimum:

$$y_0(t) = \begin{cases} C2^t + F4^t & \text{for } t \in [a, b+1] \\ C2^{b+1} + F4^{b+1} + m & \text{for } t = b+2, \end{cases}$$

where

$$C = \frac{84A\, 4^b - 3m4^a}{4^a\, 2^b\, (84\, 2^{b-a} - 10)}$$

and

$$F = \frac{-10\, A\, 2^b + 3m\, 2^a}{4^a\, 2^b\, (84\, 2^{b-a} - 10)}.$$

To determine whether this is actually a maximum or a minimum, we consider $M_{y_0}u(t) = 0$ on $[a+1, b+1]$, which simplifies to the set of equations

$$\begin{cases} u(t+1) - 6u(t) + 8u(t-1) = 0 \text{ on } t \in [a+1, b] \\ 11\, u(b+2) - 48\, u(b+1) + 64u(b) = 0. \end{cases}$$

Suppose u is a solution such that $\Delta u(b+1) = 0$. Then if $u(a) = A$,

$$u(t) = \frac{84A\, 2^b}{4^a(84\, 2^{b-a} - 10)}2^t + \frac{-10A}{4^a(84\, 2^{b-a} - 10)}4^t \text{ for } t \in [a, b+1].$$

Since u is nontrivial, $A \neq 0$. If $A > 0$, then $u(t) > 0$ for all $t \in [a, b+1]$, and if $A < 0$, then $u(t) < 0$ for all $t \in [a, b+1]$. It follows that $p(t-1)u(t)u(t-1) > 0$ for $t \in [a+1, b+1]$. Hence $M_{y_0}u(t) = 0$ is C-disfocal on $[a, b+2]$. Thus $y_0(t)$ is a proper global minimum by Corollary **??**.

9. Conclusion.

The standard necessary and sufficient conditions hold for the right focal problem, as long as careful attention is given to the definitions and the theorems at the value $t = b+1$. The motivation for looking at right focal boundary values in the variational problem comes at least partly from the interest that has developed in the relationship between positive definite quadratic forms and C-disfocality. I believe this is the first paper in which the quadratic form is based on the second variation for the right focal problem when trying to show the equivalence of C-disfocality and positive definiteness.

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