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BLOCH VARIETIES OF HIGHER-DIMENSIONAL, PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We relax the regularity conditions on potentials of higher-dimensional periodic Schrödinger operators while their resolvents may still be defined as compact operators on L^2. This enables us to define the Bloch varieties locally as the zero locus of a holomorphic map in a more general setting. We also give an asymptotic description of the Fermi curve.

1. Introduction

Given a lattice $\Gamma \subseteq \mathbb{R}^n$ $(n \geq 2)$ of full rank and a boundary condition $k \in \mathbb{C}^n$, we consider the n-dimensional periodic Schrödinger equation

$$\begin{aligned} (-\Delta + u)\psi &= \lambda\psi, \\ \psi(x+\gamma) &= \exp\left(2\pi i \langle k|\gamma\rangle\right)\psi(x) \quad \text{for all } \gamma \in \Gamma, \end{aligned} \tag{1}$$

with solutions ψ and a periodic, complex-valued potential u defined on $\mathbb{R}^n$. Here, we let $\langle\cdot|\cdot\rangle$ denote the standard Euclidean bilinear form on $\mathbb{C}^n$. We will leave the lattice Γ fixed throughout this article. Therefore, it is convenient

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to absorb the boundary condition into the Laplacian, $\Delta_k := (\nabla + 2\pi i k)^2$, to gain the equivalent equation

$$(-\Delta_k + u)\psi = \lambda\psi, \tag{2}$$

with solutions and potential defined on the torus $F := \mathbb{R}^n/\Gamma$ instead of $\mathbb{R}^n$ (one may still identify F with a compact fundamental domain of Γ in $\mathbb{R}^n$ as long as the periodic boundary conditions are observed). Indeed, it is easy to verify that solutions of (2) are obtained from solutions of (1) in a 1: 1-fashion by the transformation

$$\psi(x) \mapsto \exp\left(2\pi i \langle k|x\rangle\right) \psi(x).$$

Information about the spectrum of (2) in dependence of k is encoded in the Bloch variety

$$B(u) := \{(k,\lambda) \in \mathbb{C}^n \times \mathbb{C} : \text{ There is a nonzero } \psi \text{ such that } (-\Delta_k + u)\psi = \lambda\psi\}.$$

It is desirable to have a good analytic description of $B(u)$. The results in [5] (Theorem 4.4.2 in particular) suggest that it is possible to describe the zero set of the formal expression $(k,\lambda) \mapsto \det(\lambda + \Delta_k - u)$ by a zero set of a holomorphic function. However, we content ourselves with finding such a description only locally. To do so, we project the image of the Schrödinger operator to suitable finite dimensional spaces in a controlled way. This is done in Section 3 under the assumption that the resolvent of the Schrödinger operator, i.e. $(\lambda + \Delta_k - u)^{-1}$, can be defined as a meromorphic map into the compact operators on the Hilbert space $L^2(F)$.

To reach this prerequisite, it is necessary to impose certain regularity conditions on the potential u. Taking only L^p-spaces in account, a simple scale argument shows that the best one can hope for is for u to be in $L^{n/2}(F)$. For $n > 2$, this can indeed be achieved in the usual way, using Sobolev theory and some input from the theory of singular integral kernels. For $n = 2$, i.e. when u would be in $L^1(F)$, this fails, mainly because of the unboundedness of the Riesz potential operators in this case [8, V.1.2]. Of course, any $L^p(F)$ with $p > 1$ may be taken instead. But we look for a space with the same scale properties as $L^1(F)$. A situation such as the described one is usually salvaged by "replacing" $L^1(F)$ with the Hardy space $H^1(F)$ (see e.g. [8, VII.3.2, Corollary 1]). There is, however, no Hölder inequality for this space. Therefore we shall employ the slightly less general Zygmund space $L^1_{\log}(F)$ instead.

In Section 2 we deal with the functional analysis to define the resolvent. In Section 3 we construct the Bloch variety. We formulate the theorems of these sections for all dimensions $n \geq 2$ but we shall state most proofs only for the case $n = 2$, since, as mentioned in the previous paragraph, the simpler case $n > 2$ follows from standard arguments.

Finally, in Section 4 we examine the asymptotic properties of the zero energy part of the Bloch variety in two dimensions (see also [4, Sections 16–18]).

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2. The resolvent

For $n > 2$, this section can be dealt with the Sobolev spaces $W^{m,p}(F)$ of $L^p(F)$-functions whose weak derivatives up to order m are also in $L^p(F)$. In the case $n = 2$, however, we shall also investigate potentials from the Zygmund space $L^1_{\log}(F)$ which is defined as follows (cf. [2, Chapter 4.6]).

$$L^1_{\log}(F) := \big\{ f \in L^1(F) \colon |f| \log(2 + |f|) \in L^1(F) \big\} \quad \text{(as a set)}.$$

To make this a Banach space, we introduce the *distribution function* δ_f of $f \in L^1(F)$ and its *decreasing rearrangement* $f^\star$:

$$\begin{aligned} \delta_f &\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, & \lambda &\mapsto |\{x \in F \colon |f(x)| > \lambda\}|, \\ f^\star &\colon \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\}, & t &\mapsto \inf\{\lambda \in \mathbb{R}_{\geq 0} \colon \delta_f(\lambda) \leq t\} \end{aligned}$$

(here $|\cdot|$ denotes the measure). The decreasing rearrangement satisfies the linearity property $(af)^\star = |a| f^\star$ for a scalar a, but not the triangle inequality. However, it does satisfy the important monotonicity property $f^\star(t) \leq g^\star(t)$ whenever $|f(x)| \leq |g(x)|$ holds almost everywhere [2, Chapter 2, equation (1.14)]. To define a norm, introduce the *maximal function*

$$f^{\star\star} \colon (0, |F|] \to \mathbb{R} \cup \{\infty\}, \qquad x \mapsto \frac{1}{x} \int_0^x f^\star(t) \mathrm{d}t.$$

Then $f \in L^1_{\log}(F)$ if and only if $f^{\star\star} \in L^1((0, |F|])$ and $L^1_{\log}(F)$ becomes a Banach space with $\|f\|_{L^1_{\log}} := \|f^{\star\star}\|_1$. Expressed in terms of the *-operator, we have

$$\|f\|_{L^1_{\log}} = \int_0^{|F|} f^\star(t) \log \frac{|F|}{t} \mathrm{d}t.$$

We have the continuous embeddings $L^p(F) \subseteq L^1_{\log}(F) \subseteq L^1(F)$ for all $p > 1$. The following lemma is an immediate conclusion from the monotonicity of *:

Lemma 1. *Let $g \in L^\infty(F)$ and $f \in L^1_{\log}(F)$. Then $gf \in L^1_{\log}(F)$ and we have the inequality $\|gf\|_{L^1_{\log}} \leq \|g\|_\infty \|f\|_{L^1_{\log}}$.*

For some fine tuning of operator norms occurring later on, we localise the norms on our function spaces.

Lemma 2. *Let $1 \le p < \infty$ and define for each $\varepsilon > 0$*

$$\|f\|_{p;\varepsilon} := \sup_{x\in F} \|f\chi_\varepsilon^x\|_p, \qquad \|f\|_{L^1_{\log};\varepsilon} := \sup_{x\in F} \|f\chi_\varepsilon^x\|_{L^1_{\log}},$$

where χ_ε^x is the characteristic function of the ball $B_\varepsilon(x)$. Let $0 < \varepsilon_1 \le \varepsilon_2$, then

$$\frac{1}{m_{\varepsilon_1,\varepsilon_2}}\|f\|_{p;\varepsilon_2} \le \|f\|_{p;\varepsilon_1} \le \|f\|_{p;\varepsilon_2},$$
$$\frac{1}{m_{\varepsilon_1,\varepsilon_2}}\|f\|_{L^1_{\log};\varepsilon_2} \le \|f\|_{L^1_{\log};\varepsilon_1} \le \|f\|_{L^1_{\log};\varepsilon_2},$$

where $m_{\varepsilon_1,\varepsilon_2}$ is the minimum number of B_{ε_1}-balls required to cover a B_{ε_2}-ball.

Proof. For $L^p(F)$, this is the usual method to localise the norm (in fact, much better constants than $m_{\varepsilon_1,\varepsilon_2}$ are possible). We shall now prove its extension to the $L^1_{\log}$-case. The second inequality is clear. Concerning the proof of the first inequality, let $M_1, \dots, M_{m_{\varepsilon_1,\varepsilon_2}}$ be a partition of $B_{\varepsilon_2}(0)$ such that all M_i lie in a B_{ε_1}-ball where we denote the centre of the i-th ball by y_i. Then the monotonicity of * implies

$$\begin{aligned}\|f\chi_{\varepsilon_2}^x\|_{L^1_{\log}} &\le \sum_{i=1}^{m_{\varepsilon_1,\varepsilon_2}} \|f\chi_{x+M_i}\|_{L^1_{\log}}\\ &\le \sum_{i=1}^{m_{\varepsilon_1,\varepsilon_2}} \|f\chi_{\varepsilon_1}^{x+y_i}\|_{L^1_{\log}}\\ &\le m_{\varepsilon_1,\varepsilon_2}\sup_{x\in F}\|f\chi_{\varepsilon_1}^x\|_{L^1_{\log}}.\end{aligned}$$

□

We will also use the notations $L^{p;\varepsilon}(F)$ and $L^{1;\varepsilon}_{\log}(F)$ for the spaces $L^p(F)$ and $L^1_{\log}(F)$ with these norms, respectively. Note that the Hölder inequality and Lemma 1 also hold for $L^{p;\varepsilon}(F)$ and $L^{1;\varepsilon}_{\log}(F)$, respectively. Furthermore,

$$\lim_{\varepsilon\to 0}\|f\|_{p;\varepsilon} = 0 \text{ and } \lim_{\varepsilon\to 0}\|f\|_{L^1_{\log};\varepsilon} = 0 \quad \text{for } f \in L^p(F) \text{ and } f \in L^1_{\log}(F),$$

respectively. However, this convergence is not uniform on the whole Banach space. Therefore, given $C > 0$, we call a subset $U \subseteq L^p(F)$ or $U \subseteq L^1_{\log}(F)$ *uniformly C-bounded* if there is an $\varepsilon > 0$ such that

$$\sup_{f\in U}\|f\|_{p;\varepsilon} < C \quad \text{or} \quad \sup_{f\in U}\|f\|_{L^1_{\log};\varepsilon} < C,$$

respectively.

Next we investigate the invertibility of the operator $\lambda + \Delta_0$.

Lemma 3. *For all* $\lambda \in (-\infty, -1]$ *the inverse* $(\lambda + \Delta_0)^{-1}$ *exists. It maps* $L^q(F)$ *boundedly into* $W^{2,q}(F)$ *for all* $q > 1$*, and in the case* $n = 2$ *it maps* $L^1_{\log}(F)$ *boundedly into* $W^{2,1}(F)$*. The bound is independent of* $\lambda \in (-\infty, -1]$*.*

Proof. We prove only the second statement. The free Schrödinger operator $-\Delta_0$ has a spectrum of the form $4\pi^2\kappa^2$ where κ runs through Γ^*, the dual lattice of Γ. Hence all $\lambda \in (-\infty, -1]$ belong in the resolvent set of $-\Delta_0$, and if a bound exists, it will be independent of λ. Now consider the formal operators

$$\frac{\partial_i\partial_j}{1 - \Delta_0} \quad \text{and} \quad \frac{\partial_i\partial_j}{\Delta_0}, \qquad i, j \in \{1, 2\}.$$

We need to show that the first of these operators maps $L^1_{\log}(F)$ boundedly into $L^1_{\log}(F)$. Now, since F is compact, $\Delta_0 f \in L^1_{\log}(F)$ certainly implies $f \in L^1_{\log}(F)$, so it is sufficient to prove the boundedness of the second operator. This operator is defined by a smooth Fourier multiplier of homogeneous degree zero and is therefore bounded on the Hardy space $H^1(F)$ [7, Theorem 1.88]. We may embed $L^1_{\log}(F) \subseteq H^1(F) \subseteq L^1(F)$ continuously (see [7, Corollary 1.97]), so the assertion is proven. □

To proceed, we need to choose suitable spaces for the potentials u. As mentioned before, for $n > 2$, we may take $u \in L^{n/2}(F)$, but we cannot take $u \in L^1(F)$ in the case $n = 2$ [8, V.6.6 (b)]. The usual method to replace $L^1(F)$ with $H^1(F)$ would, as we have seen in the proof of the preceding lemma, eliminate these problems, but introduces new difficulties. Specifically, the Hölder inequality

$$\|gf\|_1 \leq \|g\|_\infty \|f\|_1$$

does not seem to have an analogue for Hardy spaces. Therefore we choose u from the slightly less general space $L^1_{\log}(F)$ instead.

Lemma 4. *There is a* $C > 0$ *such that for any uniformly* C*-bounded* $U \subseteq L^1_{\log}(F)$ *there is a* $\lambda \in (-\infty, -1]$ *such that for all* $u \in U$ *the operator* $u(\lambda + \Delta_0)^{-1}$ *maps* $L^1_{\log}(F)$ *boundedly into itself. Changing to an equivalent norm from Lemma 2 yields an explicit bound of* 1*. The same also holds for any* $n > 2$ *if* $L^1_{\log}(F)$ *is replaced by (say)* $L^{2n/(n+2)}(F)$*. In both cases, if the explicit bound is not required,* U *may be taken to be any bounded subset.*

Proof. Let G_λ be the integral kernel of $(\lambda+\Delta_0)^{-1}$ and for $\varepsilon>0$ set $G_\lambda^\varepsilon = G_\lambda \chi_\varepsilon^0$. Now let $\delta>\varepsilon>0$, then

$$\left| u(y) \int_F \left(G_\lambda \chi_\varepsilon^0\right)(y-z) f(z) \mathrm{d}z \chi_{\delta-\varepsilon}^x(y) \right|$$
$$\leq \left| u(y)\chi_{\delta-\varepsilon}^x(y) \int_F (G_\lambda\chi_\varepsilon^0)(y-z)(f\chi_\delta^x)(z)\mathrm{d}z \right|.$$

In the case $n=2$ we can use the monotonicity of *, the inequality from Lemma 1, Lemma 3 and Sobolev's theorem $W^{2,1}(F)\subseteq L^\infty(F)$ to establish from this the inequality

$$\|u(G_\lambda^\varepsilon * f)\|_{L^1_{\log};\delta-\varepsilon} \leq \left\|(u\chi_{\delta-\varepsilon}^x)\right\|_{L^1_{\log}} \|(G_\lambda^\varepsilon * (f\chi_\delta^x))\|_\infty \leq c\|u\|_{L^1_{\log};\delta}\|f\|_{L^1_{\log};\delta}$$

for a suitable $c>0$. The complementary kernel $G_\lambda(1-\chi_\varepsilon^0)$ decays exponentially, so we can diminish its contribution to $\|u(G_\lambda * f)\|$ by making $|\lambda|$ very large. Since we are in the critical case scale-wise, this does not change the operator norm $\|(\lambda-\Delta_0)^{-1}\|$. So what we have proven is that $u(\lambda+\Delta_0)^{-1}$ is a bounded operator from $L^{1;\delta}_{\log}(F)$ to $L^{1;\delta-\varepsilon}_{\log}(F)$. By Lemma 2, this is a bounded operator on $L^{1;\delta}_{\log}(F)$, and the operator norm increases by not more than a factor of R where $R\leq m_{\delta-\varepsilon,\delta}$. By setting (say) $\varepsilon:=\delta/2$, R can be bounded independently of δ. Setting $C:=1/(cR)$ and δ sufficiently small yields operators with norm smaller than 1 on $L^{1;\delta}_{\log}(F)$, for all $u\in U$. The case $n>2$ works similarly. □

The actual value that the constant C in the previous proof may not exceed can be explicitly calculated from optimal embedding constants. However, this value is not so important to us right now, as we will only deal with rather benign uniformly C-bounded sets.

Now we can extend Lemma 3 to the case of nonzero potentials and boundary conditions. We define the resulting operator on $L^2(F)$.

Lemma 5. *Let $u\in U$ be as in Lemma 4 (with sufficiently small C) and $V\subseteq\mathbb{C}^n$ be bounded. Then there is a $\lambda\in\mathbb{C}$ such that the map $(u,k)\mapsto(\lambda-\Delta_k+u)^{-1}$ is defined for all $k\in V$ and its image lies in the space of compact $L^2(F)$-endomorphisms. This map is continuous in u with the weak topology on U and the usual norm topology on $L^2(F)$, and uniformly so with respect to k.*

Proof. Lemma 4 provides us with the required λ such that $\|u(\lambda+\Delta_0)^{-1}\| < 1$. Therefore, $1 - u(\lambda + \Delta_0)^{-1}$ is boundedly invertible and we have

$$(\lambda + \Delta_0)^{-1}(1 - u(\lambda + \Delta_0)^{-1})^{-1} = (\lambda + \Delta_0 - u)^{-1}.$$

For $n = 2$, this is a continuous operator from $L^1_{\log}(F)$ to $W^{2,1}(F)$. Now, the operator $4\pi i \langle k|\nabla\rangle - 4\pi^2 k^2$ maps $W^{2,1}(F)$ boundedly into $W^{1,1}(F)$. Sobolev's theorem yields a continuous embedding of $W^{1,1}(F)$ into $L^r(F)$ with a suitable $r > 1$ such that the embedding is not critical. Hence, the operator $(\lambda + \Delta_0 - u)^{-1}(4\pi i \langle k|\nabla\rangle - 4\pi^2 k^2)$ maps $W^{2,1}(F)$ into itself, and since V is bounded, its norm can be uniformly diminished by increasing $|\lambda|$. Therefore, $1 + (\lambda + \Delta_0 - u)^{-1}(4\pi i \langle k|\nabla\rangle - 4\pi^2 k^2)$ has a continuous inverse and

$$\begin{aligned}&\left(1 + (\lambda + \Delta_0 - u)^{-1}(4\pi i \langle k|\nabla\rangle - 4\pi^2 k^2)\right)^{-1} (\lambda + \Delta_0 - u)^{-1}\\ &= (\lambda + \Delta_k - u)^{-1}\end{aligned}$$

is a continuous operator from $L^1_{\log}(F)$ to $W^{2,1}(F)$. Now, since $L^2(F) \subseteq L^1_{\log}(F)$, and furthermore $W^{2,1}(F) \subseteq L^2(F)$, this is a continuous operator on $L^2(F)$. Moreover, due to the Rellich–Kondrachov theorem [1, Theorem 6.3], and because compact operators form a two-sided ideal in the continuous linear operators of $L^2(F)$, this operator is compact. Due to the uniform C-boundedness of U, we have the well-defined series representation

$$(\lambda + \Delta_k - u)^{-1} = (\lambda + \Delta_k)^{-1} \sum_{m=0}^{\infty} (u(\lambda + \Delta_k)^{-1})^m.$$

Now consider U and $L^2(F)$ with the topologies mentioned. Let $(P_l)_{l\in\mathbb{Z}_{\geq 0}}$ be a sequence of projectors of finite rank with increasing images that exhaust a dense set. Then

$$(\lambda + \Delta_k)^{-1} \sum_{m=0}^{\infty} (P_l u P_l (\lambda + \Delta_k)^{-1})^m$$

is continuous in u and, in the limit $l \to \infty$, coincides with $(\lambda + \Delta_k - u)^{-1}$ on a dense set. Hence, $(\lambda + \Delta_k - u)^{-1}$ is continuous in $u \in U$ by uniform convergence. □

We are now able to prove this section's main theorem.

Theorem 1. *For each $k \in V$ and $u \in U$ as in the previous lemma, there is a discrete set $S(k, u) \subseteq \mathbb{C}$ such that the map $(u, k, \lambda) \mapsto (\lambda - \Delta_k + u)^{-1}$ is defined on the set $\{(u, k, \lambda) \in U \times V \times \mathbb{C} \colon \lambda \notin S(k, u)\}$ with image in the space of compact $L^2(F)$-endomorphisms. This map is continuous in u with the weak topology on U and the usual topology on $L^2(F)$.*

Proof. Due to Lemma 5, $(\lambda'+\Delta_k-u)^{-1}$ is compact on $L^2(F)$ for a suitable $\lambda'\in\mathbb{C}$. Let $\sigma(k,u)$ be the spectrum of $(\lambda'+\Delta_k-u)^{-1}$. Due to the Riesz–Schauder theorem, $\sigma(k,u)$ has at most one accumulation point, namely zero. Hence the set

$$S(k,u) := \{s\in\mathbb{C}\setminus\{\lambda'\}\colon (\lambda'-s)^{-1}\in\sigma(k,u)\}\cup\{\lambda'\}$$

is discrete. Let $\lambda\in\mathbb{C}\setminus S(k,u)$, then $(\lambda'-\lambda)^{-1}-(\lambda'+\Delta_k-u)^{-1}$ is boundedly invertible. Now,

$$\begin{aligned}&(\lambda'-\lambda)^{-1}(\lambda'+\Delta_k-u)^{-1}\left((\lambda'-\lambda)^{-1}-(\lambda'+\Delta_k-u)^{-1}\right)^{-1}\\&=(\lambda+\Delta_k-u)^{-1}\end{aligned}$$

is a compact operator on $L^2(F)$. Continuity in u follows as in Lemma 5. □

3. The Bloch variety

Let u be from a suitable function space as defined in the previous section. Due to Theorem 1, the Bloch variety $B(u)$ is well-defined as the set of pairs (k,λ) for which the resolvent of the Schrödinger operator has a pole. The compactness property allows us to apply the reverse Riesz–Schauder theorem to the resolvent to obtain a local complex analytic description of the Bloch variety. Furthermore, the deviation of the spectrum of a Schrödinger operator with potential u from the spectrum of the free Schrödinger operator can be controlled by the appropriate norm of u (dependent on n).

Proposition 1. *There is a constant $C>0$ such that for each eigenvalue λ of $-\Delta_k+u$ there is an eigenvalue λ_0 of $-\Delta_k$ satisfying $|\lambda-\lambda_0|\le C\|u\|$.*

Proof. Let λ be an eigenvalue of $-\Delta_k+u$. The assertion is evident if λ is in the spectrum of $-\Delta_k$. Now let λ be in the resolvent set of $-\Delta_k$, then $\lambda+\Delta_k$ is invertible. Let ψ be a normalised eigenvector of $-\Delta_k+u$ with eigenvalue λ. Then $\psi=(\lambda+\Delta_k)^{-1}u\psi$. Using first Lemma 3, and then Theorem 1 in the zero potential case, we find that for a suitable constant $C>0$

$$1\le C\|(\lambda+\Delta_k)^{-1}\|_{L^2\to L^2}\|u\|.$$

Now, $L^2(F)$ is a Hilbert space and Δ_k is a normal operator with discrete spectrum, so we find a specific λ_0 such that $\|(\lambda+\Delta_k)^{-1}\|=|\lambda-\lambda_0|^{-1}$ and λ_0 is an eigenvalue of $-\Delta_k$. □

For a similar but much more precise statement, see [3].

Theorem 2. *All u_0, k_0 from tuples $(u_0, k_0, \lambda_0) \in (u \mapsto B(u))$ have neighbourhoods U of u_0 in the appropriate function space and V of (k_0, λ_0) such that for all $(u, k, \lambda) \in U \times V$ there is a finite dimensional, Schrödinger-invariant subspace $W(k, u)$ of $L^2(F)$ which is not dependent on λ, such that $U \times V \cap (u \mapsto B(u))$ is the zero locus of the map*

$$(u, k, \lambda) \mapsto \det\big((\lambda - \Delta_k + u)|_{W(k,u)}\big),$$

and this map is holomorphic in λ.

Proof. Due to Theorem 1, all singularities of $(\lambda + \Delta_k - u)^{-1}$ are isolated, so we can apply the reverse Riesz-Schauder theorem (see e.g. [9, VIII.8]) and expand the resolvent in a Laurent series around any singularity λ_0. Its residue is given by the integral

$$P(k, u) = \frac{1}{2\pi i} \oint\limits_{|\lambda - \lambda_0| = c \ll 1} (\lambda + \Delta_k - u)^{-1} \mathrm{d}\lambda.$$

$P(k, u)$ is a projector to a $(-\Delta_k + u)$-invariant finite-dimensional subspace of $L^2(F)$. Let $W(k_0, u_0)$ be the range of $P(k_0, u_0)$ and pick a basis $f_1, \ldots, f_m$ of $W(k_0, u_0)$. Then clearly $\det \langle P^*(k_0, u_0) f_j | P(k_0, u_0) f_i \rangle \neq 0$. Now, due to Theorem 1, $P(k, u)$ is continuous in k and u, so there is a neighbourhood $U \times V$ of (u_0, k_0, λ_0) such that $\det \langle P^*(k, u) f_j | P(k, u) f_i \rangle \neq 0$ for all $(u, k) \in U \times V$. Thus the images $W(k, u)$ of the constant rank projections $P(k, u)$ are spanned by $P(k, u) f_1, \ldots, P(k, u) f_m$. Note that this does not necessarily mean that the multiplicity of λ_0 is preserved, rather we may get several eigenvalues spread out in a small neighbourhood of λ_0. The matrix elements A_{ij} of $\lambda + \Delta_k - u$ with respect to $P(k, u) f_1, \ldots, P(k, u) f_m$ are defined by

$$(\lambda + \Delta_k - u) P(k, u) f_i = \sum_j A_{ij} P(k, u) f_j.$$

This implies

$$\begin{aligned} &\langle P^*(k, u) f_l | (\lambda + \Delta_k - u) P(k, u) f_i \rangle = \langle f_l | (\lambda + \Delta_k - u) P(k, u) f_i \rangle \\ &= \sum_j A_{ij} \langle f_l | P(k, u) f_j \rangle = \sum_j A_{ij} \langle P^*(k, u) f_l | P(k, u) f_j \rangle, \end{aligned}$$

since $P(k, u)$ is a projector which commutes with $\lambda + \Delta_k - u$. Hence $(u, k, \lambda) \in (u \mapsto B(u))$ precisely if $(\lambda + \Delta_k - u)|_{W(k,u)}$ fails to be invertible, that is

$$\det \langle f_j | (\lambda + \Delta_k - u) P(k, u) f_i \rangle = 0.$$

□

4. The Fermi curve and its asymptotic analysis

In this section, we will investigate the asymptotic structure of the Bloch variety for fixed λ in the case $n = 2$. For this purpose we define the *Fermi curve*

$$F(u) := \{k \in \mathbb{C}^2 \colon (k, 0) \in B(u)\}.$$

Then, as a set, we have

$$B(u) = \coprod_{\lambda \in \mathbb{C}} F(u - \lambda).$$

It is therefore sufficient to consider only $F(u)$, i.e. $\lambda = 0$, for all possible u. In fact, it will be convenient to enlarge the set of Fermi curves. Therefore we introduce the *periodic Dirac equation* in an analogous manner to the Schrödinger equation:

$$\begin{pmatrix} v & \partial \\ -\bar\partial & w \end{pmatrix} \psi = \lambda \psi$$
$$\psi(x + \gamma) = \exp\left(2\pi i \langle k | x \rangle\right) \psi(x) \quad \text{for all } \gamma \in \Gamma,$$

with two periodic potentials v and w, and the usual definition of differential operators,

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}\right), \qquad \bar\partial := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}\right).$$

Here, the solution ψ has two components. Again, one can absorb the boundary condition k by replacing the differential operators with $\partial_k := \partial + \pi i k_1 + \pi k_2$ and $\bar\partial_k := \bar\partial + \pi i k_1 - \pi k_2$. We then have $\partial_k \bar\partial_k = \Delta_k / 4$. A Bloch variety, and hence a Fermi curve $F_D(v, w)$ are attached to the Dirac equation as well. Therefore, we let v have the same properties as u in the Schrödinger case and let $w \in L^\infty(F)$ and execute the analogous procedure to the one in Section 2. We omit the details (but see [6, Chapter 2]). The result is that the resolvent

$$\left(\lambda - \begin{pmatrix} v & \partial_k \\ -\bar\partial_k & w \end{pmatrix}\right)^{-1}$$

can be defined as a linear operator from $L^1_{\log}(F) \times L^2(F)$ to $W^{1,2}(F) \times W^{1,1}(F)$ or as a compact operator on $L^2(F) \times L^2(F)$ in a similar fashion as above (the order of the function spaces is reversed due to the derivatives being off-diagonal in the Dirac operator). The connexion between the Dirac and Schrödinger Fermi curves is as follows.

Proposition 2. *For a Schrödinger potential u, we have $F(u) = F_D(u/4, -1)$.*

Proof. Let $k \in F(u)$, then there is a $\psi \in W^{2,1}(F)$ from the kernel of the Schrödinger operator. The pair $(\psi_1, \psi_2) := (\psi, -\overline{\partial}_k \psi)$ lies in the kernel of the Dirac operator with potentials $(u/4, -1)$, and $\psi_1 \in W^{1,2}(F)$ due to Sobolev's theorem. Hence, $F(u) \subseteq F_D(u/4, -1)$. For the reverse inclusion, let $k \in F_D(u/4, -1)$, then there exist $\psi_1 \in W^{1,2}(F)$, $\psi_2 \in W^{1,1}(F)$ defining an element from the kernel of the Dirac operator, in particular $\psi_2 = -\overline{\partial}_k \psi_1$. It follows that $\psi_1 \in W^{2,1}(F)$. Since $\partial_k \overline{\partial}_k = \Delta_k/4$, this concludes the proof. □

Due to this proposition, it is sufficient to understand the large-k behaviour of $F_D(v, w)$ in order to understand the same problem for $F(u)$. Working with $F_D(v, w)$ instead of $F(u)$ has several advantages. The Dirac equation is a first order differential equation while the Schrödinger equation is of second order. In the Dirac equation, we have two potentials instead of only one. Finally, the theory is invariant under shifts of the boundary condition $k \in \mathbb{C}^2$ by vectors κ from the dual lattice Γ^*. In the Dirac equation we can exploit this invariance separately for ∂_k and $\overline{\partial}_k$.

In the next two propositions, we will investigate the structure of the free Fermi curve $F_D(0, 0)$ and a criterion for convergence to $F_D(0, 0)$.

Proposition 3. *Let*

$$\mathcal{R} := \left\{k \in \mathbb{C}^2 \colon k_2 - ik_1 = 0 \text{ or } k_2 + ik_1 = 0\right\},$$

then $F_D(0, 0) = \mathcal{R} + \Gamma^*$. Furthermore, $\mathcal{R}$ is a system of representatives for $F_D(0, 0)/\Gamma^*$ if the pairs of distinct points

$$k_\kappa^\pm := \frac{1}{2}(\pm\kappa_1 - i\kappa_2, \pm\kappa_2 + i\kappa_1)$$

are identified to double points (k_κ^-, k_κ^+) for all $\kappa \in \Gamma^*$, $\kappa \neq 0$.

Proof. The first part follows by considering the Fourier transform of the free Dirac equation. The rest is simple linear algebra. □

Proposition 4. *Let $U \subseteq \mathbb{C}^2$ be open and bounded and (v_n, w_n) a uniformly C-bounded sequence of potentials weakly converging to the zero potentials $(0, 0)$, with C sufficiently small. Then the sequence of Fermi curves $U \cap F_D(v_n, w_n)$ converges to $U \cap F_D(0, 0)$ (with respect to a suitable notion of convergence for bounded sets, such as the Hausdorff distance).*

Proof. Due to the equivalent of Theorem 1 for the Dirac operator, the resolvent map is continuous on uniformly C-bounded sets endowed with the weak topology, so the sequence

$$\begin{pmatrix} \lambda - v_n & \partial_k \\ -\overline{\partial}_k & \lambda - w_n \end{pmatrix}^{-1}$$

converges uniformly with respect to k. The sequence of Fermi curves $F_D(v_n, w_n)$ may be recovered from that sequence, so that $F_D(v_n, w_n)$ converges pointwise to $F_D(0,0)$ in U. Due to the regular shape of $F_D(0,0)$, the sequence $U \cap F_D(v_n, v_m)$ converges to $U \cap F_D(0,0)$ if U is bounded. □

The shape of $F_D(0,0)/\Gamma^*$ is not so regular any more, so care must be taken when trying to apply Proposition 4 to a system of representatives of $F_D(v,w)/\Gamma^*$ instead of the $F_D(v,w)$ directly, even if one takes a system of representatives nearest to $\mathcal{R}$ for each $F_D(v_n, w_n)$. For, while $F_D(v_n, w_n)$ converges to $F_D(0,0)$ near the double points $k_\kappa^\pm$ of $F_D(0,0)$ in particular, there may not be a converging sequence of double points from the systems of representatives of the $F_D(v_n, w_n)$, i.e. the double points $k_\kappa^\pm$ may split up as a result of small weak perturbations of the zero potentials. Hence, for each $\kappa \in \Gamma^*$, connected sets H_κ containing neighbourhoods of k_κ^+ and k_κ^- need to be removed from the bounded set U for the proposition to remain correct in this case. As $F_D(v_n, w_n)$ converges to $F_D(0,0)$, these neighbourhoods may be taken to be arbitrarily small.

We introduce the abbreviation $e_k(x) := \exp(2\pi i \langle k|x\rangle)$. The following lemma can be proven by a straightforward calculation.

Lemma 6. *Let $k \in \mathbb{C}^2$ and $\kappa \in \Gamma^*$. Then the following transformation properties hold:*

$$\begin{aligned} \partial_k &= \partial_{k+k_\kappa^-}, \quad \partial_{k+\kappa} = e_{-\kappa}\partial_k e_\kappa, \\ \overline{\partial}_k &= \overline{\partial}_{k+k_\kappa^+}, \quad \overline{\partial}_{k+\kappa} = e_{-\kappa}\partial_k e_\kappa. \end{aligned}$$

This implies:

Proposition 5. *We have $F_D(v,w) = F_D(e_{-\kappa}v, e_\kappa w) - k_\kappa^-$ for all pairs of potentials (v,w) and all $\kappa \in \Gamma^*$.*

Proof. Let $k \in F_D(v,w)$, then there are ψ_1, ψ_2 such that due to Lemma 6 and $k_\kappa^+ - k_\kappa^- = \kappa$

$$0 = \begin{pmatrix} v\psi_1 + \partial_k\psi_2 \\ -\overline{\partial}_k\psi_1 + w\psi_2 \end{pmatrix} = \begin{pmatrix} e_{-\kappa} v e_\kappa \psi_1 + \partial_{k+k_\kappa^-}\psi_2 \\ e_{-\kappa}(-\overline{\partial}_{k+k_\kappa^-} e_\kappa\psi_1 + e_\kappa w\psi_2) \end{pmatrix}.$$

Since $e_\kappa\psi_1$ is Γ-periodic, the proposition follows. □

Now, for $\varepsilon,\delta>0$ consider the sets

$$U^{\pm}_{\varepsilon,\delta} := \left\{ k\in\mathbb{C}^2\colon |k_1\pm ik_2|<\varepsilon,\ \|k\|>\delta^{-1},\ \underset{\substack{\kappa\in\Gamma^*\\ \kappa\neq 0}}{\forall} \left\|k-k^{\pm}_{\kappa}\right\|>\varepsilon \right\}.$$

For small ε and δ, these sets contain points near the free Fermi curve $F_D(0,0)$ for $\|k\|\gg 0$ and avoiding the neighbourhoods H_κ of k^+_κ and k^-_κ. For sufficiently small ε, we may take $U^{\pm}_{\varepsilon,\delta}$ to be a subset of $\mathbb{C}^2/\Gamma^*$ by choosing a fundamental domain. For such ε we define

$$V^{\pm}_{\varepsilon,\delta}(v,w) := U^{\pm}_{\varepsilon,\delta}\cap F_D(v,w)/\Gamma^*.$$

Now we can state the following theorem.

Theorem 3. *For all pairs of potentials (v,w) and all sufficiently small $\varepsilon>0$ there is a $\delta>0$ and a compact set*

$$K\subseteq\left\{k\in F_D(v,w)/\Gamma^*\subseteq\mathbb{C}^2/\Gamma^*\colon \|k\|\leq\delta^{-1}\right\}$$

such that we have the disjoint union

$$F_D(v,w)/\Gamma^* = V^{+}_{\varepsilon,\delta}(v,w)\cup V^{-}_{\varepsilon,\delta}(v,w)\cup K\cup\bigcup_{\kappa\in\Gamma^*_\delta}\left(H_\kappa\cap F_D(v,w)\right)/\Gamma^*,$$

where $\Gamma^*_\delta := \left\{\kappa\in\Gamma^*\colon \|k^+_\kappa\|>\delta^{-1}\right\}$ and H_κ as above.

Proof. Let $\Gamma^*_{\mathbb{C}}$ be the lattice generated by κ and $k^-_{\kappa'}$ for all $\kappa,\kappa'\in\Gamma^*$. This is a real four dimensional lattice, so it has a bounded fundamental domain. Let $\varepsilon>0$ be sufficiently small, then we choose U to be a fundamental domain of $\Gamma^*_{\mathbb{C}}$ with small balls around $k^{\pm}_\kappa$ removed:

$$U := \mathbb{C}^2/\Gamma^*_{\mathbb{C}}\setminus\bigcup_{\substack{\kappa\in\Gamma^*\\ \kappa\neq 0}} B_{\leq\varepsilon}(k^{\pm}_\kappa)/\Gamma^*.$$

The sequence $(e_{-\kappa}v, e_\kappa w)$ converges weakly to zero for $\|\kappa\|\to\infty$ due to the Riemann-Lebesgue lemma. Because of Hölder's inequality, this sequence is uniformly C-bounded for any $C>0$. Hence we find a sufficiently small $\delta>0$ such that for all $\kappa\in\Gamma^*_\delta$ the distance from $F_D(e_{-\kappa}v,e_\kappa w)$ to $F_D(0,0)$ is smaller than ε. Due to Proposition 5, this is also true for all $k\in F_D(v,w)$ of the form $k=k'+k^-_\kappa$ with $k'\in U$ and $\kappa\in\Gamma^*_\delta$, i.e. $k+\Gamma^*\in V^{\pm}_{\varepsilon,\delta}(v,w)$. By construction, the remaining $k+\Gamma^*$ are either in K or in H_κ/Γ^* for a suitable κ. □

In light of Proposition 2, we have the

Corollary 1. *Let $u \in L^1_{\log}(F)$, then the Fermi curve $F(u)$ of the Schrödinger operator with potential u can, for all $\varepsilon > 0$, be decomposed into three parts.*

(a) *A part which is "near" the set*

$$\mathcal{R} + \Gamma^* = \{k \in \mathbb{C} \colon k_2 = ik_1 \text{ or } k_2 = -ik_1\} + \Gamma^*,$$

i.e. this part is contained in a tube of width ε around $\mathcal{R} + \Gamma^*$.

(b) *A part contained in a set of thin "handles" H_κ of width ε connecting the points k^+_κ and k^-_κ.*

(c) *The remainder, which is (modulo Γ^*) contained in a compact set.*

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