

Hyman Bass and Ubiquity: Gorenstein Rings

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Dedicated to Hyman Bass.

ABSTRACT. This paper is based on a talk given by the author in October, 1997 at a conference in Columbia University in celebration of Hyman Bass's 65th birthday. The paper details some of the history of Gorenstein rings and their uses.

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Introduction

In 1963, an article appeared in *Mathematische Zeitschrift* with an interesting title, 'On the ubiquity of Gorenstein rings', and fascinating content. The author

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was Hyman Bass. The article has become one of the most-read and quoted math articles in the world. A journal survey done in 1980 [Ga] showed that it ranked third among most-quoted papers from core math journals. For every young student in commutative algebra, it is at the top of a list of papers to read. The first thing I did as a graduate student when I opened the journal to this paper was to reach for a dictionary.

DEFINITION: UBIQUITY. The state or capacity of being everywhere, especially at the same time; omnipresent.

This article will give some of the historical background of Gorenstein rings, explain a little of why they are ubiquitous and useful, and give practical ways of computing them. The property of a ring being Gorenstein is fundamentally a statement of symmetry. The ‘Gorenstein’ of Gorenstein rings is Daniel Gorenstein, the same who is famous for his role in the classification of finite simple groups. A question occurring to everyone who studies Gorenstein rings is ‘Why are they called Gorenstein rings?’ His name being attached to this concept goes back to his thesis on plane curves, written under Oscar Zariski and published in the Transactions of the American Mathematical Society in 1952 [Go]. As we shall see, they could perhaps more justifiably be called Bass rings, or Grothendieck rings, or Serre rings, or Rosenlicht rings. The usual definition now used in most textbooks goes back to the work of Bass in the ubiquity paper.

1. Plane Curves

The origins of Gorenstein rings go back to the classical study of plane curves. Fix a field k and let $f(X, Y)$ be a polynomial in the ring $k[X, Y]$. By a plane curve we will mean the ring $R = k[X, Y]/(f)$. If k is algebraically closed, Hilbert’s Nullstellensatz allows us to identify the maximal ideals of this ring with the solutions of $f(X, Y) = 0$ which lie in k , via the correspondence of solutions $(\alpha, \beta) \in k^2$ with maximal ideals $(X - \alpha, Y - \beta)$. In his thesis, Gorenstein was interested in the properties of the so-called adjoint curves to an irreducible plane curve $f = 0$. However, his main theorem had to do with the integral closure of a plane curve.

DEFINITION 1.1. Let R be an integral domain with fraction field K . The *integral closure* of R is the set of all elements of K satisfying a monic polynomial with coefficients in R .

The integral closure is a ring T , containing R , which is itself integrally closed. An important measure of the difference between R and its integral closure T is the *conductor ideal*, denoted $\mathfrak{C} = \mathfrak{C}(T, R)$.

$$\mathfrak{C} = \{r \in R \mid rT \subseteq R\}$$

It is the largest common ideal of both R and T .

EXAMPLE 1.2. Let $R = k[t^3, t^7] \cong k[X, Y]/(X^7 - Y^3)$. The element

$$t = \frac{t^7}{(t^3)^2},$$

is in the fraction field of R and is clearly integral over R . It follows that the integral closure of R is $T = k[t]$. The dimension over k of T/R can be computed by counting

2. Commutative Algebra circa 1960

The most basic commutative rings are the polynomials rings over a field or the integers, and their quotient rings. Of particular importance are the rings whose corresponding varieties are non-singular. To understand the notion of non-singularity in terms of the ring structure, the notion of regular local rings was developed.

DEFINITION-THEOREM 2.1. A *regular local ring* (S, n) is a Noetherian local ring which satisfies one of the following equivalent properties:

1. The maximal ideal n is generated by $\dim(S)$ elements.
2. Every finitely generated S -module M has a finite resolution by finitely generated free S -modules, i.e. there is an exact sequence,

$$0 \rightarrow F_k \xrightarrow{\alpha_k} F_{k-1} \xrightarrow{\alpha_{k-1}} \dots \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where the F_i are finitely generated free S -modules.

3. The residue field S/n has a finite free resolution as in (2).

These equivalences are due to Auslander-Buchsbaum [AuBu1] and Serre [Se2], and marked a watershed in commutative algebra. One immediate use of this theorem was to prove that the localization of a regular local ring is still regular. Regular local rings correspond to the localizations of affine varieties at non-singular points. The existence of a finite free resolution for finitely generated modules is of crucial importance. For example, in 1959 Auslander and Buchsbaum [AuBu2] used the existence of such resolutions to prove their celebrated result that regular local rings are UFDs.

A resolution as in (2) is said to be *minimal* if $\alpha_i(F_i) \subseteq nF_{i-1}$. In this case it is unique up to an isomorphism. The length of a minimal resolution is the projective dimension of M , denoted $\text{pd}_S(M)$.

We say a Noetherian ring is regular if it is locally at each of its primes. Every polynomial ring or power series ring over a field is regular. Other types of rings were becoming increasingly interesting and important. To explain these, some further definitions are helpful:

- A local ring (R, m) is a Noetherian commutative ring with a unique maximal ideal m .

- An m -primary ideal I is any ideal such that one of the following equivalent conditions hold:

1. R/I is dimension 0;
2. R/I is Artinian;
3. The nilradical of I is m ;
4. $\text{Supp}(R/I) = \{m\}$.

- A regular sequence x_1, \dots, x_t on an R -module M is any set of elements such that $(x_1, \dots, x_t)M \neq M$ and such that x_1 is not a zero-divisor on M , x_2 is not a zero-divisor on the module M/x_1M , and in general x_i is not a zero-divisor on $M/(x_1, \dots, x_{i-1})M$.

- The depth of a finitely generated module over a local ring R is the maximal length of a regular sequence on the module.

- M is said to be *Cohen-Macaulay* if $\text{depth}(M) = \dim(M)$. (This is the largest possible value for the depth.)

- A s.o.p. (system of parameters) for a local ring R of dimension d is any d elements x_1, \dots, x_d which generate an \mathfrak{m} -primary ideal.
- A local ring is Cohen-Macaulay iff every (equivalently one) system of parameters forms a regular sequence.
- A local ring R is regular iff the maximal ideal is generated by a s.o.p.
- The height of a prime ideal P in a commutative ring R is the supremum of integers n such that there is a chain of distinct prime ideals $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P$. The height of an arbitrary ideal is the smallest height of any prime containing the ideal.

If I has height k , then I needs at least k generators, otherwise the dimension could not drop by k . This is the statement of what is known as Krull's Height Theorem. Explicitly Krull's theorem states:

THEOREM 2.3. *Let S be a Noetherian ring and let I be an ideal generated by k elements. Then every minimal prime containing I has height at most k . In particular $\text{height}(I) \leq k$.*

The property of a ring being Cohen-Macaulay had been growing in importance in the 1950s and early 1960s. Different groups studied it under different names: semi-regular, Macaulay, even Macaulay-Cohen. In particular, Northcott and Rees had been studying the irreducibility of systems of parameters and relating it to the Cohen-Macaulay property.

- An ideal I is irreducible if $I \neq J \cap K$ for $I \subsetneq J, K$.
- Let M be a module over a local ring (R, \mathfrak{m}) . The socle of a module M , denoted by $\text{soc}(M)$, is the largest subspace of M whose R -module structure is that of a vector space, i.e., $\text{soc}(M) = \text{Ann}_M \mathfrak{m} = \{y \in M \mid my = 0\}$.

PROPOSITION 2.4. *If R is Artinian local, then (0) is irreducible iff $\text{soc}(R)$ is a 1-dimensional vector space.*

PROOF. Let V be the socle. If $\dim(V) \geq 2$ then simply pick two one dimensional subspaces of V which intersect in 0 to show 0 is not irreducible.

For the converse, the key point is that R is Artinian implies $V \subseteq R$ is essential. Any two ideals must intersect the socle, but if it is one-dimensional they then contain the socle. \square

For example, the ring $R = k[X, Y]/(X^2, Y^2)$ has a one-dimensional socle generated by XY ; every nonzero ideal in R contains this element and hence 0 is irreducible. The intersection of any two nonzero ideals must contain XY . On the other hand in $R = k[X, Y]/(X^2, XY, Y^2)$ the socle is two dimensional generated by the images of X and Y , and $XR \cap YR = 0$.

Irreducible ideals were already present in the famous proof of Emmy Noether that ideals in Noetherian commutative rings have a primary decomposition [No]. She first observed that the Noetherian property implied every ideal is a finite intersection of irreducible ideals. (For example the ideal (X^2, XY, Y^2) above is the intersection of (X, Y^2) and (X^2, Y) , which are both irreducible.) This 'irreducible decomposition' is closely related to the ideas in the theory of Gorenstein rings.

The results of Northcott and Rees are the following:

THEOREM 2.5 [NR]. *If (R, m) is a local Noetherian ring such that the ideal generated by an arbitrary system of parameters is irreducible, then R is Cohen-Macaulay.*

THEOREM 2.6 [NR]. *A regular local ring has the property that every ideal generated by a system of parameters is irreducible.*

Combining these two theorems recovers a result first due to Cohen: regular local rings are Cohen-Macaulay. In fact, as we shall see, this irreducible property characterizes Gorenstein rings. From this perspective, the theorems of Northcott and Rees say (among other things) that regular rings are Gorenstein, and Gorenstein rings are Cohen-Macaulay.

3. Gorenstein Rings

Grothendieck (1957) [Gr] developed duality theory for singular varieties. He constructed a rank one module which on the non-singular locus of an algebraic variety agreed with the differential forms of degree d (the dimension) of the variety.

Let S be a polynomial ring and R be a homomorphic image of S of dimension d . One can define a dualizing module (or *canonical module*) by setting

$$\omega_R = \text{Ext}_S^{n-d}(R, S)$$

where the dimension of S is n . When R is non-singular, this module is isomorphic to the module of differential forms of degree d on R . The best duality is not obtained unless the ring R is homologically trivial in the sense that $\text{Ext}_S^j(R, S) = 0$ for $j \neq n - d$. Such rings are exactly the Cohen-Macaulay rings.

The duality is ‘perfect’ on R -modules whose projective dimension over the regular ring S is smallest possible. (These are the Cohen-Macaulay modules.) Let M be such a module of dimension q . Then

$$\text{Ext}_R^i(M, \omega) = 0$$

for $i \neq d - q$, $\text{Ext}_R^{d-q}(M, \omega)$ is again Cohen-Macaulay and

$$\text{Ext}_R^{d-q}(\text{Ext}_R^{d-q}(M, \omega), \omega) \cong M.$$

According to Bass in the ubiquity paper, Grothendieck introduced the following definition:

DEFINITION 3.1. Let S be a regular local ring. A local ring R which is a homomorphic image of S is *Gorenstein* if it is Cohen-Macaulay and its dualizing module (or *canonical module*) $\text{Ext}_S^{n-d}(R, S)$ is free (of rank 1), where $n = \dim(S)$ and $d = \dim(R)$. We say a possibly non-local Noetherian ring R (which we assume is the homomorphic image of a regular ring) is Gorenstein if R_P is Gorenstein for all prime ideals P .

All plane curves are Gorenstein, but even more we shall see that all complete intersections are Gorenstein. This definition will only be temporary in the sense that the work of Bass made it clear how to extend the definition to all local rings, and thereby all rings, without necessarily having a canonical module.

The earliest printed reference I could find giving a definition of Gorenstein rings is in [Se1, p. 2-11]. The date given for his seminar is November 21, 1960. He

defines an algebraic variety W to be Gorenstein if it is Cohen-Macaulay and the dualizing sheaf Ω_W is locally free of rank 1. He goes on to say that for plane curves R the equality $\dim(T_Q/R_Q) = \dim(R_Q/\mathfrak{C}_Q)$ is equivalent to being Gorenstein and cites [Se3]. He further mentions that complete intersections are Gorenstein in this context. In [Se3] (in 1959) Serre had proved these claims and mentioned the papers of Rosenlicht, Samuel and Gorenstein, as well as the work of Grothendieck on the duality approach.

With the definition of Grothendieck/Serre, it is not difficult to see that complete intersections are Gorenstein. We only need to use the free resolutions of such rings. These resolutions give a powerful tool not only to prove results about Gorenstein rings but to actually compute with them.

PROPOSITION 3.2. *Let (S, m) be a regular local ring of dimension n and let $R = S/I$ be a quotient of S of dimension d . Let*

$$0 \rightarrow F_k \xrightarrow{\alpha_k} F_{k-1} \xrightarrow{\alpha_{k-1}} \dots \rightarrow F_0 \rightarrow R \rightarrow 0$$

be a minimal free S -resolution of R . Then R is Gorenstein iff $k = n - d$, and $F_k \cong S$.

PROOF. We use a famous formula of Auslander and Buchsbaum to begin the proof. If S is a regular local ring and M a finitely generated module, then

$$\text{depth}(M) + \text{pd}_S M = \dim(S)$$

Applying this formula with $R = M$ we obtain that R is Cohen-Macaulay iff $\text{depth}(R) = \dim(R)$ (by definition) iff $k = n - d$.

The module $\text{Ext}_S^{n-d}(R, S)$ can be computed from the free S -resolution of R . We simply apply $\text{Hom}_S(_, S)$ to the resolution and take homology. The $n-d$ homology is the cokernel of the transpose of α_k from $F_{k-1}^* \rightarrow F_k^*$, where $(_)^* = \text{Hom}_S(_, S)$. If R is Gorenstein, this is free of rank 1 by definition. The minimality of the resolution together with Nakayama's lemma shows that the minimal number of generators of $\text{Ext}_S^{n-d}(R, S)$ is precisely the rank of F_k . Hence it must be rank 1.

Conversely, suppose that the rank of F_k is 1. Since I kills R , it also kills $\text{Ext}_S^{n-d}(R, S)$. It follows that $\text{Ext}_S^{n-d}(R, S)$ is isomorphic to S/J for some ideal $J \supseteq I$. However, the vanishing of the other Ext groups gives that the transposed complex

$$0 \rightarrow F_0^* \rightarrow \dots \rightarrow F_k^*$$

is actually acyclic, and hence a free S -resolution of $\text{Ext}_S^{n-d}(R, S)$. Therefore

$$\text{Ext}_S^{n-d}(\text{Ext}_S^{n-d}(R, S), S) \cong R$$

by dualizing the complex back, and now the same reasoning shows that J kills R , i.e. $J \subseteq I$. Thus $\text{Ext}_S^{n-d}(R, S) \cong R$ and R is Gorenstein. \square

Observe that the end of the proof proves that the resolution of a Gorenstein quotient R of S is essentially self-dual; if you flip the free resolution you again get a free resolution of R . In particular the ranks of the free modules are symmetric around the 'middle' of the resolution.

REMARK 3.3. Proposition 3.2 is especially easy to apply when the dimension of R is 0. In that case the depth of R is 0 which forces the projective dimension to be n , so that the first condition of (3.2) is automatically satisfied; one only needs to check that the rank of the last free module in the minimal free resolution of R is exactly 1 to conclude that R is Gorenstein.

REMARK 3.4: THE GRADED CASE. If $R = \bigoplus_n R_n$ is graded with $R_0 = k$ a field, and is Noetherian, we can write $R = S/I$ with $S = k[X_1, \dots, X_n]$ and I is a homogeneous ideal of S , where the X_i are given weights. In this case the free resolution of S/I over S can be taken to be graded. If we twist¹ the free modules so that the maps have degree 0, then the resolution has the form:

$$0 \rightarrow \sum_i S(-i)^{b_{ni}} \rightarrow \dots \rightarrow \sum_i S(-i)^{b_{1i}} \rightarrow S \rightarrow S/I \rightarrow 0.$$

For example, b_{1i} is the number of minimal generators of I in degree i .

Let M be the unique homogeneous maximal ideal of S generated by the X_i . Then R is Gorenstein iff R_M is Gorenstein. This follows from the characterization above. One uses that the free resolution of R over S can be taken to be graded, and hence the locus of primes P where R_P is Cohen-Macaulay is defined by the a homogeneous ideal and the locus where the canonical module is free is also homogeneous as the canonical module is homogeneous. This means that if there exists a prime P in R such that R_P is not Gorenstein, then P must contain either the homogeneous ideal defining the non Cohen-Macaulay locus or the homogeneous ideal defining the locus where the canonical module is not free. In either case, these loci are non-empty and will also be contained in M , forcing R_M not to be Gorenstein.

To prove that complete intersections are Gorenstein, one only needs to know about the Koszul complex: Let x_1, \dots, x_n be a sequence of elements in a ring R . The tensor product of the complexes

$$0 \rightarrow R \xrightarrow{x_i} R \rightarrow 0$$

is called the *Koszul complex*² of x_1, \dots, x_n . This complex is a complex of finitely generated free R -modules of length n . If the x_i form a regular sequence, then this complex provides a free resolution of $R/(x_1, \dots, x_n)$.

COROLLARY 3.5. *Let S be a regular local ring, and let I be an ideal of height k generated by k elements. Then $R = S/I$ is Gorenstein.*

PROOF. Since S is Cohen-Macaulay, I is generated by a regular sequence and the Koszul complex provides a free S -resolution of R . The length of the resolution is k , and the last module in this resolution is S . Applying Proposition 3.2 gives that R is Gorenstein. \square

The following sequence of implications is learned by all commutative algebraists:

$$\text{regular} \Rightarrow \text{complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}.$$

¹If M is a graded S -module, then the twist $M(t)$ is the same module but with the different grading given by $M(t)_n = M_{t+n}$.

²There is another perhaps more fundamental way to think of the Koszul complex as an exterior algebra. See [BH, 1.6].

In general none of these implications is reversible. The last implication is reversible if the ring is a UFD and a homomorphic image of a Gorenstein ring, as we shall discuss later.

The language and results of free resolutions give a useful tool to analyze Gorenstein rings, especially in low codimension.

4. Examples and Low Codimension

Let S be a regular local ring. If an ideal I has height one and S/I is Gorenstein, then in particular I is unmixed. Since S is a UFD, it follows that $I = (f)$ is principal, and S/I is a complete intersection.

Let I be a height two ideal defining a Gorenstein quotient, i.e. $R = S/I$ is Gorenstein of dimension $n - 2$. The resolution for R must look like:

$$0 \rightarrow S \rightarrow S^l \rightarrow S \rightarrow S/I = R \rightarrow 0.$$

But counting ranks shows that $l = 2$, which means that I is generated by 2 elements and is a complete intersection. This recovers a result of Serre [Se1].

A natural question is whether all Gorenstein rings are complete intersections. The answer is a resounding NO, but it remains an extremely important question how to tell if a given Gorenstein ring is a complete intersection; this question plays a small but important role in the work of Wiles, for example.

In codimension three, the resolution of a Gorenstein R looks like:

$$0 \rightarrow S \rightarrow S^l \rightarrow S^l \rightarrow S \rightarrow S/I = R \rightarrow 0.$$

The question is whether $l = 3$ is forced. The answer is no. But surprisingly, the minimal number of generators must be odd.

EXAMPLE 4.1. Let $S = k[X, Y, Z]$, and let

$$I = (XY, XZ, YZ, X^2 - Y^2, X^2 - Z^2).$$

The nilradical of I is the maximal ideal $m = (X, Y, Z)$ which has height three. But I requires five generators and so is not a complete intersection. On the other hand I is Gorenstein. One can compute the resolution of S/I over S and see that the last Betti number is 1. The resolution looks like:

$$0 \rightarrow S \rightarrow S^5 \rightarrow S^5 \rightarrow S \rightarrow S/I \rightarrow 0.$$

The graded resolution is:

$$0 \rightarrow S(-5) \rightarrow S^5(-3) \rightarrow S^5(-2) \rightarrow S \rightarrow S/I \rightarrow 0.$$

This example is a special case of a famous theorem of Buchsbaum and Eisenbud [BE]. To explain their statement we first recall what Pfaffians are. Let A be a skew-symmetric matrix of size $2n$ by $2n$. Then the determinant of A is the square of an element called the Pfaffian of A (the sign is determined by convention). If A is skew symmetric and of size $2n + 1$, the ideal of Pfaffians of order $2i$ ($i \leq n$) of A is the ideal generated by the Pfaffians of the submatrices of A obtained by choosing $2i$ rows and the same $2i$ columns. The theorem of Buchsbaum and Eisenbud states:

THEOREM 4.2. *Let S be a regular local ring, and let I be an ideal in S of height 3. Set $R = S/I$. Then R is Gorenstein iff I is generated by the $2n$ -order Pfaffians of a skew-symmetric $2n+1$ by $2n+1$ alternating matrix A . In this case a minimal free resolution of R over S has the form,*

$$0 \rightarrow S \rightarrow S^{2n+1} \xrightarrow{A} S^{2n+1} \rightarrow S \rightarrow R \rightarrow 0.$$

Unfortunately, there is no structure theorem for height four ideals defining Gorenstein rings, although there has been a great deal of work on this topic. See the references in [VaVi] and [KM].

Thinking back to the first example of monomial plane curves, it is natural to ask when the ring $k[t^{n_1}, \dots, t^{n_k}]$ is Gorenstein.

DEFINITION 4.3. A semigroup $T = \langle 0, n_1, \dots, n_k \rangle$ is said to be *symmetric* if there is a value $c \notin T$ such that $m \in T$ iff $c - m \notin T$.

THEOREM 4.4 [HeK1]. *The monomial curve $R = k[t^{n_1}, \dots, t^{n_k}]$ is Gorenstein iff $T = \langle 0, n_1, \dots, n_k \rangle$ is symmetric.*

It follows that every semigroup T generated by two relatively prime integers a and b is symmetric, since this is a plane curve, hence Gorenstein. For such examples, the value of c as in (4.3) is $c = (a-1)(b-1) - 1$. In general c must be chosen one less than the least power of t in the conductor. This example can be used to give many examples of Gorenstein curves which are not complete intersections or Pfaffians. The easiest such example is $T = \langle 0, 6, 7, 8, 9, 10 \rangle$. The ring $R = k[t^6, t^7, t^8, t^9, t^{10}]$ is Gorenstein since T is symmetric. The conductor is everything from t^{12} and up, and one can take $c = 11$. The defining ideal when we represent R as a quotient of a polynomial ring in 5 variables has height 4 and requires 9 generators. The minimal resolution over the polynomial ring S has the form

$$0 \rightarrow S \rightarrow S^9 \rightarrow S^{16} \rightarrow S^9 \rightarrow S \rightarrow R \rightarrow 0.$$

See [Br] for a discussion of symmetric semigroups. For a non-Gorenstein example, we can take $R = k[t^3, t^4, t^5]$. The semi-group is not symmetric. Writing R as a quotient of a polynomial ring in three variables, the defining ideal is a prime ideal of height 2 requiring 3 generators. From the theorem of Serre explained earlier, one knows that this cannot be Gorenstein since it would have to be a complete intersection being height two. Alternatively, one could use an amazing result due to Kunz [Ku], that almost complete intersections are never Gorenstein. Almost complete intersection means minimally generated by one more element than a complete intersection. There are non-trivial examples of ideals generated by two more elements than their height which define Gorenstein ideals; the Pfaffian ideal of 5 generators gives such an examples, while whole classes were constructed in [HU2] (see also [HM]).

Where does ubiquity fit into this? We need another historical thread before the tapestry is complete. This last thread concerns the theory of injective modules.

5. Injective Modules and Matlis Duality

There is a ‘smallest’ injective module containing an R -module M , denoted $E_R(M)$, or just $E(M)$. It is called the *injective hull* of M . (It is the largest

essential extension of M .) Any module both essential over M and injective must be isomorphic to the injective hull of M .

THEOREM 5.1 (MATLIS, [Ma]). *Let R be a Noetherian commutative ring. Every injective module is a direct sum of indecomposable injective modules, and the nonisomorphic indecomposable injective modules are exactly (up to \cong) $E(R/P)$ for P prime.*

Let (R, m) be a 0-dimensional Noetherian local ring. There is only one indecomposable injective module, E , the injective hull of the residue field of R . Matlis proved that the length of E , that is the number of copies of the residue field $k = R/m$ in a filtration whose quotients are k , is the same as that of R . Matlis extended these ideas and came up with what is now called Matlis duality.

THEOREM 5.2. *Let (R, m) be a Noetherian local ring with residue field k , $E = E_R(k)$, and completion \widehat{R} . Then there is a 1–1 arrow reversing correspondence from finitely generated \widehat{R} -modules M to Artinian R -modules N given by $M \rightarrow M^\vee$ and $N \rightarrow N^\vee$ where $(\)^\vee := \text{Hom}_R(\ , E)$. Furthermore $M^{\vee\vee} \cong M$ and $N^{\vee\vee} \cong N$. Restricting this correspondence to the intersection of Artinian modules and finitely generated modules, i.e. to modules of finite length, preserves length.*

The work of Matlis allows one to understand the 0-dimensional commutative Noetherian rings which are self-injective.

THEOREM 5.3. *Let (R, m) be a 0-dimensional local ring with residue field k and $E = E_R(k)$. The following are equivalent.*

1. R is injective as an R -module.
2. $R \cong E$.
3. (0) is an irreducible ideal.
4. The socle of R is 1-dimensional.
5. R is Gorenstein in the sense of Grothendieck/Serre.

PROOF. Clearly (2) implies (1). On the other hand, if R is injective it must be a sum of copies of E by the theorem of Matlis. Since R is indecomposable, it is then isomorphic to E .

Every 0-dimensional local ring is an essential extension of its socle V . For given any submodule of R , that is an ideal I of R , there is a least power n of m such that $m^n I = 0$. Then $m^{n-1} I \subseteq V \cap I$ and is nonzero. Since the length of R and E agree, it then is clear that R is self-injective iff its socle is 1-dimensional, i.e., iff 0 is irreducible. This shows the equivalence of (2), (3) and (4).

To see the equivalence of (5), first observe that R is the homomorphic image of a regular local ring S , using the Cohen structure theorem. R is Cohen-Macaulay because it is 0 dimensional and of course has depth 0. So it just needs to be shown that $\text{Ext}_S^n(R, S)$ is isomorphic to R , where $\dim(S) = n$. Equivalently, as shown in Proposition 3.2 we need to prove that the rank of the last free module in a minimal free S resolution of R is exactly one. This rank is the dimension of $\text{Tor}_S^n(k, R)$ by computing this Tor using the minimal free resolution of R . However, k has a free S -resolution by a Koszul complex since the maximal ideal of S is generated by a regular sequence as S is regular. Using the Koszul complex to compute this Tor yields the isomorphism

$$\text{Tor}_S^n(k, R) \cong \text{soc}(R).$$

Thus R is Gorenstein iff the socle is 1-dimensional. \square

A natural question from Theorem 5.3 and from the second theorem of Northcott and Rees is how to calculate the socle of a system of parameters in a regular local ring (R, m) . For if one takes a regular local ring and quotients by a system of parameters, then 0 is irreducible by Northcott and Rees, and the quotient ring is therefore a self-injective ring. It turns out there is an excellent answer to how to calculate socles.

THEOREM 5.4. *Let (R, m) be a regular local ring of dimension d , and write $m = (x_1, \dots, x_d)$. Let f_1, \dots, f_d be a system of parameters. Write $(x_1, \dots, x_d)A = (f_1, \dots, f_d)$ for some d by d matrix A , and put $\Delta = \det(A)$. Then the image of Δ in $R/(f_1, \dots, f_d)$ generates the socle of this Artinian algebra.*

Another way to state this theorem is $(f_1, \dots, f_d) :_R m = (f_1, \dots, f_d, \Delta)$. Here $I :_R J = \{r \in R \mid rJ \subseteq I\}$. Theorem 5.4 gives a very useful computational device for computing socles. One easy example is to compute a generator for the socle of $R/(x_1^{n_1}, \dots, x_d^{n_d})$. In this case the matrix A can be taken to be a diagonal matrix whose i th entry along the diagonal is $x_i^{n_i-1}$, and $\Delta = \prod_i x_i^{n_i-1}$. Of course, when we lift the socle generator back to R it is unique only up to a unit multiple plus an element of the ideal we mod out.

In the case when R is complete, regular, and contains a field, the Cohen Structure Theorem gives that R is isomorphic with a formal power series over a field. In this case the theory of residues can be used to give another description of the socle:

THEOREM 5.5. *Let $R = k[[x_1, \dots, x_d]]$ be a formal power series ring over a field k of characteristic 0, and let f_1, \dots, f_d be a system of parameters in R . Let J be the Jacobian matrix whose (i, j) entry is $\frac{\partial f_i}{\partial x_j}$. Set $\Delta = \det(J)$. Then Δ generates the socle of $R/(f_1, \dots, f_d)$.*

If the f_i happen to be homogeneous polynomials, then this Theorem follows directly from the previous one as Euler's formula can be used to express the f_i in terms of the x_j using only units times the rows of the Jacobian matrix. However, in the nonhomogenous case, this second theorem is extremely useful.

REMARK 5.6. In the graded case in which $R = k[R_1]$ is 0-dimensional and a homomorphic image of a polynomial ring $S = k[X_1, \dots, X_n]$, one can calculate the degree of the socle generator for R by using the free resolution of R as an S -module. The resolution will be of the form

$$0 \rightarrow S(-m) \rightarrow \dots \rightarrow \sum_i S(-i)^{b_{1i}} \rightarrow S \rightarrow R \rightarrow 0.$$

The last module will be a copy of S twisted by an integer m since R is Gorenstein. The degree of the socle of R is then exactly $m - n$. For example, in (4.1) we saw that the last twist is $S(-5)$ in the resolution of the ring $R = k[X, Y, Z]/I$ where $I = (X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$. Hence the socle of R sits in degree $2 = 5 - 3$. In fact the socle can be taken to be any quadric not in I .

6. Ubiquity

In the early 1960s, Bass had been studying properties of rings with finite injective dimension. In particular, he related irreducibility to finite injective dimension. The following is from Bass [Ba1] (1962):

THEOREM 6.1. *Let R be a commutative Noetherian local ring. The following conditions are equivalent:*

1. R is Cohen-Macaulay and every system of parameters generates an irreducible ideal.
2. $\text{id}_R(R) < \infty$.
3. $\text{id}_R(R) = \dim(R)$.

One can see in retrospect how the theme of irreducibility, injective dimension and complete intersections were coming together. Most of the work discussed above was available by the late 1950s. The Auslander-Buchsbaum work was done in 1957-58, Grothendieck's work on duality in 1957, Serre's work in 1957-1960, Matlis's work on injective modules in 1957, and the work of Northcott and Rees on irreducible systems of parameters was done in 1957. The work of Rosenlicht and Gorenstein was done about 1952. According to Bass, Serre pointed out to him that the rings of finite injective dimension were, at least in the geometric context, simply the Gorenstein rings of Grothendieck. To quote Bass's paper he remarks, 'Gorenstein rings, it is now clear, have enjoyed such a variety of manifestations as to justify, perhaps, a survey of their relevance to various situations and problems.' The rest, as one says, is history.

Bass put all this together in the ubiquity paper in 1963 [Ba2]. It remains one of the most read papers in commutative algebra.

THEOREM 6.2 UBIQUITY. *Let (R, m) be a Noetherian local ring. The following are equivalent.*

1. If R is the homomorphic image of a regular local ring, then R is Gorenstein in the sense of Grothendieck/Serre.
2. $\text{id}_R(R) < \infty$.
3. $\text{id}_R(R) = \dim(R)$.
4. R is Cohen-Macaulay and some system of parameters generates an irreducible ideal.
5. R is Cohen-Macaulay and every system of parameters generates an irreducible ideal.
6. If $0 \rightarrow R \rightarrow E^0 \rightarrow \dots \rightarrow E^h \rightarrow \dots$ is a minimal injective resolution of R , then for each $h \geq 0$, $E^h \cong \sum_{\text{height}(p)=h} E_R(R/p)$.

By Theorem 2.5 of Northcott and Rees one can remove the assumption that R be Cohen-Macaulay in (5). We can now take any of these equivalent properties to be the definition of Gorenstein in the local case. The usual one chosen is the second, that the ring have finite injective dimension over itself. In the non-local case we say R is Gorenstein if R_P is Gorenstein for every prime P in R . If R has finite Krull dimension this is equivalent to saying R has finite injective dimension over itself.

EXAMPLE 6.3. Let (R, m) be a one dimensional Gorenstein local ring. The equivalent conditions above say that the injective resolution of R looks like

$$0 \rightarrow R \rightarrow K \rightarrow E \rightarrow 0$$

where K is the fraction field of R (this is the injective hull of R) and E is the injective hull of the residue field $k = R/m$. This gives a nice description of E as K/R . If we further assume that $R = k[[t]]$, a regular local ring, then one can identify K/R with the inverse powers of t . This module is an essential extension of the residue field k which sits in K/R as the R -span of t^{-1} . This example generalizes to higher dimensions in the form of the so-called ‘inverse systems’ of Macaulay. See section 8.

Moreover in dimension one for Noetherian reduced rings (this means the ring has no nonzero nilpotent elements) with finite integral closure, Bass was able to generalize the results of Gorenstein and Rosenlicht to arbitrary Gorenstein rings:

THEOREM 6.4. *Let S be a regular local ring and let $R = S/I$ be a 1-dimensional Gorenstein domain. Assume that the integral closure of R is a finite R -module (this is automatic for rings essentially of finite type over a field, or for complete local rings). Let T be the integral closure of R and let \mathfrak{C} be the conductor. Then*

$$\lambda(R/\mathfrak{C}) = \lambda(T/R).$$

PROOF. Let ω be the canonical module of R , namely $\text{Ext}_S^{n-1}(R, S)$ where $n = \dim(S)$. Consider the exact sequence, $0 \rightarrow R \rightarrow T \rightarrow D \rightarrow 0$, where $D \cong T/R$. Applying $(\)^* = \text{Hom}_R(\ , R)$ we get the sequence,

$$D^* \rightarrow T^* \rightarrow R^* \rightarrow \text{Ext}_R^1(D, R) \rightarrow \text{Ext}_R^1(T, R).$$

Since T is a Cohen-Macaulay of dimension 1, the last Ext vanishes. As D is torsion $D^* = 0$. Moreover $R^* \cong R$ and $T^* \cong \mathfrak{C}$, the isomorphism given by sending a homomorphism to its evaluation at a fixed nonzero element. It follows that $R/\mathfrak{C} \cong \text{Ext}_R^1(D, R)$, and it only remains to prove that the length of $\text{Ext}_R^1(D, R)$ is the same as the length of D . This follows since R is Gorenstein. We can use the injective resolution of R as in Example 6.3 to find that

$$\text{Ext}_R^1(D, R) \cong \text{Hom}_R(D, E).$$

By Matlis duality the length of this last module is the same as the length of D . \square

REMARK. In fact the converse to Proposition 6.4 is also true.

7. Homological Themes

One of the remarks Bass makes in the ubiquity paper is the following: ‘It seems conceivable that, say for A local, there exist finitely generated $M \neq 0$ with finite injective dimension only if A is Cohen-Macaulay. The converse is true for if $(B = A$ modulo a system of parameters) and M is a finitely generated non-zero B -injective module then $\text{inj dim}_A(M) < \infty$.’

This became known as Bass’s conjecture, one of a celebrated group of ‘homological conjectures’. The search for solutions to these problems was a driving force behind commutative algebra in the late 1960s and 1970s, and continues even today.

However, Bass's conjecture has now been solved positively, by Peskine and Szpiro [PS1] in the geometric case, by Hochster [Ho1] for all local rings containing a field, and by Paul Roberts [Rob1] in mixed characteristic.

The following is due to Peskine and Szpiro [PS1, (5.5) and (5.7)] based on their work on the conjecture of Bass:

THEOREM 7.1. *Let (R, m) be a local ring. The following are equivalent:*

1. R is Gorenstein.
2. \exists an ideal I such that $\text{id}_R(R/I) < \infty$.
3. \exists an m -primary ideal I such that $\text{pd}_R(R/I) < \infty$ and I is irreducible.

Other homological themes were introduced in the ubiquity paper which are now part of the standard landscape. A module M of a local ring R is said to be a k th syzygy if there is an exact sequence,

$$0 \rightarrow M \rightarrow F_{k-1} \rightarrow \dots \rightarrow F_0$$

where the F_i are finitely generated free R -modules. It is important to know intrinsically when a given module is a k th syzygy. For Gorenstein local rings there is a good answer due to Bass [Ba2, Theorem 8.2]:

THEOREM 7.2. *Let R be a Gorenstein local ring, and $k \geq 2$. A finitely generated R -module M is a k th syzygy iff M is reflexive and $\text{Ext}_R^i(M^*, R) = 0$ for all $i \leq k - 1$.*

The idea of the proof is to take a resolution of the dual module $M^* := \text{Hom}_R(M, R)$ and then dualize it, using that $M \cong M^{**}$ since M is reflexive. A modern version of this result was given by Auslander and Bridger [AB], see for example [EG, Theorem 3.8].

Other important homological objects introduced by Bass in ubiquity were the Bass numbers, $\mu^i(p, M) = \dim_{k(p)} \text{Ext}_{R_p}^i(k(p), M_p)$, where $k(p) = R_p/pR_p$ are the residue fields of the localizations at prime ideals of R . These numbers play a very important role in understanding the injective resolution of M , and are the subject of much work. For example, another equivalent condition for a local Noetherian ring (R, m) to be Gorenstein is that R be Cohen-Macaulay and the d th Bass number, $\mu^d(m, R) = 1$. This was proved by Bass. Vasconcelos conjectured that one could delete the hypothesis that R be Cohen-Macaulay. This was proved by Paul Roberts in 1983 [Rob2]. Another line of work inspired by the work of Bass is that of Avramov and Foxby, who have developed and studied the concept of homomorphisms between rings being Gorenstein. For example see [AvF] and the references there.

8. Inverse Powers and 0-dimensional Gorenstein Rings

The study of Gorenstein rings can be approached by first trying to understand the 0-dimensional Gorenstein rings. These are all the 0-dimensional Artinian rings which are injective as modules over themselves. Equivalently they are exactly the 0-dimensional Artinian rings with a 1-dimensional socle. Given any local Gorenstein ring of arbitrary dimension, one can always mod out the ideal generated by a system of parameters and obtain a 0-dimensional Gorenstein ring. In the 0-dimensional case, one has the formidable results of Matlis to help.

EXAMPLE 8.1. Let $R = k[X_1, \dots, X_n]$, or a power series ring $k[[X_1, \dots, X_n]]$. An injective hull of the residue field $k = R/(X_1, \dots, X_n)$ is given by the *inverse powers*,

$$E = k[X_1^{-1}, \dots, X_n^{-1}]$$

with the action being determined by

$$X_i(X_1^{-a_1} \dots X_n^{-a_n}) = X_1^{-a_1} \dots X_i^{-a_i+1} \dots X_n^{-a_n}$$

if $a_i \geq 1$. If $a_i = 0$, then the product is 0. Notice the copy of k inside E is exactly the span of 1.

EXAMPLE 8.2. To obtain the injective hull of the residue field of a graded quotient

$$k[X_1, \dots, X_n]/I$$

one simply takes $\text{Hom}_S(S/I, E)$. This follows easily from the fact that this module is injective over S/I (using Hom-tensor adjointness) and is also essential over k , since it can be identified with a submodule of E which is already essential over k . This is naturally identified with the elements of E killed by I . One sees that the graded structure of E is simply that of R upside down.

R is 0-dimensional Gorenstein local iff $R \cong E(k)$. In this case, $R \cong \text{Hom}_S(R, E)$ which means that the annihilator of R in the inverse powers is cyclic. The converse also holds. The m -primary ideals I such that $R = S/I$ are Gorenstein are EXACTLY the annihilators of single elements in the inverse powers. (See Proposition 8.4 below.)

EXAMPLE 8.3. Let $F = X^{-2} + Y^{-2} + Z^{-2}$. The set of elements in S killed by F is the ideal

$$I = (X^2 - Y^2, Y^2 - Z^2, XY, XZ, YZ).$$

This is the height 3 Gorenstein ideal that we considered in Example 4.1.

It is not difficult to prove that if F is a quadratic form in X^{-1}, Y^{-1}, Z^{-1} , then the rank of F as a quadratic form is 3 iff the corresponding ideal has 5 generators, while if the rank is at most 2, the corresponding ideal is a complete intersection generated by 3 polynomials. See [Ei, p. 551, exer. 21.6].

In characteristic 0, there is an important form of inverse powers using the ring of differential operators. Let k be a field of characteristic 0 and let $T = k[y_1, \dots, y_n]$ be a polynomial ring over k . A polynomial differential operator with constant coefficients is an operator on T of the form

$$D = \sum_{i_1, \dots, i_r} a_{i_1, \dots, i_r} \left(\frac{\partial}{\partial y_1}\right)^{i_1} \dots \left(\frac{\partial}{\partial y_n}\right)^{i_n}$$

with $a_{i_1, \dots, i_r} \in k$. We think of D as an element in a new polynomial ring $S = k[x_1, \dots, x_n]$ acting on T by letting x_i act as $\frac{\partial}{\partial y_i}$. Since the partials commute with each other this makes sense. Fix $f \in T$, and let $I = I_f$ be the ideal of all elements $D \in S$ such that $Df = 0$. Then I is an ideal in S primary to (x_1, \dots, x_n) such that S/I is 0-dimensional Gorenstein. Furthermore, all such 0-dimensional Gorenstein quotients of S arise in this manner. The point is that we can identify T with the injective hull of the residue field of S . See [Ei, exer. 21.7]. This point of view has been very important for work on the Hilbert scheme of 0-dimensional Gorenstein

quotients of fixed dimension. See for example the paper of Iarrobino [Ia] and its references.

There is a related way to construct all 0-dimensional Gorenstein local rings containing a field. Start with a power series ring $S = k[[X_1, \dots, X_n]]$. Choose an arbitrary system of parameters f_1, \dots, f_n in S and an arbitrary element $g \notin (f_1, \dots, f_n)$. Set $I = (f_1, \dots, f_n) :_S g$. Then $R = S/I$ is a 0-dimensional Gorenstein ring. Conversely, all 0-dimensional Gorenstein rings are of this form. The proof follows at once from the following proposition:

PROPOSITION 8.4. *Let R be a 0-dimensional Gorenstein ring, and let $J \subseteq R$. Then R/J is Gorenstein iff $J = 0 :_R g$ for some nonzero element $g \in R$. Precisely, $gR = 0 :_R J$.*

PROOF. Since R is Gorenstein, it is isomorphic to its own injective hull. It follows that the injective hull of R/J is $\text{Hom}_R(R/J, R) \cong 0 :_R J$, and R/J is Gorenstein iff $0 :_R J \cong R/J$ iff $0 :_R J = gR$ and $0 :_R g = J$. \square

For example, to construct the Gorenstein quotient $k[X, Y, Z]/I$ where $I = (X^2 - Y^2, Y^2 - Z^2, XY, XZ, YZ)$, we can do the following. First take the complete intersection quotient $T = k[X, Y, Z]/(X^3, Y^3, Z^3)$. Set $g = X^2Y^2 + X^2Z^2 + Y^2Z^2$. Then $I = (X^3, Y^3, Z^3) : g$ and is therefore Gorenstein. ALL Gorenstein quotients R/K such that $(X^3, Y^3, Z^3) \subseteq K$ arise in this fashion: $K = (X^3, Y^3, Z^3) : f$ for some $f \in k[X, Y, Z]$ and f is unique up to an element of (X^3, Y^3, Z^3) and a unit multiple.

Understanding 0-dimensional Gorenstein rings is extremely important if one uses Gorenstein rings as a tool. For a great many problems one can reduce to a 0-dimensional Gorenstein ring. A natural question in this regard is the following: let (R, m) be a Noetherian local ring. When does there exist a descending sequence of m -primary ideals I_n such that R/I_n are Gorenstein for all n (equivalently I_n are irreducible) and such that $I_n \subseteq m^n$? Melvin Hochster completely answered this question. He called such rings *approximately Gorenstein*.

THEOREM 8.5 [Ho3, 1.2, 1.6]. *A local Noetherian ring (R, m) is approximately Gorenstein if and only if its m -adic completion is approximately Gorenstein. An excellent local ring R (so, for example, a complete local Noetherian ring) with $\dim(R) \geq 1$ is approximately Gorenstein if and only if the following two conditions hold:*

1. $\text{depth}(R) \geq 1$.
2. If $P \in \text{Ass}(R)$ and $\dim(R/P) = 1$, then $R/P \oplus R/P$ is not embeddable in R .

9. Hilbert Functions

The *Hilbert function* of a graded ring over a field is the function $H(n) = \dim_k R_n$. The generating function for the Hilbert function is called the Hilbert series, i.e. the series

$$F(R, t) = \sum_{n \geq 0} (\dim_k R_n) t^n$$

If R is generated by one-forms and is Noetherian, we can always write

$$F(R, t) = \frac{h_0 + h_1 t + \dots + h_l t^l}{(1-t)^d}$$

where $d = \dim R$, the Krull dimension of R . The vector of integers (h_0, h_1, \dots, h_l) is called the h -vector of R . For example, if R is 0-dimensional, then $h_i = H(i)$.

Let $R = k[R_1]$. We can write $R = S/I$ where $S = k[X_1, \dots, X_n]$ and I is a homogeneous ideal of S . In this case the free resolution of S/I over S can be taken to be graded, as in Section 3 above. The resolution has the form

$$0 \rightarrow \sum_i S(-i)^{b_{ni}} \rightarrow \dots \rightarrow \sum_i S(-i)^{b_{1i}} \rightarrow S \rightarrow S/I \rightarrow 0.$$

Fixing a degree n allows one to compute the dimension of $(S/I)_n$ as the alternating sum $\sum_{j,i} (-1)^j (S(-i)^{b_{ji}})_n$. Each term in this sum is easily computable as the dimension of the forms of a certain degree in the polynomial ring.

To do this calculation, it is more convenient to write

$$F_i = S(-c_{1i}) \oplus \dots \oplus S(-c_{ni}).$$

Then the Hilbert series for S/I is exactly

$$\frac{\sum_{i=0}^t (-1)^i (t^{c_{1i}} + \dots + t^{c_{ni}})}{(1-t)^n}.$$

The free resolution of a Gorenstein ring is essentially symmetric. The dual of the resolution of a Cohen-Macaulay ring always gives a free resolution of $\text{Ext}_S^{n-d}(R, S)$, the canonical module (Here S is a regular local ring of dimension n and R is Cohen-Macaulay of dimension d). When R is further assumed to be Gorenstein the canonical module is isomorphic to R , and the flipped resolution is isomorphic to the original resolution. This basic duality should make the following theorem no surprise:

THEOREM 9.1. *A 0-dimensional graded Gorenstein ring $R = R_0 \oplus R_1 \oplus \dots \oplus R_t$ (with $k = R_0$ a field and $R_t \neq 0$) has symmetric Hilbert function, i.e. $H(i) = H(t-i)$ for all $i = 0, \dots, t$.*

One sees this by considering the pairing $R_i \times R_{t-i} \rightarrow R_t = k$. Since R is essential over the socle, it is not difficult to show that this pairing is perfect and identifies R_i with the dual of R_{t-i} .

EXAMPLE 9.2. Let us compute the Hilbert function of the Example 4.1. The ideal is defined by five quadrics in three variables, and every cubic is in the defining ideal. Hence the Hilbert function is $(1, 3, 1)$.

EXAMPLE 9.3. Let f_1, \dots, f_n be a homogeneous system of parameters in $S = k[X_1, \dots, X_n]$ of degrees m_1, \dots, m_n respectively generating an ideal I . The Hilbert function can be computed from the graded free resolution of S/I , and this is just the Koszul complex. The various graded free modules in this resolution depend only upon the degrees of the f_i , and so the Hilbert function is the same as that of $X_1^{m_1}, \dots, X_n^{m_n}$. This is easily computed as above. The socle is generated in degree $\sum_i (m_i - 1)$.

This symmetry and the fact the socle is one-dimensional is behind the classical applications of the Gorenstein property outlined in [EGH]. That article details nine versions of the so-called Cayley-Bacharach Theorem, beginning with the famous theorem of Pappus proved in the fourth century A.D., and ending with a common generalization of all of them: that polynomial rings are Gorenstein! It is worth repeating one of these avatars here and hopefully hooking the reader to look at [EGH]. The following theorem was proved by Chasles in 1885:

THEOREM 9.4. *Let $X_1, X_2 \subseteq \mathbf{P}^2$ be cubic plane curves meeting in exactly nine points. If $X \subseteq \mathbf{P}^2$ is any cubic containing eight of these points, then it contains the ninth as well.*

PROOF. First note that X_1 and X_2 meet in the maximum number of points by Bézout's theorem. We interpret the theorem algebraically. Let $S = k[X, Y, Z]$ be the homogeneous coordinate ring of the projective plane. X_1 and X_2 correspond to the vanishing loci of two homogeneous cubics F_1, F_2 . The ideal $I = (F_1, F_2) = Q_1 \cap \dots \cap Q_9$ where Q_i is defined by two linear forms. These correspond to the nine points in which X_1 and X_2 meet. Assume that another cubic, G , is contained in $Q_1 \cap \dots \cap Q_8$. We want to prove that $G \in Q_9$, i.e. that $G \in I$.

We proceed to cut down to Artinian quotients by killing a general linear form. Set $T = S/(\ell)$, where ℓ is a general linear form. Of course T is simply a polynomial ring in two variables. We write f_1, f_2, g for the images of F_1, F_2, G in this new ring. $R = T/(f_1, f_2)$ is a complete intersection whose Hilbert function is $(1, 2, 3, 2, 1)$. The socle of $T/(f_1, f_2)$ sits in degree 4. If $G \notin I$, then $g \notin (f_1, f_2)$, and g sits in degree 3. Since R is Gorenstein, gR must contain the socle. As g sits in degree three, the Hilbert function of the ideal gR is $(0, 0, 0, 1, 1)$ and hence R/gR has dimension seven. The Hilbert function of R/gR must be $(1, 2, 3, 1)$. However, by assumption (F_1, F_2, G) is contained in the eight linear ideals Q_1, \dots, Q_8 . When we cut by a general linear form, it follows that the dimension must be *at least* 8! This contradiction proves that $G \in I$. \square

A beautiful result of Richard Stanley shows that the symmetry of the Hilbert function is not only necessary but even a sufficient condition for a Cohen-Macaulay graded domain to be Gorenstein.

THEOREM 9.5 [St3]. *If $R = k[R_1]$ is a Cohen-Macaulay domain of dimension d , then R is Gorenstein iff*

$$F(R, t) = (-1)^{dt} F(R, 1/t)$$

for some $l \in \mathbf{Z}$.

EXAMPLE 9.6. The converse is NOT true if the ring is not a domain. For instance, $I = (X^3, XY, Y^2)$ has Hilbert function $(1, 2, 1)$ and satisfies the equation $F(t) = t^2 F(1/t)$, but is not Gorenstein.

It is an open problem posed by Stanley to characterize which sequences of integers can be the h-vector of a Gorenstein graded algebra. Symmetry is a necessary but not sufficient condition. Stanley gives an example of a Gorenstein 0-dimensional graded ring with Hilbert function $(1, 13, 12, 13, 1)$. The lack of unimodality is the interesting point in this example.

Stanley's theorem was at least partially motivated, as far as I can tell, by the uses he makes of it. One of them is to prove that invariants of tori are Gorenstein. Invariants of group actions provide a rich source of Gorenstein rings, as we will discuss in the next section.

10. Invariants and Gorenstein rings

There are a great many sources of Gorenstein rings. Complete intersections are the most prevalent. After all every finitely generated k -algebra which is a domain is birationally a complete intersection, hence birationally Gorenstein. The Gorenstein property behaves well under flat maps, and often fibers of such maps are Gorenstein. For example, if K is a finitely generated field extension of a field k and L is an arbitrary field extension of k , then $K \otimes_k L$ is Gorenstein.

A huge source of Gorenstein rings comes from invariants of groups. We fix notation and definitions. Let G be a closed algebraic subgroup of the general linear group $GL(V)$, where V is a finite dimensional vector space over a ground field k . We obtain a natural action of G on the polynomial ring $k[V] = S$, and we denote the ring of invariant polynomials by $R = S^G$. Thus,

$$S^G = \{f \in k[V] \mid g(f) = f \text{ for all } g \in G\}.$$

We will focus on groups G which are linearly reductive. This means that every G -module V is a direct sum of simple modules. Equivalently, every G -submodule of V has a G -stable complement. Examples of linearly reductive groups include finite groups whose order is invertible, tori, the classical groups in characteristic 0, and semisimple groups in characteristic 0. When G is linearly reductive, there is a retraction $S \rightarrow S^G$ called the *Reynolds operator* which is S^G -linear. It follows that the ring of invariants R must be Noetherian. For a chain of ideals in R , when extended to S will stabilize, and then applying the retraction stabilizes the chain in R . But even more is true: a famous theorem of Hochster and Roberts [HoR] says that the ring of invariants is even Cohen-Macaulay.

THEOREM 10.1. *Let S be a Noetherian k -algebra which is regular (e.g. a polynomial ring over k) and let G be a linearly reductive linear algebraic group acting k -rationally on S . Then S^G is Cohen-Macaulay.*

A natural question is to ask when S^G is Gorenstein. The following question was the focus of several authors (see [Ho2, St1, St2, Wa] and their references).

QUESTION 10.2. Let k be an algebraically closed field and let G be a linearly reductive algebraic group over k acting linearly on a polynomial ring S over k such that the det is the trivial character. Is S^G Gorenstein?

Thus, for example, when G is not only in $GL(V)$, but in $SL(V)$, the ring of invariants should be Gorenstein. This question was motivated because of the known examples: it was proved for finite groups whose order is invertible by Watanabe [Wa], for an algebraic torus by Stanley [St1], or for connected semisimple groups by a result of Murthy [Mu] that Cohen-Macaulay rings which are UFDs are Gorenstein, provided they have a canonical module. However, this question has a negative solution. Friedrich Knop in 1989 [Kn] gave a counterexample to Question 10.2 and has basically completed described when the ring of invariants is Gorenstein.

Murthy's result quoted in the above paragraph is straightforward based on the theory of the canonical module. The canonical module of an integrally closed Noetherian ring R is a rank 1 reflexive module which therefore lives in the class group. If R is a UFD, the class group is trivial and so is the canonical module. But this means that it is a free R module, which together with R being Cohen-Macaulay gives Gorenstein.³ Griffith has shown in [Gr, Theorem 2.1] that if S is a local ring which is a UFD, which is finite over a regular local ring R in such a way that the extension of fraction fields is Galois, then S is necessarily a complete intersection. See also [AvB].

11. Ubiquity and Module Theory

In trying to recapture the material that led Bass to write his ubiquity paper it seems that his choice of the word 'ubiquity' came from that fact that the Gorenstein property was arising in totally different contexts from many authors; from Northcott and Rees' work on irreducible systems of parameters and Cohen-Macaulay rings, from the work of Gorenstein and Rosenlicht on plane curves and complete intersections, from the work of Grothendieck and Serre on duality, and from his own work on rings of finite injective dimension. They were indeed ubiquitous.

But there are other ways in which they are ubiquitous. Every complete local domain or finitely generated domain over a field is birationally a complete intersection; hence up to integral closure every such ring is Gorenstein.

Every Cohen-Macaulay ring R with a canonical module ω is actually a Gorenstein ring up to nilpotents by using the idealization idea of Nagata. Specifically, we form a new ring consisting of 2 by 2 matrices whose lower left hand corner is 0, whose diagonal is a constant element of R , and whose upper left component is an arbitrary element of ω_R . Pictorially, elements look like

$$\begin{pmatrix} r & y \\ 0 & r \end{pmatrix}$$

where $r \in R$ and $y \in \omega_R$. It is easy to see this ring S is commutative, ω is an ideal of S , $S/\omega \cong R$, and $\omega^2 = 0$. What is amazing is that S is Gorenstein! Up to 'radical' this transfers questions about Cohen-Macaulay rings to questions about Gorenstein rings.

Even if a local ring R is not Cohen-Macaulay, we can still closely approximate it by a Gorenstein ring if it is the homomorphic image of a regular local ring S (e.g. in the geometric case or the complete case). Write $R = S/I$ and choose a maximal regular sequence $f_1, \dots, f_g \in I$ where $g = \text{height}(I)$. Then the ring $T = S/(f_1, \dots, f_g)$ is Gorenstein, being a complete intersection, T maps onto R , and $\dim(T) = \dim(R)$.

Regular rings are the most basic rings in the study of commutative rings. However, Gorenstein rings are the next most basic, and as the examples above demonstrate, one can approximate arbitrary local commutative rings quite closely by Gorenstein rings. Moreover, the module theory over Gorenstein rings is as close to that of regular rings as one might hope.

³A partial converse to Murthy's result was given by Ulrich [U]. His construction uses the theory of liaison, or linkage, in which Gorenstein ideals play an extremely important role. See for example, [PS2] or [HU1].

Recall that finitely generated modules over a regular local ring are of finite projective dimension. This characterizes regular local rings, so we cannot hope to achieve this over non-regular rings. However there are two ‘approximation’ theorems for finitely generated modules over Gorenstein local rings which show that up to Cohen-Macaulay modules, we can still approximate such modules by modules of finite projective dimension. The first theorem goes back to Auslander and Bridger [AB].

THEOREM 11.1. *Let (R, m) be a Gorenstein local ring, and let M be a finitely generated R -module of dimension equal to the dimension of R . Then there exists an exact sequence,*

$$0 \rightarrow C \rightarrow M \oplus F \rightarrow Q \rightarrow 0$$

where F is finitely generated and free, C is a Cohen-Macaulay module of maximal dimension, and Q is a module of finite projective dimension.

A second and similar theorem in spirit is due to Auslander and Buchweitz [ABu]:

THEOREM 11.2. *Let (R, m) be a Gorenstein local ring, and let M be a finitely generated R -module. Then there is an exact sequence*

$$0 \rightarrow Q \rightarrow C \rightarrow M \rightarrow 0$$

such that C is a finitely generated Cohen-Macaulay module of maximal dimension and Q has finite projective dimension.

To put this in context, any Cohen-Macaulay module of maximal dimension over a regular local ring is free, so both of these results recover that over a regular local ring finitely generated modules have finite projective dimension. The point is that over a Gorenstein ring, the study of modules often reduces to the study of modules of finite projective dimension and Cohen-Macaulay modules of maximal dimension. This is certainly the best one can hope for.

The modules C in the (11.2) are called Cohen-Macaulay approximations of M and are a topic of much current interest. Theorem 11.2 can be thought of as a generalization of an argument known as Serre’s trick, which he used in the study of projective modules. Let M be a finitely generated module over a ring R and choose generators for $\text{Ext}_R^1(M, R)$. If there are n such generators, then one can use the Yoneda definition of Ext to create an exact sequence $0 \rightarrow R^n \rightarrow N \rightarrow M \rightarrow 0$ where $\text{Ext}_R^1(N, R) = 0$. The point is after dualizing the dual of R^n maps onto $\text{Ext}_R^1(M, R)$. It turns out one can continue this process and at the next stage obtain a short exact sequence, $0 \rightarrow P \rightarrow N_1 \rightarrow M \rightarrow 0$ where now $\text{Ext}_R^1(N_1, R) = \text{Ext}_R^2(N_1, R) = 0$ and P has projective dimension at most 1. Continuing until the dimension d of R we eventually obtain a sequence $0 \rightarrow Q \rightarrow C \rightarrow M \rightarrow 0$ where Q has finite projective dimension and $\text{Ext}_R^1(C, R) = \text{Ext}_R^2(C, R) = \dots = \text{Ext}_R^d(C, R) = 0$. If R is assumed to be local and Gorenstein, the duality forces C to be Cohen-Macaulay, which gives Theorem 11.2.

Gorenstein rings are now part of the basic landscape of mathematics. A search for the word ‘Gorenstein’ in MathSci Net reveals around 1,000 entries. There are many offshoots which were not mentioned in this article. The ubiquity paper and all of its manifestations are indeed ubiquitous!

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