An aperiodic set of 13 Wang tiles *

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Abstract

A new aperiodic tile set containing only 13 tiles over 5 colors is presented. Its construction is based on a technique recently developed by J. Kari. The tilings simulate behavior of sequential machines that multiply real numbers in balanced representations by real constants.

1 Introduction

Wang tiles are unit square tiles with colored edges. A *tile set* is a finite set of Wang tiles. We consider *tilings* of the infinite Euclidean plane using arbitrarily many copies of the tiles in the given tile set. The tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiles may not be rotated. A tiling is *valid* if everywhere the contiguous edges have the same color.

Let T be a finite tile set, and $f : \mathbb{Z}^2 \to T$ a tiling. Tiling f is *periodic* with period $(a, b) \in \mathbb{Z}^2 - \{(0, 0)\}$ iff $f(x, y) = f(x + a, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. If there exists a periodic valid tiling with tiles of T, then there exists a *doubly periodic* valid tiling, i.e. a tiling f such that, for some $a, b > 0$, $f(x, y) = f(x + a, y) = f(x, y + b)$ for all $(x, y) \in \mathbb{Z}^2$. A tile set T is called *aperiodic* iff (i) there exists a valid tiling, and (ii) there does not exist any periodic valid tiling.

R. Berger in his well known proof of the undecidability of the tiling problem [2] refuted Wang's conjecture that no aperiodic set exists, and constructed the first aperiodic set containing 20426 tiles, he shortly reduced it to 104 tiles. Between 1966 and 1978 progressively smaller aperiodic sets were found by Knuth, Läuchli, Robinson, Penerose and finally a set of 16 tiles by R. Ammann. An excellent discussion of these and related results is in chapters 10 and 11 of [3]. There was no further progress until recently, when J. Kari developed a completely new method of constructing aperiodic sets. His method also provides short and elegant correctness arguments, that is proofs that the constructed set admits a tiling but admits no periodic one. He used his method to construct an aperiodic set consisting of 14 tiles over 6 colors. We will use his method with an additional small trick to improve this to 13 tiles over 5 colors.

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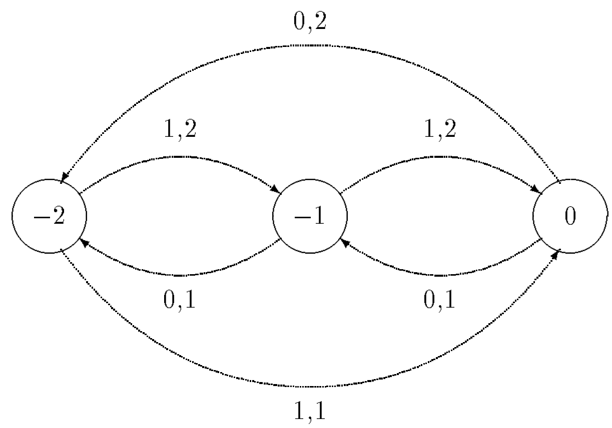


Figure 1: Sequential machine M_3.

2 Balanced representation of numbers

For an arbitrary real number r we denote by $\lfloor r \rfloor$ the integer part of r, i.e. the largest integer that is not greater than r, and by $\{r\}$ the fractional part $r - \lfloor r \rfloor$. In proving that our tile set can be used to tile the plane we use *Beatty sequences* of numbers. Given a real number α its bi-infinite Beatty sequence is the integer sequence $A(\alpha)$ consisting of the integral parts of the multiples of α. In other words, for all $i \in \mathbb{Z}$,

$$A(\alpha)_i = \lfloor i \cdot \alpha \rfloor.$$

Beatty sequences were introduced by S.Beatty [1] in 1926.

We use sequences obtained by computing the differences of consecutive elements of Beatty sequences. Define, for every $i \in \mathbb{Z}$,

$$B(\alpha)_i = A(\alpha)_i - A(\alpha)_{i-1}.$$

The bi-infinite sequence $B(\alpha)_i$ will be called the *balanced* representation of α. The balanced representations consist of at most two different numbers: If $k \le \alpha \le k+1$ then $B(\alpha)$ is a sequence of k's and $(k+1)$'s. Moreover, the averages over finite subsequences approach α as the lengths of the subsequences increase. In fact, the averages are as close to α as they can be: The difference between $l \cdot \alpha$ and the sum of any l consecutive elements of $B(\alpha)$ is always smaller than one.

For example,

$$B(1.5) = \ldots, 121212\ldots, \; B(\tfrac{1}{3}) = \ldots 001001\ldots, \text{ and } B(\tfrac{8}{3}) = \ldots 233233\ldots.$$

Now, we introduce sequential machines which define mappings on bi-infinite strings. We will use them to implement multiplication of numbers in balanced representation and later show that they are isomorphic to set of tiles.

A *sequential machine* is a 4-tuple $M = (K, \Sigma, \Delta, \gamma)$ where K is the set of states, Σ is the input alphabet, Δ is the output alphabet, and $\gamma \subseteq K \times \Sigma \times \Delta \times K$ is the transition set. Sequential machine M can be represented by a labeled directed graph with nodes K and an edge from node q to node p labeled a, b for each transition (q, a, b, p) in γ.

Machine M computes a relation $\rho(M)$ between bi-infinite sequences of letters. A bi-infinite sequence x over set S is a function $x : \mathbb{Z} \to S$. We will abbreviate $x(i)$ by x_i. Bi-infinite sequences x and y over input and output alphabets, respectively, are in relation $\rho(M)$ if and only if there is a bi-infinite sequence s of states of M such that, for every $i \in \mathbb{Z}$, there is a transition from s_{i-1} to s_i labeled by x_i,y_i.

For a given positive rational number $q = \frac{n}{m}$, let us construct a sequential machine (nondeterministic Mealy machine) M_q that multiplies balanced representations $B(\alpha)$ of real numbers by q. The states of M_q will represent all possible values of $q\lfloor r \rfloor - \lfloor qr \rfloor$ for $r \in \mathbb{R}$. Because

$$q\lfloor r \rfloor - 1 \leq qr - 1 < \lfloor qr \rfloor \leq qr < q(\lfloor r \rfloor + 1),$$

we have

$$-q < q\lfloor r \rfloor - \lfloor qr \rfloor < 1.$$

Because the possible values of $q\lfloor r \rfloor - \lfloor qr \rfloor$ are multiples of $\frac{1}{m}$, they are among the $n + m - 1$ elements of

$$S = \{-\frac{n-1}{m}, -\frac{n-2}{m}, \ldots, \frac{m-2}{m}, \frac{m-1}{m}\}.$$

S is the state set of M_q.

The transitions of M_q are constructed as follows: There is a transition from state $s \in S$ with input symbol a and output symbol b into state $s + qa - b$, if such a state exists. If there is no state $s + qa - b$ in S then no transition from s with label a, b is needed. After reading input $\ldots\ B(\alpha)_{i-2}\ B(\alpha)_{i-1}$ and producing output $\ldots\ B(q\alpha)_{i-2}\ B(q\alpha)_{i-1}$, the machine is in state

$$s_{i-1} = qA(\alpha)_{i-1} - A(q\alpha)_{i-1} \in S.$$

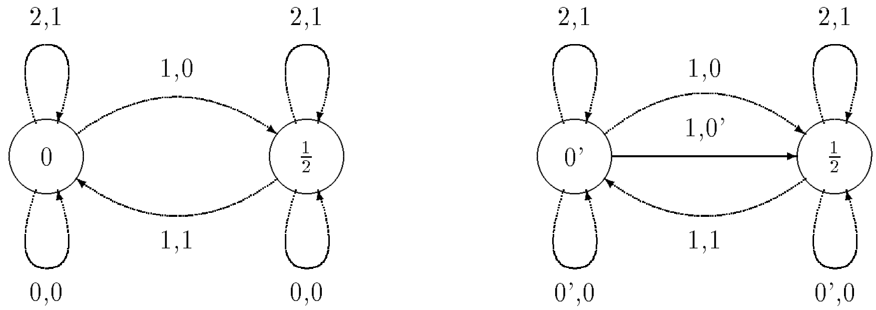
On the next input symbol $B(\alpha)_i$ the machine outputs $B(q\alpha)_i$ and moves to state

$$\begin{aligned} s_{i-1} + qB(\alpha)_i - B(q\alpha)_i &= qA(\alpha)_{i-1} + qB(\alpha)_i - (A(q\alpha)_{i-1} + B(q\alpha)_i) \\ &= qA(\alpha)_i - A(q\alpha)_i \\ &= s_i \in S \end{aligned}$$

The sequential machine was constructed in such a way that the transition is possible. This shows that if the balanced representation $B(\alpha)$ is a sequence of input letters and $B(q\alpha)$ is over output letters, then $B(\alpha)$ and $B(q\alpha)$ are in relation $\rho(M_q)$.

Sequential machine M_3 in Fig. 1 is constructed in this fashion for multiplying by 3, using input symbols $\{0, 1\}$ and output symbols $\{1, 2\}$. This means that $B(\alpha)$ and $B(2\alpha)$ are in relation $\rho(M_2)$ for all real numbers α satisfying $0 \leq \alpha \leq 1$ and $1 \leq 3\alpha \leq 2$, that is, for all $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Similarly, $M_{1/2}$, shown in Fig. 2(a), is constructed for input symbols $\{0, 1, 2\}$ and output symbols $\{0, 1\}$, so that $B(\alpha)$ and $B(\frac{1}{2}\alpha)$ are in relation $\rho(M_{1/2})$ for all $\alpha \in [0, 2]$.

Our intention is to iterate sequential machines M_3 and $M_{\frac{1}{2}}$ without allowing $M_{\frac{1}{2}}$ to be used more than twice in a row. To assure this, we modify $M_{\frac{1}{2}}$ by introducing new input/output symbol $0'$ and changing its diagram to $M'_{\frac{1}{2}}$ as shown in Fig. 2(b). We also change the state 0 to $0'$ to make the sets of states of M_3 and $M'_{\frac{1}{2}}$ disjoint. That allows us to view the union of M_3 and $M'_{\frac{1}{2}}$ as one sequential machine M.



(a) $M_{\frac{1}{2}}$ (b) $M'_{\frac{1}{2}}$

Figure 2: Sequential machines $M_{\frac{1}{2}}$ and $M'_{\frac{1}{2}}$.

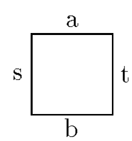


Figure 3: A tile corresponding to the transition $s \xrightarrow{a,b} t$

3 Sequential machines and tile sets

There is a one-to-one correspondence between tile sets and sequential machines which translates the properties of tile sets to properties of computations of sequential machines.

A finite tile set T over set of colors C_{EW} on east-west edges and set of colors C_{NS} on north-south edges is represented by sequential machine $M = (C_{EW}, C_{NS}, C_{NS}, \gamma)$ where $(s, a, b, t) \in \gamma$ iff there is a tile in T whose left, top, bottom and right edges are colored s, a, b and t, respectively, as shown in Fig. 3. Obviously, bi-infinite sequences x and y are in the relation $\rho(M)$ iff there exists a row of tiles, with matching vertical edges, whose upper edges form sequence x and lower edges sequence y. So there is a one-to-one correspondence between valid tilings of the plane, and bi-infinite iterations of the sequential machine on bi-infinite sequences.

Clearly, the two conditions for T being aperiodic can be translated to conditions on computations of M. Clearly, set T is aperiodic if (i) there exists a bi-infinite computation of M, and (ii) there is no bi-infinite word w over C_{NS} such that $(w, w) \in [\rho(M)\,]^+$, where ρ^+ denotes the transitive closure of ρ.

There are a few other techniques that can be used to design small sequential machines computing simple functions on integers in balanced representation. If sequential machine M computes $f(x)$, we can modify it to compute $f(x)+n$ for arbitrary integer n by simply replacing outputs $m, m+1$ by outputs $m+n, m+n+1$, respectively. Another modification is to interchange the outputs m and $m+1$. That will change

$f(x)$ to $2m + 1 - f(x)$. Both these modifications are "free" they do not increase the size of the machine. We tried to use this in building an even smaller aperiodic set but without success.

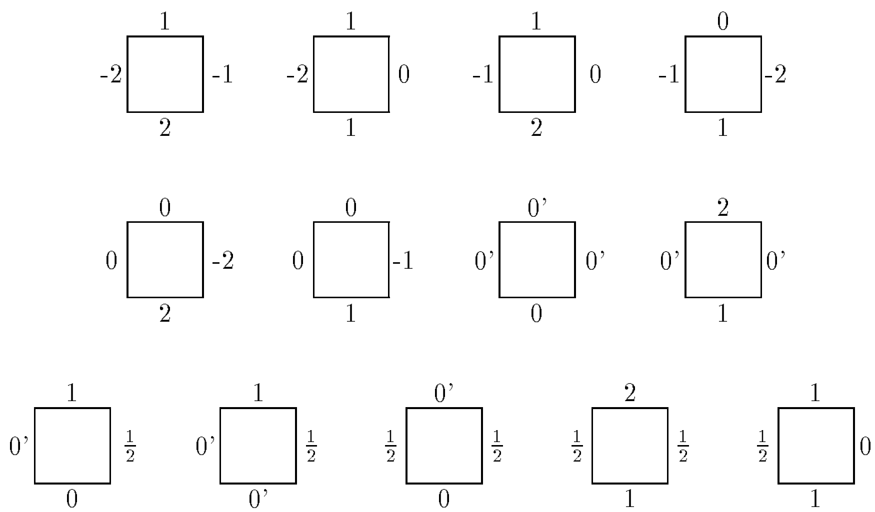


Figure 4: An aperiodic set of 13 Wang tiles

4 Aperiodic sets of tiles

We say that the tile in Fig. 3 multiplies by q if $aq + s = b + t$. In other words, the tile multiplies the number on its upper edge by q, adds the "carry" from the left edge, and splits the result between the lower edge and the "carry" to the right.

Let us denote by T_3 and by $T_{\frac{1}{2}}$ the tile sets representing the sequential machines M_3 and $M'_{\frac{1}{2}}$, respectively. Therefore, T_3 and $T_{\frac{1}{2}}$ multiply by 3 and by 1/2, respectively. The tile set $T = T_3 \cup T_{\frac{1}{2}}$, consisting of 13 tiles, is shown in Fig. 4.

Now, we proceed to prove that T is an aperiodic set of tiles.

Lemma 1 *Tile set T admits uncountably many valid tilings of the plane.*

Proof. From the input sequence $B(\alpha)$ for any $\alpha \in [\frac{1}{3}, 2]$, the sequential machine M computes output $B(3\alpha)$ if $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and output $B(\frac{\alpha}{2})$ if $\alpha \in [\frac{2}{3}, 2]$. In the later case, if $\alpha \in [\frac{4}{3}, 2]$ then output $B(\frac{\alpha}{2}) \in [\frac{2}{3}, 1]$ can be encoded in alphabet $\{0', 1\}$ and the second application of M computes $B(\frac{\alpha}{4}) \in [\frac{1}{3}, \frac{1}{2}]$ represented in alphabet $\{0, 1\}$. In any case, the machine M can be applied again using the previous output as input, and this may be repeated arbitrarily many times.

On the other hand, if $\alpha \in [\frac{1}{3}, 2]$ there is input $B(\frac{\alpha}{3})$ or $B(2\alpha)$, that is in relation $\rho(M)$ with $B(\alpha)$. Input sequence $B(\frac{\alpha}{3})$ is used for $\alpha \geq 1$, and $B(2\alpha)$ for $\alpha \leq 1$. This can be repeated many times so M can be iterated also backwards. Hence, for every bi-infinite $B(\alpha), \alpha \in [\frac{1}{3}, 2]$, there is a bi-infinite iteration yielding a tiling of the plane. □

Lemma 2 *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f : \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period a and vertical period b. The colors on the vertical edges of the tiles in T_3 and $T_{1/2}$ are disjoint from each other, so on each row of tiles all tiles are from the same set T_x.

It is easy to see that the tile set $T_{1/2}$ alone cannot tile the plane: Label 0 does not appear on the upper edge of any tile in $T_{1/2}$, so tiles with 0 on the lower edge cannot be used. Similarly, tiles with label 2 are not used because 2 appears only on the upper edges of tiles. Clearly the remaining two tiles do not admit a valid tiling.

We can assume without loss of generality that in row b the tiles are from T_3. Let n_i denote the sum of colors on the upper edges of tiles $f(1,i), f(2,i), \ldots, f(a,i)$. Because the tiling is horizontally periodic with period a, the "carries" on the left edge of $f(1,i)$ and the right edge of $f(a,i)$ are equal.

Therefore $n_{i-1} = q_i n_i$, where $q_i = 3$ if tiles from T_3 are used in row i and $q_i = \frac{1}{2}$ if tiles from $T_{1/2}$ are used. Because the vertical period of tiling is b,

$$n_b = n_0 = q_1 q_2 \ldots q_b \cdot n_b \ .$$

Since tiles from T_3 are used for $i = b$, there are no 0's on the upper edges of the b'th row and thus $n_b \neq 0$. Hence, $q_1 q_2 \ldots q_b = 1$. This contradicts the fact that no nonempty product of 3's and $\frac{1}{2}$'s can be 1. □

Corollary. *The tile set T is aperiodic.*

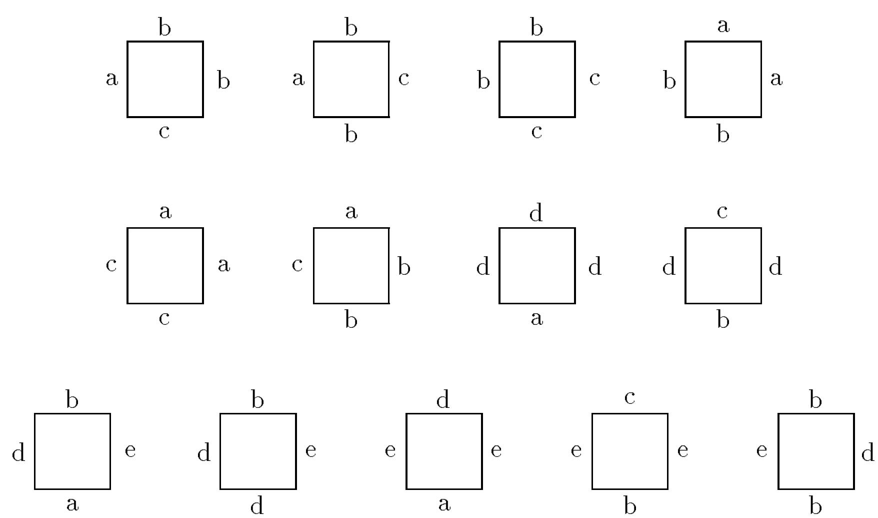


Figure 5: An aperiodic set of 13 tiles over 5 colors

Since no rotation of given Wang tiles is allowed when coloring the plane, we can obviously replace the colors in one tile set by the max of the number of states and the number of input/output symbol, that is $\max(5,4) = 5$. One of the aperiodic sets which we obtain is shown in Fig. 5.

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